DESIGN OF IIR DIGITAL FILTERS - PART 1

Solution 14.1

(a) Applying the Laplace transform to both sides of the differential equation we obtain

$$Y_a(s) \left[s + 0.9 \right] = X_a(s)$$

or

$$\frac{Y_a(s)}{X_a(s)} = \frac{1}{s + 0.9} = H(s)$$

Thus

$$H_a(j\Omega) = \frac{1}{j\Omega + 0.9}$$

and

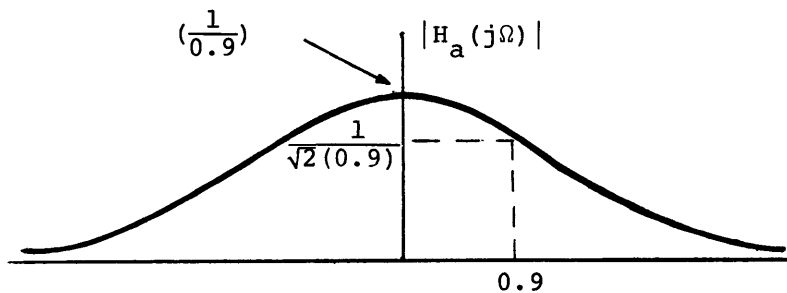$$\left| H_a(j\Omega) \right|^2 = \frac{1}{\Omega^2 + (0.9)^2}$$

Figure S14.1-1

(b) Applying the z-transform to both sides of the difference equation we obtain

$$Y(z) \left[\frac{z - 1}{T} \right] + 0.9 \ Y(z) = X(z)$$

or

$$H(z) = \frac{Y(z)}{X(z)} = T \Big/ \left[z + (0.9T - 1) \right]$$

with $T = \frac{10}{9}$,

$$H(z) = \frac{10/9}{z} = \frac{10}{9} z^{-1}$$

and

$$\left| H(e^{j\omega}) \right| = \frac{10}{9}$$

For this particular choice for T the frequency response is constant, independent of frequency in contrast to the analog filter, which is a lowpass filter. While this is a particularly severe example of the effect of transforming an analog filter to a digital filter by replacing derivatives by differences, it emphasizes the fact that the frequency response of the resulting digital filter will in general be severely distorted from that of the original analog filter.

(c) From the system function determined in part (b), the pole is at $z = 1 - 0.9T$. Assuming T to be positive, the pole is outside of the unit circle for $T \geq 20/9$.

Solution 14.2

$$h(n) = h_a(nT) = e^{-0.9nT} u(n)$$

$$= \left(e^{-0.9T}\right)^n u(n)$$

Thus

$$H(z) = \frac{1}{1 - e^{-0.9T}z^{-1}}$$

The frequency response is given by

$$H(e^{j\omega}) = \frac{1}{1 - e^{-0.9T}e^{-j\omega}}$$

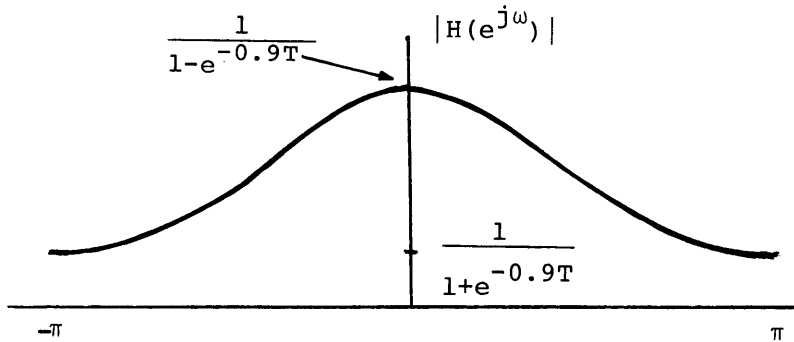The magnitude of which is sketched in figure S14.2-1.

Figure S14.2-1

Thus the digital filter approximates a lowpass filter. Since the pole is located at $z = e^{-0.9T}$ the pole is inside the unit circle and hence the system is stable for $T > 0$.

Solution 14.3

(a) The step response of the analog filter is the integral of its impulse response, i.e.

S14.2

$$s_a(t) = \int_{-\infty}^{t} h_a(\tau)\, d\tau$$

Hence for $t < 0$ $s_a(t) = 0$ and for $t \geq 0$

$$s_a(t) = \int_{0}^{t} e^{-0.9\tau}\, d\tau = \frac{1}{0.9} \left[1 - e^{-0.9t} \right]$$

(b) $s(n) = s_a(nT) = \frac{1}{0.9} \left[1 - e^{-0.9nT} \right] u(n)$

(c) Let $S(z)$ denote the z-transform of $s(n)$. Then, since the z-transform of a unit step is $\dfrac{1}{1 - z^{-1}}$, $H(z)$ is given by

$$H(z) = (1 - z^{-1})\, S(z)$$

From (b),

$$S(z) = \frac{1}{0.9} \left[\frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-0.9T}\, z^{-1}} \right]$$

and

$$H(z) = \frac{1}{0.9} \left[1 - \frac{(1 - z^{-1})}{1 - e^{-0.9T}\, z^{-1}} \right]$$

$$= \frac{1}{0.9} \left[\frac{z^{-1}(1 - e^{-0.9T})}{1 - e^{-0.9T}\, z^{-1}} \right]$$

Solution 14.4

The most straightforward procedure is to expand $H_a(s)$ in a partial-fraction expansion and utilize the relationship between $H_a(s)$ and $H(z)$ indicated on chalkboard (c) of lecture 14. Thus,

$$H_a(s) = \frac{A_1}{s + 1} + \frac{A_2}{s + 2}$$

where

$$A_1 = H_a(s)(s + 1) \Big|_{(s = -1)} = -1$$

$$A_2 = H_a(s)(s + 2) \Big|_{(s = -2)} = 2$$

Then

$$H(z) = \frac{-1}{1 - e^{-T}\, z^{-1}} + \frac{2}{1 - e^{-2T}\, z^{-1}} = \frac{1 + z^{-1}(e^{-2T} - 2e^{-T})}{(1 - e^{-T}\, z^{-1})(1 - e^{-2T}\, z^{-1})}$$

Solution 14.5[*]

(a) Since $\Omega_c < \frac{\pi}{T}$ there is no aliasing introduced in obtaining the digital filter from the analog filter. Thus the frequency response $H_d(e^{j\omega})$ of

the digital filter is most easily obtained using equation 5.8. Thus

$$H_d(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} H_a(\frac{j\omega}{T} + j\frac{2\pi}{T}k)$$

and since

$$H_a(j\Omega) = 0 \qquad\qquad |\Omega| \geq \frac{\pi}{T} \, ,$$

$$\begin{aligned}
H_a(e^{j\omega}) &= \frac{1}{T} H_a(\frac{j\omega}{T}) & |\omega| &\leq \pi \\
&= \frac{1}{T} (\frac{j\omega}{T}) \, e^{-j\frac{\omega}{T}\tau} & |\omega| &\leq \Omega_c T \\
&= \frac{1}{T^2} \, j\omega \, e^{-j\frac{\omega}{T}\tau} & |\omega| &\leq \Omega_c T
\end{aligned}$$

We note that this has the same shape as the analog frequency response (because there is no aliasing) but with a scaling of the frequency axis.

(b) If $h_d(n) = \hat{h}_d(n - n_\tau)$, $H_d(e^{j\omega})$ and $\hat{H}_d(e^{j\omega})$ are related by

$$H_d(e^{j\omega}) = e^{-j\omega n_\tau} \hat{H}_d(e^{j\omega})$$

From (a) it follows, then, that $n_\tau = \frac{\tau}{T}$. For n_τ an integer, τ must be an integer multiple of the sampling period T.

MIT OpenCourseWare
http://ocw.mit.edu

Resource: Digital Signal Processing
Prof. Alan V. Oppenheim

The following may not correspond to a particular course on MIT OpenCourseWare, but has been provided by the author as an individual learning resource.

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.