THE DISCRETE-TIME FOURIER TRANSFORM

1. Lecture 4 - 44 minutes

Frequency Response

$e^{j\omega n}\longrightarrow H(e^{j\omega})e^{j\omega n}$

$H(e^{j\omega})=\sum_{n=-\infty}^{+\infty}h(n)e^{-j\omega n}$

Properties.
1) fcn. of continuous variable ω

2) periodic - period 2π

$H(e^{j\omega})=H(e^{j(\omega+2\pi k)})$

$h(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}H(e^{j\omega})e^{j\omega n}d\omega$

$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_k h(k)e^{-j\omega k}\right]e^{j\omega n}d\omega$

$=\frac{1}{2\pi}\sum_k h(k)\int_{-\pi}^{\pi}e^{j\omega(n-k)}d\omega$

$\quad n\neq k \quad 0$
$\quad n=k \quad 2\pi$

$=h(n)$

Fourier Transform

$X(e^{j\omega})=\sum_{n=-\infty}^{+\infty}x(n)e^{-j\omega n}$

$x(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})e^{j\omega n}d\omega$

$=\lim_{\Delta\omega\to 0}\sum_k\left[X(e^{jk\Delta\omega})\frac{\Delta\omega}{2\pi}\right]e^{jk\Delta\omega n}$

Convolution Property

$x(n)*h(n)\longrightarrow X(e^{j\omega})H(e^{j\omega})$

a.

$\longrightarrow\boxed{LSI}\longrightarrow$

$e^{j\omega_0 n}\longrightarrow H(e^{j\omega_0})e^{j\omega_0 n}$

$\sum_k A_k e^{j\omega_k n}$
$\longrightarrow\sum_k A_k H(e^{j\omega_k})e^{j\omega_k n}$

$x(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})e^{j\omega n}d\omega$

$\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})H(e^{j\omega})e^{j\omega n}d\omega=y(n)$

$Y(e^{j\omega})=X(e^{j\omega})H(e^{j\omega})$

Symmetry Properties
$x(n)$ real

$X(e^{j\omega})=X^*(e^{-j\omega})$

$X(e^{j\omega})=\sum_{n=-\infty}^{+\infty}x(n)e^{-j\omega n}$

$X^*(e^{-j\omega})=\sum_{n=-\infty}^{+\infty}\underset{x(n)}{x^*(n)}e^{-j\omega n}$

$X(e^{j\omega})=X_R(e^{j\omega})+jX_I(e^{j\omega})$

$X^*(e^{-j\omega})=X_R(e^{j\omega})-jX_I(e^{j\omega})$

$X_R(e^{j\omega})=X_R(e^{-j\omega})$ even

$X_I(e^{j\omega})=-X_I(e^{-j\omega})$ odd

$|X(e^{j\omega})|$ even

$\angle X(e^{j\omega})$ odd

b.

$x_A(t)\boxed{\times T}\tilde{x}_A(t)$

$\tilde{x}_A(t)=x_A(t)\cdot\sum_{-\infty}^{+\infty}u_o(t-nT)$

$=\sum_{-\infty}^{+\infty}x_A(nT)u_o(t-nT)$

$\tilde{X}_A(j\Omega)=X_A(j\Omega)*P(j\Omega)$

$=\frac{1}{T}\sum_{r=-\infty}^{+\infty}X_A(j\Omega+j\frac{2\pi r}{T})$

$\tilde{X}_A(j\Omega)=\int_{-\infty}^{+\infty}\tilde{x}_A(t)e^{-j\Omega t}dt$

$=\sum_{n=-\infty}^{+\infty}x_A(nT)\int_{-\infty}^{+\infty}e^{-j\Omega t}u_o(t-nT)dt$

$=\sum_{n=-\infty}^{+\infty}x_A(nT)e^{-jn\Omega T}$

$\tilde{x}_A(t)\longrightarrow\boxed{C|D}\longrightarrow x(n)=x_A(nT)$

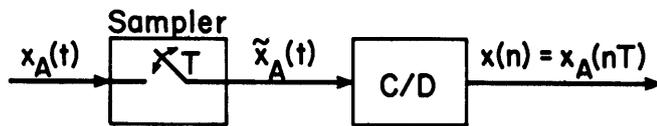$\tilde{X}_A(j\Omega)=\frac{1}{T}\sum_{r=-\infty}^{+\infty}X_A(j\Omega+\frac{2\pi r}{T})$

$=\sum_{n=-\infty}^{+\infty}x_A(nT)e^{-jn\Omega T}$

$X(e^{j\omega})=\sum_n x_A(nT)e^{-j\omega n}$

$X(e^{j\omega})=\tilde{X}_A(j\Omega)\big|_{\Omega T=\omega}$
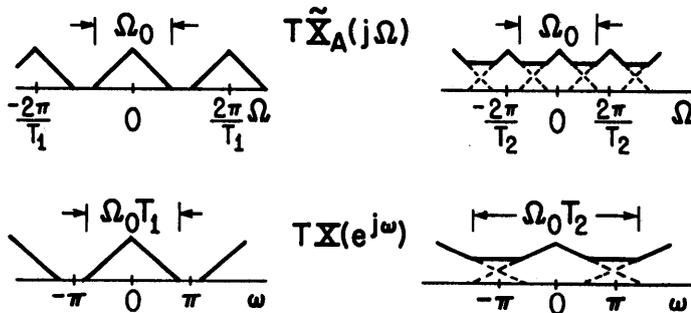
$=\frac{1}{T}\sum_{n=-\infty}^{+\infty}X_A\left(j\frac{\omega}{T}+j\frac{2\pi r}{T}\right)$

c.

Sampler

$x_A(t) \to$ [×T] $\to \tilde{x}_A(t) \to$ [C/D] $\to x(n) = x_A(nT)$

Relationship in the time domain for a continous-time signal, its samples, and the resulting sequence

$T = T_1$

$T = T_2 = 2T_1$

d.

$T = T_1$ $X_A(j\Omega)$ $T = T_2 = 2T_1$
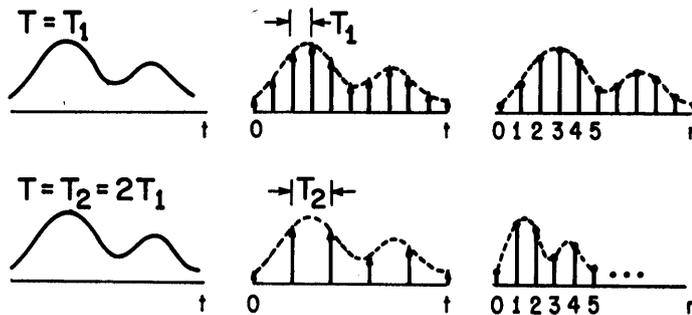
Relationship among the Fourier transforms of a continous-time signal, its samples, and the resulting sequence.

$T\tilde{X}_A(j\Omega)$

$TX(e^{j\omega})$

e.

2. Comments

In the previous lecture we developed the notion of the frequency response of LSI systems and discussed the determination of the frequency response from the unit sample response. We begin this lecture by developing an inverse relation by interpreting the expression for the frequency response as a Fourier series expansion, the coefficients for which are the unit sample response values. We then consider the generalization of the frequency response representation of LSI systems to the Fourier transform representation of sequences. Throughout this set of lectures we will exploit various Fourier transform properties. Many of these properties are derived in a similar manner and it is useful to understand the style in which they are

derived rather than simply memorizing the properties. In the lecture the convolution property and some symmetry properties were discussed. A number of other important properties are presented in the text and should be carefully reviewed.

The lecture concludes with a discussion of the relationships between continuous-time and discrete-time Fourier transforms. In particular you should be aware from your background in continuous-time linear system theory of the form of the Fourier transform of a sampled time function. Converting a sampled time function to a sequence introduces in essence a "time" normalization since the spacing of sequence values is always interpreted to be unity. Equivalently this time normalization corresponds in the Fourier domain to a frequency normalization.

3. Reading

Text: Review section 2.6 (page 39). Read sections 2.7, 2.8, 2.9 and sections 3.1 through 3.5.

4. Problems

Problem 4.1

Determine the Fourier transform of each of the sequences below:
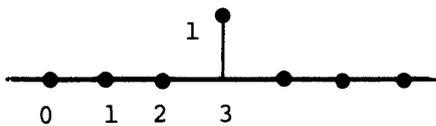
(a)

$$x(n) = \delta(n - 3)$$

Figure P4.1-1

(b)

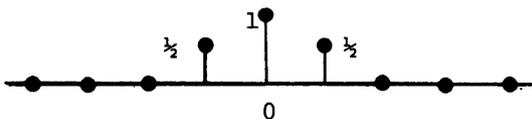$$x(n) = \frac{1}{2} \delta(n + 1) + \delta(n) + \frac{1}{2} \delta(n - 1)$$

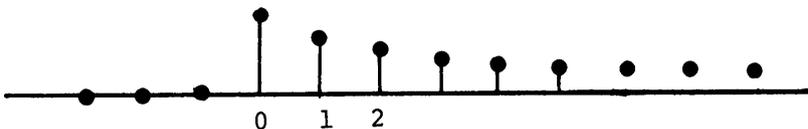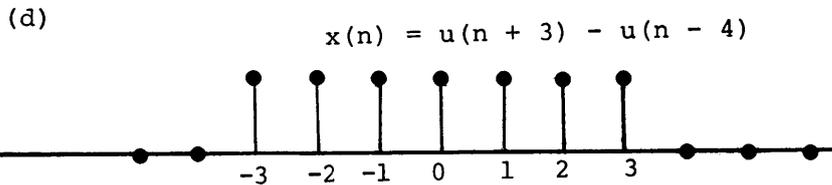Figure P4.1-2

(c)

$$x(n) = a^n u(n) \quad 0 < a < 1$$

Figure P4.1-3

(d)

$$x(n) = u(n + 3) - u(n - 4)$$

Figure P4.1-4

Problem 4.2

(a) Consider a linear shift-invariant system with unit-sample response $h(n) = \alpha^n u(n)$, where α is real and $0 < \alpha < 1$. If the input is $x(n) = \beta^n u(n)$, $0 < |\beta| < 1$, determine the output $y(n)$ in the form $y(n) = (k_1\alpha^n + k_2\beta^n) u(n)$ by explicitly evaluating the convolution sum.

(b) By explicitly evaluating the transforms $X(e^{j\omega})$, $H(e^{j\omega})$ and $Y(e^{j\omega})$ corresponding to $x(n)$, $h(n)$, and $y(n)$ specified in part (a), show that

$$Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega})$$

Problem 4.3

Let $x(n)$ and $X(e^{j\omega})$ represent a sequence and its transform. Do not assume that $x(n)$ is real or that $x(n)$ is zero for $n < 0$. Determine in terms of $X(e^{j\omega})$ the transform of each of the following:

(a) $k\, x(n)$

(b) $x(n - n_0)$ where n_0 is an integer

(c) $n\, x(n)$

Problem 4.4

Let $h_a(t)$ denote the impulse response of a linear time-invariant continuous-time filter and $h_d(n)$ the unit-sample response of a linear shift-invariant discrete-time filter. $h_a(t)$ is given by

$$h_a(t) = \begin{cases} ae^{-at} & t \geq 0 \qquad a > 0 \\ 0 & t < 0 \end{cases}$$

(a) Determine the analog filter frequency response and sketch its magnitude.

(b) If $h_d(n) = c\, h_a(nT)$, determine the digital filter frequency response and determine the value of c so that the digital filter has unity gain at $\omega = 0$. Sketch the magnitude of $H_d(e^{j\omega})$.

4.4

Problem 4.5

One context in which digital filters are frequently used is in filtering bandlimited analog data as shown below:
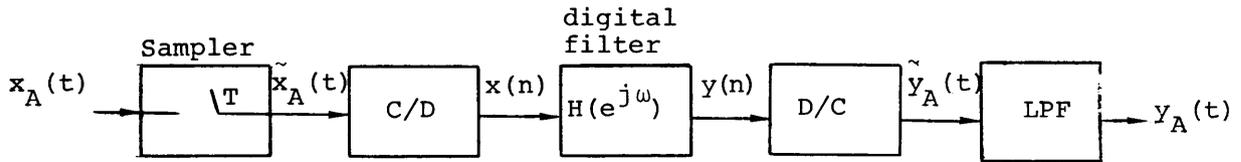

Figure P4.5-1

Assume that the sampling period T is sufficiently small to prevent aliasing. The system labelled D/C is a discrete to continuous time converter, i.e. the inverse of a continuous to discrete time converter. The system labelled LPF is an ideal lowpass filter with a gain of T in the passband and cutoff frequency of $\frac{\pi}{T}$ rad/sec. The overall system is equivalent to a continous-time filter. Indicated below are two choices for T and the digital filter frequency response $H(e^{j\omega})$. For each of these, sketch the frequency response of the overall continuous-time system.
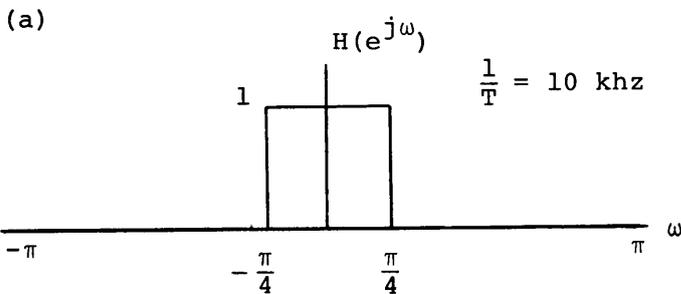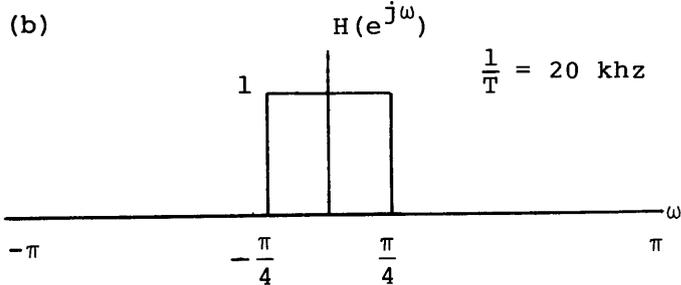
(a)

Figure P4.5-2

(b)

Figure P4.5-3

Problem 4.6[*]

In Sec. 1.6 of the text, a number of symmetry properties of the Fourier transform are stated. All these properties follow in a relatively straight-forward way from the transform pair. Following is a list of some of the

properties stated. Prove that each is true. In carrying out the proof you may use the definition of the transform pair as given in Eqs. (1.19) and (1.20) of the text and any previous property in the list. For example, in proving property 3 you may use properties 1 and 2.

Sequence	Fourier Transform
1. $x^*(n)$	$X^*(e^{-j\omega})$
2. $x^*(-n)$	$X^*(e^{j\omega})$
3. Re $x(n)$	$X_e(e^{j\omega})$
4. j Im $[x(n)]$	$X_0(e^{j\omega})$
5. $x_e(n)$	Re $[X(e^{j\omega})]$
6. $x_0(n)$	j Im $[X(e^{j\omega})]$

Problem 4.7*

$f(n)$ and $g(n)$ are causal and stable sequences with Fourier transforms $F(e^{j\omega})$ and $G(e^{j\omega})$, respectively. Show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{j\omega})\ G(e^{j\omega})\ d\omega = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{j\omega})\ d\omega\right\}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})\ d\omega\right\}$$

Problem 4.8

In the design of either analog or digital filters, we often approximate a specified magnitude characteristic, without particular regard to the phase. For example, standard design techniques for lowpass and bandpass filters are derived from consideration of the magnitude characteristics only.

In many filtering problems, one would ideally like the phase characteristics to be zero or linear. For causal filters it is impossible to have zero phase. However, for many digital filtering applications, it is not necessary that the unit-sample response of the filter be zero for n < 0 if the processing is not to be carried out in real time.

One technique commonly used in digital filtering when the data to be filtered are of finite duration and stored, for example, on a disc or magnetic tape, is to process the data forward and then backward through the same filter.

Let h(n) be the unit-sample response of a causal filter with an arbitrary phase characteristic. Assume that h(n) is real and denote its Fourier transform by $H(e^{j\omega})$. Let x(n) be the data we want to filter. The filtering operation is performed as follows.

(a) Method A:

i.

$x(n)$ → [$h(n)$] → $g(n)$

ii.

$g(-n)$ → [$h(n)$] → $r(n)$
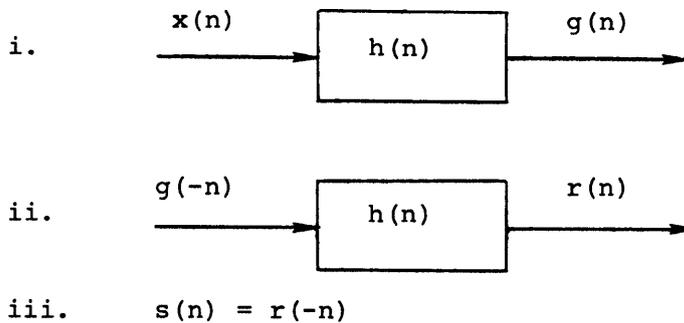
iii. $s(n) = r(-n)$

Figure P4.8-1

(1) Determine the overall unit-sample response $h_1(n)$ that relates $x(n)$ and $s(n)$, and show that it has a zero-phase characteristic.

(2) Determine $|H_1(e^{j\omega})|$ and express it in terms of $H(e^{j\omega})$ and $\arg[H(e^{j\omega})]$.

(b) Method B: Process $x(n)$ through the filter $h(n)$ to get $g(n)$. Also process $x(n)$ backward through $h(n)$ to get $r(n)$. The output $y(n)$ is then taken as the sum $g(n)$ plus $r(-n)$.

i.

$x(n)$ → [$h(n)$] → $g(n)$

ii.

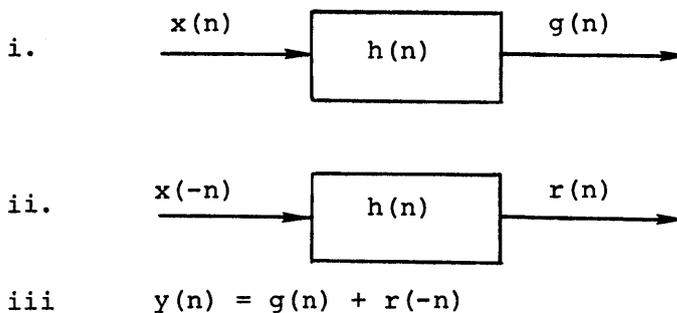$x(-n)$ → [$h(n)$] → $r(n)$

iii $y(n) = g(n) + r(-n)$

Figure P4.8-2

This composite set of operations can be represented by a filter, with input $x(n)$, output $y(n)$, and unit-sample response $h_2(n)$.

(1) Show that the composite filter $h_2(n)$ has a zero-phase characteristic.
(2) Determine $|H_2(e^{j\omega})|$ and express it in terms of $|H(e^{j\omega})|$ and $\arg[H(e^{j\omega})]$.

c. Suppose that we are given a sequence of finite duration, on which we would like to perform a bandpass, zero-phase filtering operation. Furthermore, assume that we are given the bandpass filter $h(n)$, with frequency response as specified in Figure P4.8-3, which has the magnitude characteristic that we desire but linear phase. To achieve zero phase, we could use either method (A) or (B). Determine and sketch $|H_1(e^{j\omega})|$ and $|H_2(e^{j\omega})|$. From these results which method would you

use to achieve the desired bandpass filtering operation? Explain why.
More generally, if h(n) has the desired magnitude but a nonlinear phase
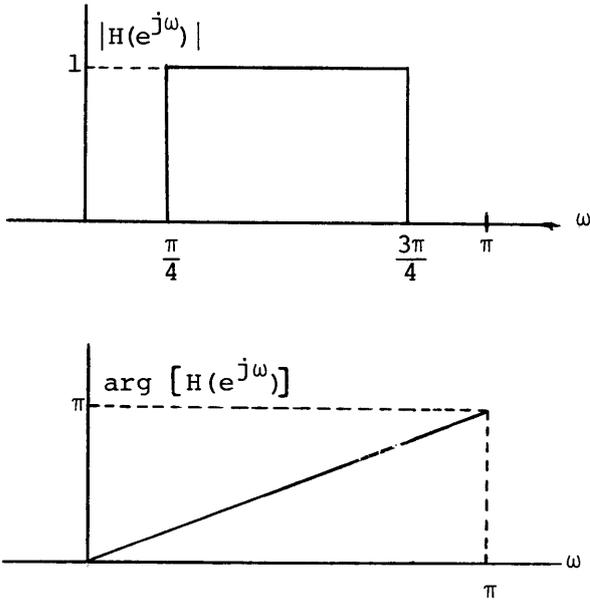characteristic which method is preferable to achieve a zero phase
characteristic?

Figure P4.8-3

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