9

Fourier Transform Properties

The Fourier transform is a major cornerstone in the analysis and representation of signals and linear, time-invariant systems, and its elegance and importance cannot be overemphasized. Much of its usefulness stems directly from the properties of the Fourier transform, which we discuss for the continuous-time case in this lecture. Many of the Fourier transform properties might at first appear to be simple (or perhaps not so simple) mathematical manipulations of the Fourier transform analysis and synthesis equations. However, in this and later lectures, as we discuss issues such as filtering, modulation, and sampling, it should become increasingly clear that these properties all have important interpretations and meaning in the context of signals and signal processing.

The first property that we introduce in this lecture is the symmetry property, specifically the fact that for time functions that are real-valued, the Fourier transform is conjugate symmetric, i.e., $X(-\omega) = X^*(\omega)$. From this it follows that the real part and the magnitude of the Fourier transform of real-valued time functions are even functions of frequency and that the imaginary part and phase are odd functions of frequency. Because of this property of conjugate symmetry, in displaying or specifying the Fourier transform of a real-valued time function it is necessary to display the transform only for positive values of ω.

A second important property is that of time and frequency scaling, specifically that a linear expansion (or contraction) of the time axis in the time domain has the effect in the frequency domain of a linear contraction (expansion). In other words, linear scaling in time is reflected in an inverse scaling in frequency. As we discuss and demonstrate in the lecture, we are all likely to be somewhat familiar with this property from the shift in frequencies that occurs when we slow down or speed up a tape recording. More generally, this is one aspect of a broader set of issues relating to important trade-offs between the time domain and frequency domain. As we will see in later lectures, for example, it is often desirable to design signals that are both narrow in time and narrow in frequency. The relationship between time and frequency scaling is one indication that these are competing requirements; i.e., attempting to make a signal narrower in time will typically have the effect of broadening its Fourier transform.

Duality between the time and frequency domains is another important property of Fourier transforms. This property relates to the fact that the analysis equation and synthesis equation look almost identical except for a factor of $1/2\pi$ and the difference of a minus sign in the exponential in the integral. As a consequence, if we know the Fourier transform of a specified time function, then we also know the Fourier transform of a signal whose functional form is the same as the form of this Fourier transform. Said another way, the Fourier transform of the Fourier transform is proportional to the original signal reversed in time. One consequence of this is that whenever we evaluate one transform pair we have another one for free. As another consequence, if we have an effective and efficient algorithm or procedure for implementing or calculating the Fourier transform of a signal, then exactly the same procedure with only minor modification can be used to implement the inverse Fourier transform. This is in fact very heavily exploited in discrete-time signal analysis and processing, where explicit computation of the Fourier transform and its inverse play an important role.

There are many other important properties of the Fourier transform, such as Parseval's relation, the time-shifting property, and the effects on the Fourier transform of differentiation and integration in the time domain. The time-shifting property identifies the fact that a linear displacement in time corresponds to a linear phase factor in the frequency domain. This becomes useful and important when we discuss filtering and the effects of the phase characteristics of a filter in the time domain. The differentiation property for Fourier transforms is very useful, as we see in this lecture, for analyzing systems represented by linear constant-coefficient differential equations. Also, we should recognize from the differentiation property that differentiating in the time domain has the effect of emphasizing high frequencies in the Fourier transform. We recall in the discussion of the Fourier series that higher frequencies tend to be associated with abrupt changes (for example, the step discontinuity in the square wave). In the time domain we recognize that differentiation will emphasize these abrupt changes, and the differentiation property states that, consistent with this result, the high frequencies are amplified in relation to the low frequencies.

Two major properties that form the basis for a wide array of signal processing systems are the convolution and modulation properties. According to the convolution property, the Fourier transform maps convolution to multiplication; that is, the Fourier transform of the convolution of two time functions is the product of their corresponding Fourier transforms. For the analysis of linear, time-invariant systems, this is particularly useful because through the use of the Fourier transform we can map the sometimes difficult problem of evaluating a convolution to a simpler algebraic operation, namely multiplication. Furthermore, the convolution property highlights the fact that by decomposing a signal into a linear combination of complex exponentials, which the Fourier transform does, we can interpret the effect of a linear, time-invariant system as simply scaling the (complex) amplitudes of each of these exponentials by a scale factor that is characteristic of the system. This "spectrum" of scale factors which the system applies is in fact the Fourier transform of the system impulse response. This is the underlying basis for the concept and implementation of filtering.

The final property that we present in this lecture is the modulation property, which is the dual of the convolution property. According to the modulation property, the Fourier transform of the product of two time functions is proportional to the convolution of their Fourier transforms. As we will see in a later lecture, this simple property provides the basis for the understanding and interpretation of amplitude modulation which is widely used in communication systems. Amplitude modulation also provides the basis for sampling, which is the major bridge between continuous-time and discrete-time signal processing and the foundation for many modern signal processing systems using digital and other discrete-time technologies.

We will spend several lectures exploring further the ideas of filtering, modulation, and sampling. Before doing so, however, we will first develop in Lectures 10 and 11 the ideas of Fourier series and the Fourier transform for the discrete-time case so that when we discuss filtering, modulation, and sampling we can blend ideas and issues for both classes of signals and systems.

Suggested Reading

Section 4.6, Properties of the Continuous-Time Fourier Transform, pages 202–212

Section 4.7, The Convolution Property, pages 212–219

Section 6.0, Introduction, pages 397–401

Section 4.8, The Modulation Property, pages 219–222

Section 4.9, Tables of Fourier Properties and of Basic Fourier Transform and Fourier Series Pairs, pages 223–225

Section 4.10, The Polar Representation of Continuous-Time Fourier Transforms, pages 226–232

Section 4.11.1, Calculation of Frequency and Impulse Responses for LTI Systems Characterized by Differential Equations, pages 232–235

TRANSPARENCY
9.1
Analysis and synthesis equations for the continuous-time Fourier transform.

CONTINUOUS – TIME FOURIER TRANSFORM

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\, e^{j\omega t}\, d\omega \qquad \text{synthesis}$$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \qquad \text{analysis}$$

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$$

$$X(\omega) = Re\,\{X(\omega)\} + j\, Im\,\{X(\omega)\}$$

$$= |X(\omega)|\, e^{j\measuredangle X(\omega)}$$

TRANSPARENCY
9.2
Symmetry properties of the Fourier transform.

PROPERTIES OF THE FOURIER TRANSFORM

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$$

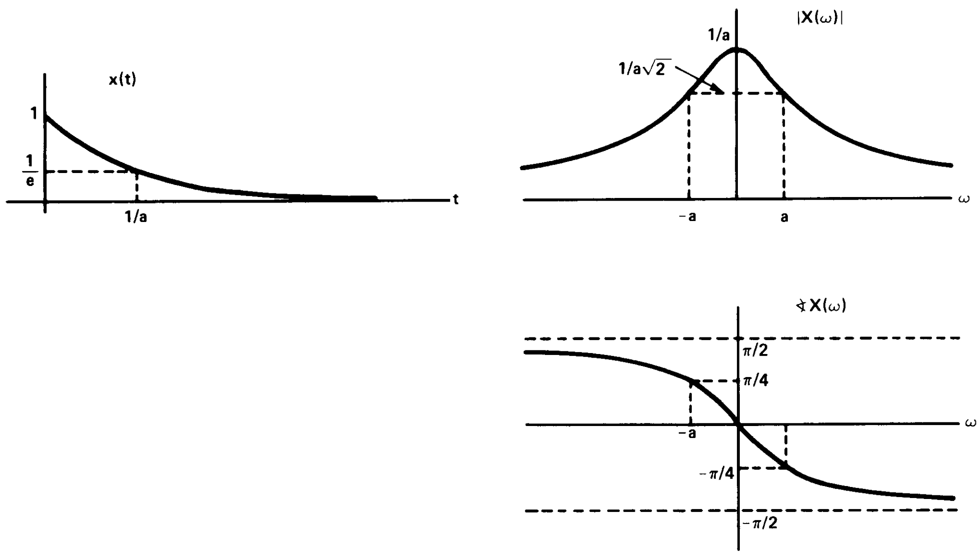
Symmetry:

$$x(t)\ \text{real} \Rightarrow X(-\omega) = X^*(\omega)$$

$$\left.\begin{aligned} Re\, X(\omega) &= Re\, X(-\omega) \\ |X(\omega)| &= |X(-\omega)| \end{aligned}\right\}\ \text{even}$$

$$\left.\begin{aligned} Im\, X(\omega) &= -Im\, X(-\omega) \\ \measuredangle X(\omega) &= -\measuredangle X(-\omega) \end{aligned}\right\}\ \text{odd}$$

Example 4.7: $e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a+j\omega} \quad a > o$



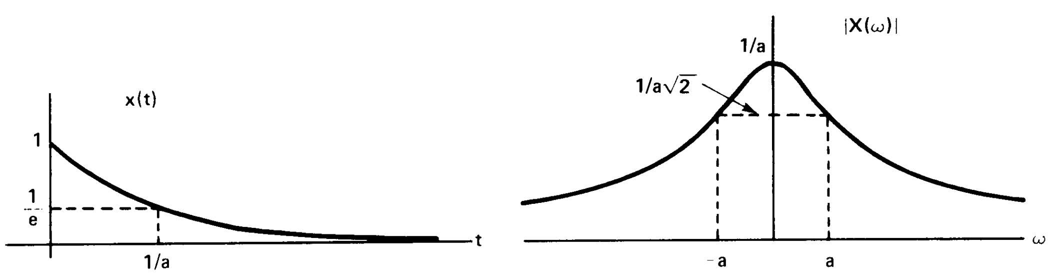
TRANSPARENCY 9.3
The Fourier transform for an exponential time function illustrating the property that the Fourier transform magnitude is even and the phase is odd.

Time and frequency scaling:

$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

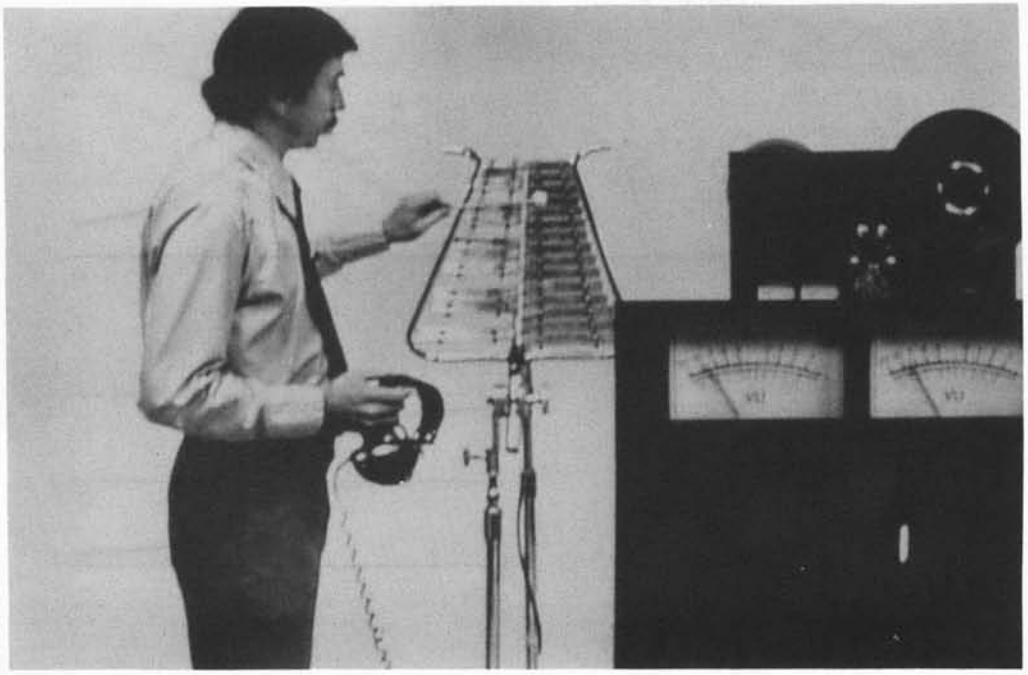
Example:

$$e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a+j\omega} = \frac{1}{a} \frac{1}{1 + j\left(\frac{\omega}{a}\right)}$$



TRANSPARENCY 9.4
The property of time and frequency scaling for the Fourier transform.

DEMONSTRATION 9.1
Time and frequency scaling. A glockenspiel note is recorded and then replayed at twice and half speed.



TRANSPARENCY 9.5
The property of duality.

CONTINUOUS – TIME FOURIER TRANSFORM

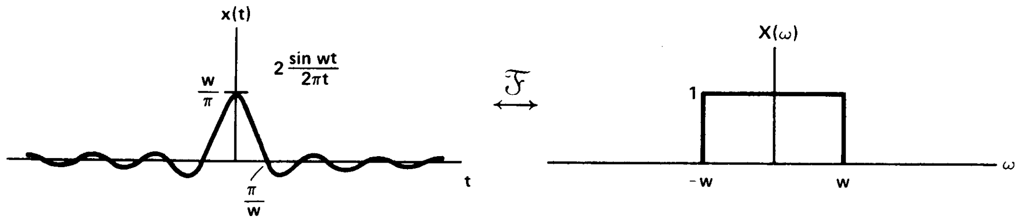
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\, e^{j\omega t}\, d\omega \qquad \text{synthesis}$$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \qquad \text{analysis}$$

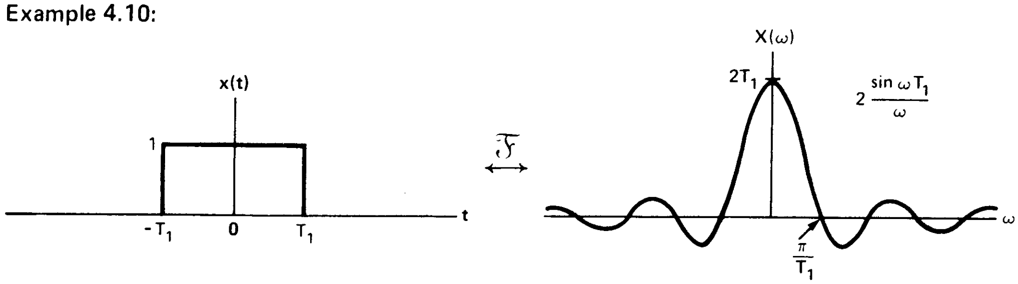
$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$$

$$X(\omega) = Re\left\{X(\omega)\right\} + j\ Im\left\{X(\omega)\right\}$$
$$= |X(\omega)|\, e^{j \measuredangle X(\omega)}$$

Example 4.11:



Example 4.10:



TRANSPARENCY 9.6
Illustration of the duality property.

Parseval's relation:

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(\omega)|^2 d\omega$$

$$\frac{1}{T_0} \int_{T_0} |\tilde{x}(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

TRANSPARENCY 9.7
Parseval's relation for the Fourier transform and the Fourier series.

TRANSPARENCY 9.8
Some additional properties of the Fourier transform.

Time shifting:

$$x(t-t_0) \overset{\mathfrak{F}}{\longleftrightarrow} e^{-j\omega t_0} X(\omega)$$

Differentiation:

$$\frac{dx(t)}{dt} \overset{\mathfrak{F}}{\longleftrightarrow} j\omega X(\omega)$$

Integration:

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathfrak{F}}{\longleftrightarrow} \frac{1}{j\omega} X(\omega) + \pi X(0)\, \delta(\omega)$$

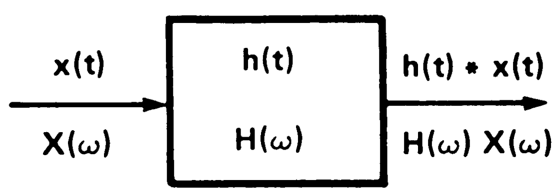
Linearity:

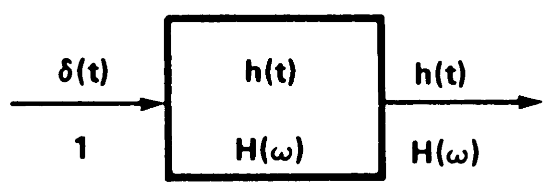
$$ax_1(t) + bx_2(t) \overset{\mathfrak{F}}{\longleftrightarrow} aX_1(\omega) + bX_2(\omega)$$

TRANSPARENCY 9.9
Transparencies 9.9 and 9.10 illustrate the convolution property and its interpretation for LTI systems. This transparency indicates the response to an impulse.

CONVOLUTION PROPERTY

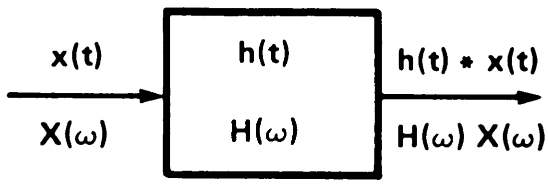
$$h(t) * x(t) \overset{\mathfrak{F}}{\longleftrightarrow} H(\omega)\, X(\omega)$$

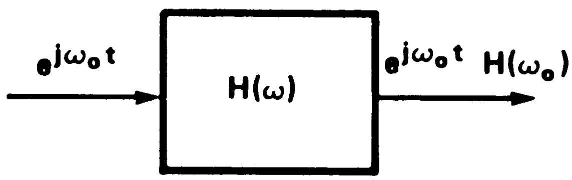




CONVOLUTION PROPERTY

$$h(t) * x(t) \overset{\mathcal{F}}{\longleftrightarrow} H(\omega)\,X(\omega)$$



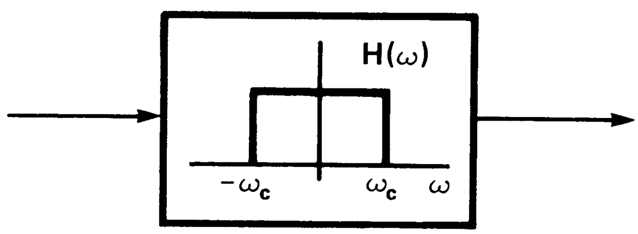


$H(\omega)$ = **frequency response**

TRANSPARENCY 9.10
Response to a complex exponential.

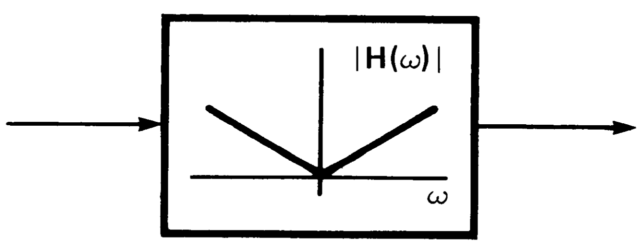
FILTERING

Ideal lowpass filter:



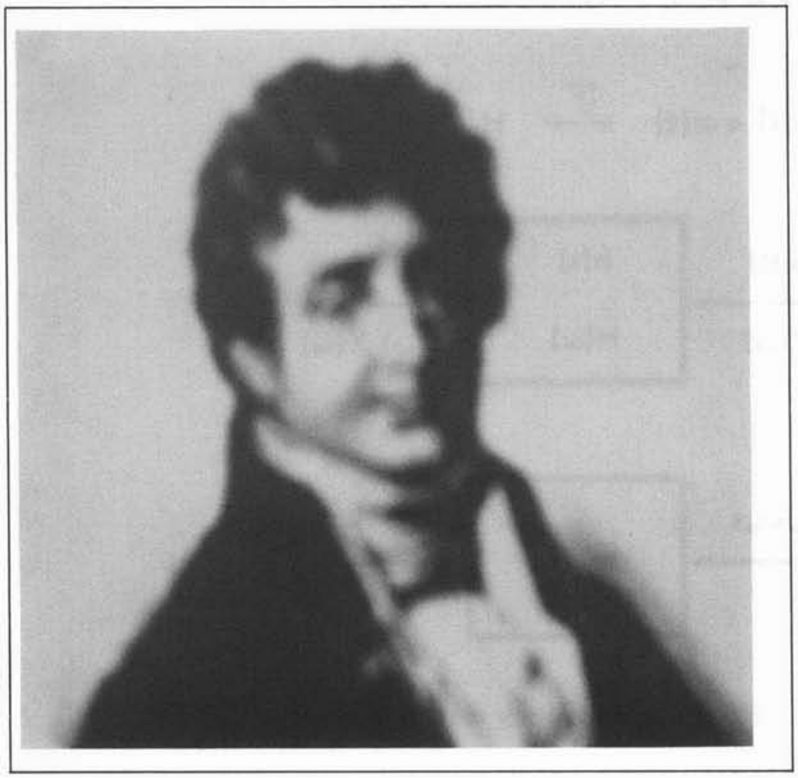
Differentiator:

$$y(t) = \frac{dx(t)}{dt} \Rightarrow H(\omega) = j\omega$$



TRANSPARENCY 9.11
Filtering as an illustration of the interpretation of the convolution property for LTI systems.

DEMONSTRATION
9.2
Lowpass filtering of an image.



DEMONSTRATION
9.3
Effect of differentiating an image.

MODULATION PROPERTY

$$s(t)\,p(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}\left[S(\omega) * P(\omega)\right]$$

Modulation: $$s(t)\,p(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}\left[S(\omega) * P(\omega)\right]$$

Convolution: $$s(t) * p(t) \overset{\mathcal{F}}{\longleftrightarrow} S(\omega)\,P(\omega)$$

TRANSPARENCY 9.12
The modulation property for the Fourier transform.

MARKERBOARD 9.1