PROBLEM 6.1

Part a

From Fig. 6P.1 we see the geometric relations

$$r' = r, \theta' = \theta - \Omega t, \quad z' = z, t' = t \tag{a}$$

There is also a set of back transformations

$$r = r', \quad \theta = \theta' + \Omega t', \quad z = z', \quad t = t' \tag{b}$$

Part b

Using the chain rule for partial derivatives

$$\frac{\partial\psi}{\partial t'} = \left(\frac{\partial\psi}{\partial r}\right)\left(\frac{\partial r}{\partial t'}\right) + \left(\frac{\partial\psi}{\partial\theta}\right)\left(\frac{\partial\theta}{\partial t'}\right) + \left(\frac{\partial\psi}{\partial z}\right)\left(\frac{\partial z}{\partial t'}\right) + \left(\frac{\partial\psi}{\partial t}\right)\left(\frac{\partial t}{\partial t'}\right) \tag{c}$$

From (b) we learn that

$$\frac{\partial r}{\partial t'} = 0, \ \frac{\partial\theta}{\partial t'} = \Omega, \ \frac{\partial z}{\partial t'} = 0, \ \frac{\partial t}{\partial t'} = 1 \tag{d}$$

Hence,

$$\frac{\partial\psi}{\partial t'} = \frac{\partial\psi}{\partial t} + \Omega\frac{\partial\psi}{\partial\theta} \tag{e}$$

We note that the remaining partial derivatives of ψ are

$$\frac{\partial\psi}{\partial r'} = \frac{\partial\psi}{\partial r}, \quad \frac{\partial\psi}{\partial\theta'} = \frac{\partial\psi}{\partial\theta}, \quad \frac{\partial\psi}{\partial z'} = \frac{\partial\psi}{\partial z} \tag{f}$$

PROBLEM 6.2

Part a

The geometric transformation laws between the two inertial systems are

$$x_1' = x_1 - Vt, \ x_2' = x_2, \ x_3' = x_3, \ t' = t \tag{a}$$

The inverse transformation laws are

$$x_1 = x_1' + Vt', \ x_2 = x_2', \ x_3 = x_3', \ t = t' \tag{b}$$

The transformation of the magnetic field when there is no electric field present in the laboratory faame is

$$\bar{B}' = \bar{B} \tag{c}$$

Hence the time rate of change of the magnetic field seen by the moving observer is

$$\frac{\partial B'}{\partial t'} = \frac{\partial B}{\partial t'} = \left(\frac{\partial B}{\partial x_1}\right)\left(\frac{\partial x_1}{\partial t'}\right) + \left(\frac{\partial B}{\partial x_2}\right)\left(\frac{\partial x_2}{\partial t'}\right) + \left(\frac{\partial B}{\partial x_3}\right)\left(\frac{\partial x_3}{\partial t'}\right) + \left(\frac{\partial B}{\partial t}\right)\left(\frac{\partial t}{\partial t'}\right) \tag{d}$$

PROBLEM 6.2 (Continued)

From (b) we learn that

$$\frac{\partial x_1}{\partial t'} = V, \ \frac{\partial x_2}{\partial t'} = 0, \ \frac{\partial x_3}{\partial t'} = 0, \ \frac{\partial t}{\partial t'} = 1 \tag{e}$$

While from the given field we learn that

$$\frac{\partial B}{\partial x_1} = kB_o \cos kx_1, \ \frac{\partial B}{\partial x_2} = \frac{\partial B}{\partial x_3} = \frac{\partial B}{\partial t} = 0 \tag{f}$$

Combining these results

$$\frac{\partial B'}{\partial t'} = \frac{\partial B}{\partial t'} = V\frac{\partial B}{\partial x_1} = VkB_o \cos kx_1 \tag{g}$$

which is just the convective derivative of B.

Part b

Now (b) becomes

$$x_1 = x_1', \ x_2 = x_2' + Vt, \ x_3 = x_3', \ t = t' \tag{h}$$

When these equations are used with (d) we learn that

$$\frac{\partial B'}{\partial t'} = \frac{\partial B}{\partial t'} = V\frac{\partial B}{\partial x_2} + \frac{\partial B}{\partial t} = 0 \tag{i}$$

because both $\frac{\partial B}{\partial x_2}$ and $\frac{\partial B}{\partial t}$ are naught. The convective derivative is zero.

PROBLEM 6.3

Part a

The quasistatic magnetic field transformation is

$$\bar{B}' = \bar{B} \tag{a}$$

The geometric transformation laws are

$$x = x' + Vt', \ y = y', \ z = z', \ t = t' \tag{b}$$

This means that

$$\begin{aligned}\bar{B}' = \bar{B}(t,x) = \bar{B}(t', x' + Vt') &= \bar{i}_y B_o \cos(\omega t' - k(x' + Vt')) \\ &= \bar{i}_y B_o \cos[(\omega - kV)t' - kx']\end{aligned} \tag{c}$$

From (c) it is possible to conclude that

$$\omega' = \omega - kV \tag{d}$$

Part b

If $\omega' = 0$ the wave will appear stationary in time, although it will still have a spacial distribution; it will not appear to move.

PROBLEM 6.3 (Continued)

$$\omega' = 0 = \omega - kV; \; V = \omega/k = v_p \tag{e}$$

The observer must move at the phase velocity v_p to make the wave appear stationary.

PROBLEM 6.4

These three laws were determined in an inertial frame of reference, and since there is no a priori reason to prefer one inertial frame more than another, they should have the same form in the primed inertial frame.

We start with the geometric laws which relate the coordinates of the two frames

$$\bar{r}' = \bar{r} - \bar{v}_r t, \; t = t', \; \bar{r} = \bar{r}' + \bar{v}_r t' \tag{a}$$

We recall from Chapter 6 that as a consequence of (a) and the definitions of the operators

$$\nabla = \nabla', \; \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \bar{v}_r \cdot \nabla', \; \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \bar{v}_r \cdot \nabla \tag{b}$$

In an inertial frame of reference moving with the velocity $\bar{v}_r$ we expect the equation to take the same form as in the fixed frame. Thus,

$$\rho' \frac{\partial \bar{v}'}{\partial t'} + \rho'(\bar{v}' \cdot \nabla')\bar{v}' + \nabla' p' = 0 \tag{c}$$

$$\frac{\partial \rho'}{\partial t'} + \nabla' \cdot \rho' \bar{v}' = 0 \tag{d}$$

$$p' = p'(\rho') \tag{e}$$

However, from (b) these become

$$\rho' \frac{\partial \bar{v}'}{\partial t} + \rho'(\bar{v}' + \bar{v}_r) \cdot \nabla(\bar{v}' + \bar{v}_r) + \nabla p' = 0 \tag{f}$$

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot \rho'(\bar{v}' + \bar{v}_r) = 0 \tag{g}$$

$$p' = p'(\rho') \tag{h}$$

where we have used the fact that $\bar{v}_r \cdot \nabla \rho' = \nabla \cdot (\bar{v}_r \rho')$. Comparison of (1)-(3) with (f)-(h) shows that a self consistent transformation that leaves the equations invariant in form is

$$\rho' = \rho; \; p' = p; \; \bar{v}' = \bar{v} - \bar{v}_r$$

PROBLEM 6.5

Part a

$$\rho'(\bar{r}',t') = \rho(\bar{r},t) = \rho_o(1-\frac{r}{a}) = \rho_o(1-\frac{r'}{a}) \quad (a)$$

$$\bar{J}' = \rho'\bar{v}' = 0 \quad (b)$$

Where we have chosen $\bar{v}_r = v_o\bar{i}_z$ so that

$$\bar{v}' = \bar{v} - \bar{v}_r = 0 \quad (c)$$

Since there are no currents, there is only an electric field in the primed frame

$$\bar{E}' = (\rho_o/\varepsilon_o)(\frac{r'}{2} - \frac{r'^2}{3a})\bar{i}_r \quad (d)$$

$$\bar{H}' = 0, \ \bar{B}' = \mu_o\bar{H}' = 0 \quad (e)$$

Part b

$$\rho(r,t) = \rho_o(1-\frac{r}{a}) \quad (f)$$

This charge distribution generates an electric field

$$\bar{E} = (\rho_o/\varepsilon_o)(\frac{r}{2} - \frac{r^2}{3a})\bar{i}_r \quad (g)$$

In the stationary frame there is an electric current

$$\bar{J} = \rho\bar{v} = \rho_o(1-\frac{r}{a})v_o\bar{i}_z \quad (h)$$

This current generates a magnetic field

$$\bar{H} = \rho_o v_o(\frac{r}{2} - \frac{r^2}{3a})\bar{i}_\theta \quad (i)$$

Part c

$$\bar{J} = \bar{J}' - \rho'\bar{v}_r = \rho_o(1-\frac{r'}{a})v_o\bar{i}_z \quad (j)$$

$$\bar{E} = \bar{E}' - \bar{v}_r \times \bar{B}' = \bar{E}' = (\rho_o/\varepsilon_o)(\frac{r'}{2} - \frac{r'^2}{3a})\bar{i}_r \quad (k)$$

$$\bar{H} = \bar{H}' + \bar{v}_r \times \bar{D}' = v_o\rho_o(\frac{r'}{2} - \frac{r'^2}{3a})\bar{i}_\theta \quad (l)$$

If we include the geometric transformation $r' = r$, (j), (k), and (l) become (h), (g), and (i) of part (b) which we derived without using transformation laws. The above equations apply for $r<a$. Similar reasoning gives the fields in each frame for $r>a$.

PROBLEM 6.6

Part a

In the frame rotating with the cylinder

$$\bar{E}'(r') = \frac{K}{r'}\,\bar{i}_r \qquad \text{(a)}$$

$$\bar{H}' = 0, \; \bar{B}' = \mu_o\bar{H}' = 0 \qquad \text{(b)}$$

But then since $r' = r$, $\bar{v}_r(r) = r\omega\bar{i}_\theta$

$$\bar{E} = \bar{E}' - \bar{v}_r \times \bar{B}' = \bar{E}' = \frac{K}{r}\,\bar{i}_r \qquad \text{(c)}$$

$$V = \int_a^b \bar{E}\cdot d\bar{\ell} = \int_a^b \frac{K}{r}\,dr = K\ln(b/a) \qquad \text{(d)}$$

$$\bar{E} = \frac{V}{\ln(b/a)}\frac{1}{r}\,\vec{i}_r = \bar{E}' = \frac{V}{\ln(b/a)}\frac{1}{r'}\,\vec{i}_r \qquad \text{(e)}$$

The surface charge density is then

$$\sigma_a' = \vec{i}_r\cdot\varepsilon_o\bar{E}' = \frac{\varepsilon_o V}{\ln(b/a)}\frac{1}{a} = \sigma_a \qquad \text{(f)}$$

$$\sigma_b' = -\vec{i}_r\cdot\varepsilon_o\bar{E}' = -\frac{\varepsilon_o V}{\ln(b/a)}\frac{1}{b} = \sigma_b \qquad \text{(g)}$$

Part b

$$\bar{J} = \bar{J}' + \bar{v}_r\,\rho' \qquad \text{(h)}$$

But in this problem we have only surface currents and charges

$$\bar{K} = \bar{K}' + \bar{v}_r\,\sigma' = \bar{v}_r\,\sigma' \qquad \text{(i)}$$

$$\bar{K}(a) = \frac{a\omega\varepsilon_o V}{a\ln(b/a)}\,\vec{i}_\theta = \frac{\omega\varepsilon_o V}{\ln(b/a)}\,\vec{i}_\theta \qquad \text{(j)}$$

$$\bar{K}(b) = -\frac{b\omega\varepsilon_o V}{b\ln(b/a)}\,\vec{i}_\theta = -\frac{\omega\varepsilon_o V}{\ln(b/a)}\,\vec{i}_\theta \qquad \text{(k)}$$

Part c

$$\bar{H} = -\frac{\omega\varepsilon_o V}{\ln(b/a)}\,\vec{i}_z \qquad \text{(l)}$$

Part d

$$\bar{H} = \bar{H}' + \bar{v}_r \times \bar{D}' = \bar{v}_r \times \bar{D}' \qquad \text{(m)}$$

$$\bar{H} = r'\omega\left(\frac{\varepsilon_o V}{\ln(b/a)}\frac{1}{r'}\right)(\vec{i}_\theta \times \vec{i}_r) \qquad \text{(n)}$$

PROBLEM 6.6 (Continued)

$$\bar{H} = -\frac{\omega\varepsilon_o V}{\ln(b/a)}\vec{i}_z \qquad (o)$$

This result checks with the calculation of part (c).

PROBLEM 6.7

Part a

The equation of the top surface is

$$f(x,y,t) = y - a\sin(\omega t)\cos(kx) + d = 0 \qquad (a)$$

The normal to this surface is then

$$\bar{n} = \frac{\nabla f}{|\nabla f|} \approx ak\sin(\omega t)\sin(kx)\bar{i}_x + \bar{i}_y \qquad (b)$$

Applying the boundary condition $\bar{n}\cdot\bar{B} = 0$ at each surface and keeping only linear terms, we learn that

$$h_y(x,d,t) = -ak\sin(\omega t)\sin(kx)\frac{\Lambda}{\mu_o d} \qquad (c)$$

$$h_y(x,0,t) = 0 \qquad (d)$$

We look for a solution for $\bar{h}$ that satisfies

$$\nabla\times\bar{h} = 0, \quad \nabla\cdot\bar{h} = 0 \qquad (e)$$

Let $\bar{h} = \nabla\psi$, $\nabla^2\psi = 0$ (f)

Now we must make an intelligent guess for a Laplacian ψ using the periodicity of the problem and the boundary condition $h_y = \partial\psi/\partial y = 0$ at $y = 0$. Try

$$\psi = \frac{A}{k}\cosh(ky)\sin(kx)\sin(\omega t) \qquad (g)$$

$$\vec{h} = A\sin(\omega t)[\cos(kx)\cosh(ky)\bar{i}_x + \sin(kx)\sinh(ky)\bar{i}_y] \qquad (h)$$

Equation (c) then requires the constant A to be

$$A = \frac{-ak\quad\Lambda}{\sinh(kd)\mu_o d} \qquad (i)$$

Part b

$$\nabla\times\bar{E} = \bar{i}_x\left(\frac{\partial E}{\partial y}z\right) - \bar{i}_y\left(\frac{\partial E}{\partial y}z\right) = -\frac{\partial\bar{B}}{\partial t} \qquad (j)$$

$$\frac{\partial\bar{B}}{\partial t} = \omega\mu_o A\cos(\omega t)[\cos(kx)\cosh(ky)\bar{i}_x + \sin(kx)\sinh(ky)\bar{i}_y] \qquad (k)$$

PROBLEM 6.7 (Continued)

$$\bar{E} = -\omega\mu_o \frac{A}{k}\cos(\omega t)[\cos(kx)\sinh(ky)]\bar{i}_z \tag{l}$$

Now we check the boundary conditions. Because $\bar{v}(y=0) = 0$

$$\bar{n} \times \bar{E} = (\bar{n}\cdot\bar{v})\bar{B} = 0 \quad (y=0) \tag{m}$$

But $\bar{E}(y=0) = 0$, so (m) is satisfied.

If a particle is on the top surface, its coordinates x,y,t must satisfy (a). It follows that

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \bar{v}\cdot\nabla f = 0 \tag{n}$$

Since $\bar{n} = \frac{\nabla f}{|\nabla f|}$ we have that

$$(\bar{n}\cdot\bar{v}) = \frac{-1}{|\nabla f|}\frac{\partial f}{\partial t} \approx a\omega\cos(\omega t)\cos(kx) \tag{o}$$

Now we can check the boundary condition at the top surface

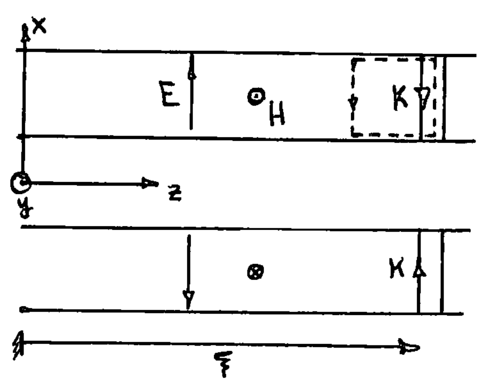
$$\bar{n}\times\bar{E} = -\omega\mu_o\frac{A}{k}\cos(\omega t)\cos(kx)\sinh(kd)[\bar{i}_x - ak\sin(\omega t)\sin(kx)\bar{i}_y] \tag{p}$$

$$(\bar{n}\cdot\bar{v})\bar{B} = a\omega\cos(\omega t)\cos(kx)\Big[\frac{-\sinh(kd)}{ak}\mu_o A\bar{i}_x + \mu_o A\sin(\omega t)\sin(kx)\sinh(kd)\bar{i}_y\Big] \tag{q}$$

Comparing (p) and (q) we see that the boundary condition is satisfied at the top surface.

PROBLEM 6.8

Part a



Since the plug is perfectly conducting we expect that the current I will return as a surface current on the left side of the plug. Also E', H' will be zero in the plug and the transformation laws imply that E,H will then also be zero.

Using ampere's law

$$\bar{H} = \begin{cases} \dfrac{-I}{2\pi r}\,\bar{i}_\theta & 0 < z < \xi \\ 0 & \xi < z \end{cases} \tag{a}$$

PROBLEM 6.8 (Continued)

Also we know that

$$\nabla\cdot\bar{E} = 0,\ \nabla \times \bar{E} = -\frac{\partial\bar{B}}{\partial t} = 0 \qquad 0 < z < \xi \tag{b}$$

We choose a simple Laplacian $\bar{E}$ field consistent with the perfectly conducting boundary conditions

$$\bar{E} = \frac{K}{r}\vec{i}_r \tag{c}$$

K can be evaluated from

$$\oint_C \bar{E}''\cdot d\bar{\ell} = -\frac{d}{dt}\int_S \bar{B}\cdot da \tag{d}$$

If we use the deforming contour shown above which has a fixed left leg at z = z and a moving right leg in the conductor. The notation $\bar{E}''$ means the electric field measured in a frame of reference which is stationary with respect to the local element of the deforming contour. Here

$$\bar{E}''(z) = \bar{E}(z),\quad \bar{E}''(\xi+\Delta) = \bar{E}'(\xi+\Delta) = 0 \tag{e}$$

$$\oint \bar{E}''\cdot d\bar{\ell} = -\int_a^b E(z,r)dr = -K\ln(b/a) \tag{f}$$

The contour contains a flux

$$\int_S \bar{B}\cdot d\bar{a} = (\xi-z)\int_a^b \mu_o H_\theta dr = -\mu_o\frac{I}{2\pi}\ln(b/a)(\xi-z) \tag{g}$$

So that

$$-K\ln(b/a) = -\frac{d}{dt}\int_S \bar{B}\cdot d\bar{a} = +\mu_o\frac{I}{2\pi}\ln(b/a)\frac{d\xi}{dt} \tag{h}$$

Since $v = \frac{d\xi}{dt}$,

$$\bar{E} = \begin{cases} -\dfrac{v\mu_o I}{2\pi}\dfrac{1}{r}\vec{i}_r & 0 < z < \xi \\ 0 & \xi < z \end{cases} \tag{j}$$

Part b

The voltage across the line at z = 0 is

$$V = -\int_a^b E_r dr = \frac{v\mu_o I}{2\pi}\ln(b/a) \tag{k}$$

PROBLEM 6.8 (Continued)

$$I(R + \frac{v\mu_o}{2\pi} \ln(b/a)) = V_o \qquad (l)$$

$$I = \frac{V_o}{R + \frac{v\mu_o}{2\pi} \ln(b/a)} \qquad (m)$$

$$V = \left[\frac{1}{\frac{2\pi R}{v\mu_o \ln(b/a)} + 1}\right] V_o \qquad (n)$$

$$\bar{H} = \begin{cases} \dfrac{-V_o}{R + \frac{v\mu_o}{2\pi} \ln(b/a)} \dfrac{1}{2\pi r} \vec{i}_\theta & 0 < z < \xi \\ 0 & \xi < z \end{cases} \qquad (o)$$

$$\bar{E} = \begin{cases} -\left[\dfrac{1}{\frac{2\pi R}{v\mu_o} + (\ln b/a)}\right] \dfrac{V_o}{r} \vec{i}_r & 0 < z < \xi \\ 0 & \xi < z \end{cases} \qquad (p)$$

Part c

Since $\bar{E} = 0$ to the right of the plug the voltmeter reads zero. The terminal voltage V is not zero because of the net change of magnetic flux in the loop connecting these two voltage points.

Part d

Using the results of part (b)

$$P_{in} = VI = \frac{v\mu_o \ln(b/a)}{2\pi} \left[\frac{1}{R + \frac{v\mu_o}{2\pi} \ln(b/a)}\right]^2 V_o^2$$

$$\frac{dW_m}{dt} = v \int_a^b \frac{\mu_o}{2} H^2(r) 2\pi r \, dr$$

$$= \frac{1}{2} [\frac{v\mu_o \ln(b/a)}{2\pi} (\frac{1}{R + \frac{v\mu_o}{2\pi} \ln(b/a)})^2 V_o^2]$$

PROBLEM 6.8 (Continued)

There is a net electrical force on the block, the mechanical system that keeps the block traveling at constant velocity receives power at the rate

$$\frac{1}{2}\frac{v\mu_o \ln(b/a)}{2\pi}\left[\frac{1}{R + \frac{v\mu_o \ln(b/a)}{2\pi}}\right]^2 V_o^2$$

from the electrical system.

Part e

$$L(x) = \int \frac{\mu_o H(r,I)x\,dr}{I} = \frac{\mu_o}{2\pi}\ln\,(b/a)x$$

$$f^e = \frac{\partial W_m'}{\partial x}\ ;\ W_m' = \frac{1}{2}L(x)i^2$$

$$f^e = \frac{1}{2}\frac{\partial L}{\partial x}i^2 = \frac{1}{2}\frac{\mu_o}{2\pi}\ln(b/a)i^2$$

The power converted from electrical to mechanical is then

$$\bar{f}_e'\frac{\bar{d}x}{dt} = f_e\ v = \frac{1}{2}\frac{\mu_o v}{2\pi}\ln(b/a)\ \left[\frac{V_o}{R + \frac{v\mu_o}{2\pi}\ln(b/a)}\right]^2$$

as predicted in Part (d).

PROBLEM 6.9

The surface current circulating in the system must remain

$$K = \frac{B_o}{\mu_o} \tag{a}$$

Hence the electric field in the finitely conducting plate is

$$E' = \frac{B_o}{\mu_o\sigma_s} \tag{b}$$

But then

$$E = E' - \bar{V}\times\bar{B} \tag{c}$$

$$= B_o\left(\frac{1}{\mu_o\sigma_s} - v\right)$$

v must be chosen so that E = 0 to comply with the shorted end, hence

$$v = \frac{1}{\mu_o\sigma_s} \tag{d}$$

PROBLEM 6.10

Part a

Ignoring the effect of the induced field we must conclude that

$$\bar{E} = 0 \tag{a}$$

everywhere in the stationary frame. But then

$$E' = \bar{E} + \bar{V} \times \bar{B} = \bar{V} \times \bar{B} \tag{b}$$

Since the plate is conducting

$$\bar{J}' = \bar{J} = \sigma\bar{V} \times \bar{B} \tag{c}$$

The force on the plate is then

$$F = \int \bar{J} \times B \, dv = DWd(\sigma\bar{V} \times \bar{B}) \times \bar{B} \tag{d}$$

$$F_x = - DWd\,\sigma v\,B_o^2 \tag{e}$$

Part b

$$M \frac{dv}{dt} + (DWd\sigma B_o^2)v = 0 \tag{f}$$

$$v = v_o\, e^{-\frac{DWd\sigma B_o^2\, t}{M}} \tag{g}$$

Part c

The additional induced field must be small. From (e)

$$J' \simeq \sigma B_o v_o \tag{h}$$

Hence $K' \simeq \sigma B_o d v_o$ (i)

The induced field then has a magnitude

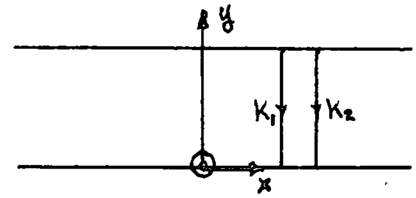
$$\frac{B'}{B_o} \simeq \frac{\mu_o K'}{B_o} = \mu_o \sigma d v_o << 1 \tag{j}$$

$$\sigma d << \frac{1}{\mu_o v_o} \tag{k}$$

It must be a very thin plate or a poorly conducting one.

PROBLEM 6.11

Part a



The condition $\frac{i}{W} << H_o$ means that the field induced by the current can be ignored. Then the magnetic field in the stationary frame is

$$\vec{H} \simeq -H_o \vec{i}_z \quad \text{everywhere outside the perfect conductors} \tag{a}$$

The surface currents on the sliding conductor are such that

$$K_1 + K_2 = i/W \tag{b}$$

The force on the conductor is then

$$F = \int \bar{J} \times \bar{B}\, dv = [(K_1 + K_2)\vec{i}_y \times B_o \vec{i}_z]WD$$

$$= \mu_o H_o di\, \vec{i}_x \tag{c}$$

Part b

The circuit equation is

$$Ri + \frac{d\lambda}{dt} = V_o \tag{d}$$

$$\frac{d\lambda}{dt} \simeq \mu_o H_o dv \tag{e}$$

Since $F = M \frac{dv}{dt}$ (f)

$$\left(\frac{MR}{\mu_o H_o d}\right) \frac{dv}{dt} + (\mu_o H_o d)v = V_o \tag{g}$$

$$v = \frac{V_o}{\mu_o H_o d}\left(1 - e^{-\frac{(\mu_o H_o d)^2}{MR} t}\right)u_{-1}(t) \tag{h}$$

PROBLEM 6.12

Part a

We assume the simple magnetic field

$$\bar{H} = \begin{cases} -\frac{i}{D}\vec{i}_3 & 0 < x_1 < x \\ 0 & x < x_1 \end{cases} \tag{a}$$

$$\lambda(x) = \int \bar{B}\cdot\overline{da} = \frac{\mu_o Wx}{D} i \tag{b}$$

Part b

$$L(x) = \frac{\lambda(x,i)}{i} = \frac{\mu_o Wx}{D} \tag{c}$$

PROBLEM 6.12 (Continued)

Since the system is linear

$$W_m'(i,x) = \frac{1}{2} L(x)i^2 = \frac{1}{2} \frac{\mu_o Wx}{D} i^2 \tag{d}$$

Part c

$$f^e = \frac{\partial W_m'}{\partial x} = \frac{1}{2} \frac{\mu_o W}{D} i^2 \tag{e}$$

Part d

The mechanical equation is

$$M \frac{dx^2}{dt^2} + B \frac{dx}{dt} = \frac{1}{2} \frac{\mu_o W}{D} i^2 \tag{f}$$

The electrical circuit, equation is

$$\frac{d\lambda}{dt} = \frac{d}{dt} \left(\frac{\mu_o Wx}{D} i\right) = V_o \tag{g}$$

Part e

From (f) we learn that

$$\frac{dx}{dt} = \frac{\mu_o W}{2BD} i^2 = \text{const} \tag{h}$$

while from (g) we learn that

$$\frac{\mu_o Wi}{D} \frac{dx}{dt} = V_o \tag{i}$$

Solving these two simultaneously

$$\frac{dx}{dt} = \left[\frac{DV_o^2}{2\mu_o WB}\right]^{1/3} \tag{j}$$

Part f

From (e)

$$i = \sqrt{\frac{2BD}{\mu_o W} \frac{dx}{dt}} = \left(\frac{D}{\mu_o W}\right)^{2/3} (2B)^{1/3} V_o^{\ 1/3} \tag{k}$$

Part g

As in part (a)

$$\bar{H} = \begin{cases} -\dfrac{i(t)\vec{i}_3}{D} & 0 < x_1 < x \\ 0 & x < x_1 \end{cases} \tag{l}$$

Part h

The surface current $\bar{K}$ is

PROBLEM 6.12 (Continued)

$$\bar{K} = -\frac{i(t)}{D}\,\vec{i}_2 \tag{m}$$

The force on the short is

$$\bar{F} = \int \bar{J} \times \bar{B}\, dv = DW\, \bar{K} \times \left(\frac{\mu_o \bar{H}_1 + \mu_o \bar{H}_2}{2}\right) \tag{n}$$

$$= \frac{\mu_o W}{2D} i^2(t)\vec{i}_1$$

Part i

$$\nabla \times \bar{E} = \frac{\partial E_2}{\partial x_1}\vec{i}_3 = -\frac{\partial \bar{B}}{\partial t} = \frac{\mu_o}{D}\frac{di}{dt}\,\vec{i}_3 \tag{o}$$

$$\bar{E}_2 = \left[\frac{\mu_o}{D} x \frac{di}{dt} + C\right]\vec{i}_3 \tag{p}$$

$$= \left[\frac{\mu_o}{D} x \frac{di}{dt} - \frac{V(t)}{W}\right] i_3$$

Part j

Choosing a contour with the right leg in the moving short, the left leg fixed at $x_1 = 0$

$$\oint_C \vec{E}' \cdot d\vec{\ell} = -\frac{d}{dt}\int_S \bar{B} \cdot d\bar{a} \tag{q}$$

Since $E' = 0$ in the short and we are only considering quasistatic fields

$$\oint \vec{E}' \cdot d\vec{\ell} = V(t) = W\, x\, \mu_o \frac{\partial H_o}{\partial t} + W \frac{dx}{dt}\mu_o H_o \tag{r}$$

$$= \frac{d}{dt}\left(\frac{\mu_o W x}{D}\, i(t)\right) \tag{s}$$

Part k

$$\bar{n} \times (\bar{E}^b) = v_n\, \bar{B}^b \tag{t}$$

Here

$$\bar{n} = i_1,\; v_n = \frac{dx}{dt},\; \bar{B}^b = -\frac{\mu_o i}{D}\,\vec{i}_3 \tag{u}$$

$$\bar{E}_b = \left(\frac{\mu_o x}{D}\frac{di}{dt} - \frac{V(t)}{W}\right)\vec{i}_2 = \left(-\frac{dx}{dt}\,\frac{\mu_o W}{D}\, i\right)\vec{i}_2 \tag{v}$$

$$-\frac{dx}{dt}\frac{\mu_o W}{D}\, i = \left(\frac{dx}{dt}\right)\left(-\frac{\mu_o i}{D}\right) \tag{w}$$

Part l

Equations (n) and (e) are identical. Equations (s) and (g) are identical if $V(t) = V_o$. Since we used (e) and (g) to solve the first part we would get the same answer using (n) and (s) in the second part.

PROBLEM 6.12 (Continued)

Part m

Since $\frac{di}{dt} = 0$,

$$\bar{E}_2(x) = -\frac{V(t)}{w}\vec{i}_y = -\frac{V_o}{W}\vec{i}_y \qquad (x)$$

PROBLEM 6.13

Part a

$$K\frac{d^2\psi}{dt^2} = T_1^e(\psi) + T_2(\psi) \qquad (a)$$

Part b

$$J_1 = \frac{i_1\vec{i}_r}{D2\alpha r}\ ;\ \bar{F}_1 = \bar{J}_1 \times \bar{B} = -\frac{\mu_o H_o i_1}{D2\alpha R}\vec{i}_\theta \qquad (b)$$

Similarly

$$\bar{F}_2 = -\frac{\mu_o H_o i_2}{D2\alpha R}\vec{i}_\theta \qquad (c)$$

Part c

$$T_1^e = [\int(\bar{r}\times\bar{f})dv]_z = -\mu_o H_o(R_2-R_1)i_1 \qquad (d)$$

$$T_2^e = -\mu_o H_o(R_2-R_1)i_2 \qquad (e)$$

Part d

$$v_1 = E_1(R_2-R_1);\ v_2 = E_2(R_2-R_1) \qquad (f)$$

Part e

$$J_1 = J_1' = \sigma E_1' = \sigma(\bar{E}_1+\bar{V}\times\bar{B}) = \sigma(E_1+R\mu_o H_o\frac{d\psi}{dt}) \qquad (g)$$

$$E_1 = \frac{1}{\sigma}\frac{i_1}{2\alpha DR} - R\mu_o H_o\frac{d\psi}{dt} \qquad (h)$$

$$v_1 = \frac{1}{\sigma}\frac{(R_2-R_1)}{2\alpha RD}i_1 - \mu_o H_o R(R_2-R_1)\frac{d\psi}{dt} \qquad (i)$$

$$v_2 = \frac{1}{\sigma}\frac{R_2-R_1}{2\alpha RD}i_2 - \mu_o H_o R(R_2-R_1)\frac{d\psi}{dt} \qquad (j)$$

Part f

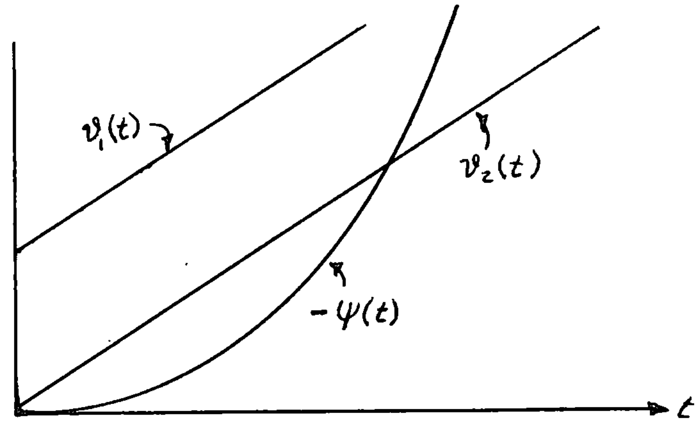
$$K\frac{d^2\psi}{dt^2} = -\mu_o H_o(R_2-R_1)i_0\ u_{-1}(t) \qquad (k)$$

$$\psi(t) = -\frac{\mu_o H_o}{2K}(R_2-R_1)i_0\ t^2\ u_{-1}(t) \qquad (l)$$

$$v_2(t) = (\mu_o H_o(R_2-R_1))^2\frac{R}{K}i_0 t\ u_{-1}(t) \qquad (m)$$

PROBLEM 6.13 (Continued)

$$v_1(t) = [\frac{1}{\sigma}\frac{(R_2-R_1)}{2\alpha RD} + (\mu_o H_o(R_2-R_1)^2 \frac{R}{K} t]i_o u_{-1}(t) \qquad (n)$$



Part g

$$K\frac{d^2\psi}{dt^2} = -\mu_o H_o(R_2-R_1)i_1 \qquad (o)$$

$$= -\frac{\mu_o H_o(R_2-R_1)\sigma 2\alpha RD}{(R_2-R_1)}[v_1+\mu_o H_o R(R_2-R_1)\frac{d\psi}{dt}]$$

$$\frac{d^2\psi}{dt^2} + K_1\frac{d\psi}{dt} = -K_2 v_1(t) \qquad (p)$$

$$K_1 = [(\mu_o H_o R)^2\, 2\alpha D(R_2-R_1)\sigma]/K \qquad (q)$$

$$K_2 = \frac{\mu_o H_o 2\alpha DR\sigma}{K}$$

Find the particular solution

$$\psi_p(\omega,t) = R_e[\frac{-jK_2 v_o}{\omega^2-K_1 j\omega}e^{j\omega t}] \qquad (r)$$

$$= \frac{K_2 v_o}{\omega\sqrt{K_1^2+\omega^2}}\sin(\omega t+\tan^{-1}\frac{K_1}{\omega})u_{-1}(t) \qquad (s)$$

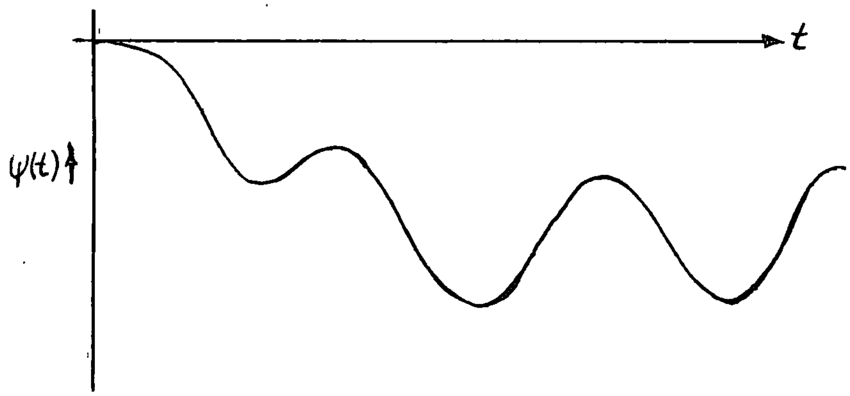
$$\psi(t) = A + \frac{B}{K_1}e^{-K_1 t} + \psi_p(\omega,t) \qquad (t)$$

We must choose A and B so that

$$\psi(0) = 0;\ \frac{d\psi}{dt}(0) = 0 \qquad (u)$$

$$A = -\frac{K_2}{K_1\omega}v_o \qquad B = +\frac{K_2\omega}{(K_1^2+\omega^2)}v_o \qquad (v)$$

PROBLEM 6.13(Continued)



Part h

The secondary terminals are constrained so that $v_2=-i_2R_2$. Thus, (j) becomes

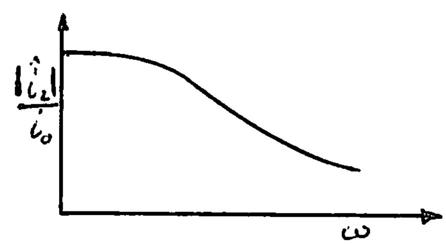
$$\frac{d\psi}{dt} = \frac{R_3}{RK_4}\, i_2;\; R_3 = R_o + \frac{1}{\sigma}\,\frac{(R_2-R_1)}{2\,RD\alpha};\; K_4 = \mu_o H_o (R_2-R_1) \qquad \text{(w)}$$

Then, it follows from (a), (d) and (e) that

$$\frac{di_2}{dt} + \frac{RK_4^2}{KR_3}\, i_2 = -\frac{K_4^2 R i_o}{KR_3} \cos \omega t$$

from which it follows that

$$\frac{|\hat{i}_2|}{i_o} = \frac{K_4^2 R}{KR_3\sqrt{\omega^2+\left(\frac{RK_4^2}{KR_3}\right)^2}}$$



PROBLEM 6.14

Part a

The electric field in the moving laminations is

$$E' = \frac{J'}{\sigma} = \frac{J}{\sigma} = \frac{i}{\sigma A}\,\vec{i}_z \qquad \text{(a)}$$

The electric field in the stationary frame is

$$\vec{E} = \vec{E}' - \bar{V}\times\bar{B} = \left(\frac{i}{\sigma A} + r\omega B_y\right)\vec{i}_z \qquad \text{(b)}$$

$$B_y = -\frac{\mu_o N i}{S} \qquad \text{(c)}$$

$$V = \left(\frac{2D}{\sigma A} - \frac{\mu_o 2Dr\omega N}{S}\right) i \qquad \text{(d)}$$

PROBLEM 6.14 (Continued)

Now we have the V-i characteristic of the device. The device is in series with an inductance and a load resistor $R_t = R_L + R_{int}$.

$$[R_t + \frac{2D}{\sigma A} - \frac{\mu_o 2DrN}{S}\omega]i + \frac{\mu_o N^2 aD}{S}\frac{di}{dt} = 0 \tag{e}$$

Part b

Let

$$R_1 = R_t + \frac{2D}{\sigma A} - \frac{2D\mu_o rN\omega}{S}, \quad L = \frac{\mu_o N^2 aD}{S} \tag{f}$$

$$i = I_o e^{-(R_1/L)t} \tag{g}$$

$$P_d = i^2/R_L = \frac{I_o^2}{R_L}\left[e^{-(R_1/L)t}\right]^2$$

If

$$R_1 = R_t + \frac{2D}{\sigma A} - \frac{2D\mu_o rN\omega}{S} < 0 \tag{h}$$

the power delivered is unbounded as $t \to \infty$.

Part c

As the current becomes large, the electrical nonlinearity of the magnetic circuit will limit the exponential growth and determine a level of stable steady state operation (see Fig. 6.4.12).

PROBLEM 6.15

After the switch is closed, the armature circuit equation is*

$$(R_L + R_a)i_L + L_a\frac{di_L}{dt} = G\dot{\theta}i_f \tag{a}$$

Since $G\dot{\theta}i_f$ is a constant and $i_L(0) = 0$ we can solve for the load current and shaft torque

$$i_L(t) = \frac{G\dot{\theta}i_f}{(R_L+R_a)}\left(1-e^{-\frac{(R_L+R_a)}{L_a}t}\right)u_{-1}(t) \tag{b}$$

$$T^e(t) = i_L(t)\,Gi_f$$

$$= \frac{(Gi_f)^2\,\dot{\theta}}{(R_L+R_a)}\left(1-e^{-\frac{(R_L+R_a)}{L_a}t}\right)u_{-1}(t) \tag{c}$$

* Note: $\dot{i}_a = -i_L$

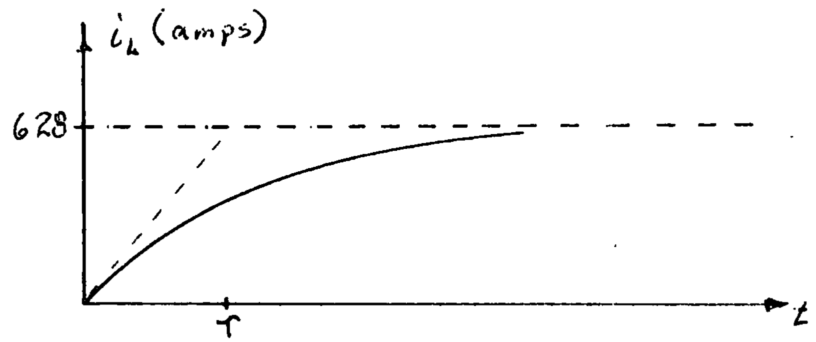
PROBLEM 6.15 (Continued)

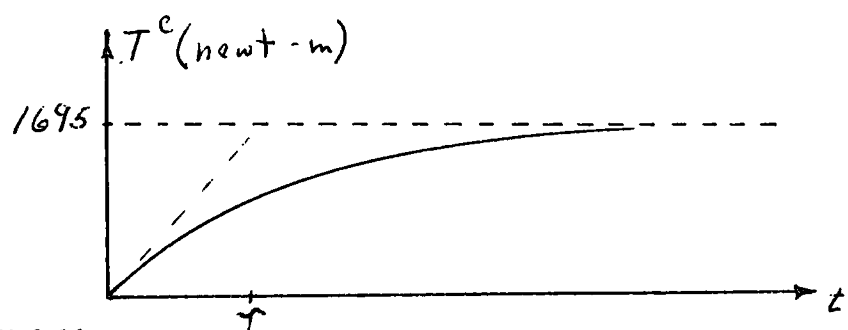
From the data given

$$\tau = L_a/R_L+R_a \simeq 2.5 \times 10^{-3} \text{ sec} \quad (d)$$

$$i_{L_{max}} = \frac{G\dot{\theta}i_f}{R_L+R_a} \quad 628 \text{ amps} \quad (e)$$

$$T_{max} = \frac{(Gi_f)^2\dot{\theta}}{R_L+R_a} \simeq 1695 \text{ newton-meters} \quad (f)$$





PROBLEM 6.16

Part a

With S_1 closed the equation of the field circuit is

$$R_f i_f + L_f \frac{di_f}{dt} = V_f \quad (a)$$

Since $i_f(0) = 0$

$$i_f(t) = \frac{V_f}{R_f}\left(1-e^{-\frac{R_f}{L_f}t}\right)u_{-1}(t) \quad (b)$$

Since the armature circuit is open

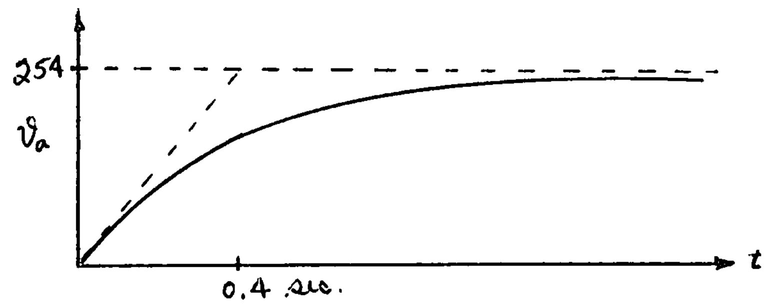
$$V_a = G\dot{\theta}i_f = \frac{V_f\, G\dot{\theta}}{R_f}\left(1-e^{-\frac{R_f}{L_f}t}\right)u_{-1}(t) \quad (c)$$

PROBLEM 6.16 (Continued)

From the given data

$$\tau = L_f/R_f = 0.4 \text{ sec}$$

$$V_{a_{max}} = \frac{V_f\, G\dot{\theta}}{R_f} = 254 \text{ volts}$$



Part b

Since there is no coupling of the armature circuit to the field circuit i_f is still given by (b).

Because S_2 is closed, the armature circuit equation is

$$(R_L+R_a)V_L + L_a \frac{dV_L}{dt} = R_L G\dot{\theta} i_f \tag{d}$$

Since the field current rises with a time constant

$$\tau = 0.4 \text{ sec} \tag{e}$$

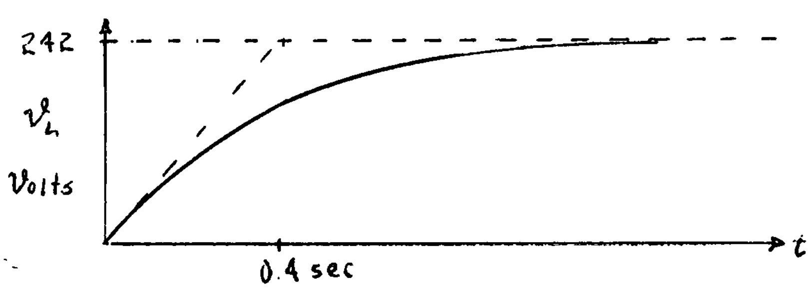
while the time constant of the armature circuit is

$$\tau = L_a/R_L+R_a = 0.0025 \text{ sec} \tag{f}$$

we will only need the particular solution for $V_L(t)$

$$V_L(t) \approx \frac{R_L\, G\dot{\theta}}{R_L+R_a} i_f = \left(\frac{R_L}{R_L+R_a}\right) G\dot{\theta}\, \frac{V_f}{R_f} (1-e^{-\frac{R_f}{L_f}t})u_{-1}(t) \tag{g}$$

$$V_{L_{max}} = \left(\frac{R_L}{R_L+R_a}\right) \left(\frac{G\dot{\theta}}{R_f}\right) V_f = 242 \text{ volts} \tag{h}$$



PROBLEM 6.17

The equation of motion of the shaft is

$$J_r \frac{d\omega}{dt} + \frac{T_o}{\omega_o} \omega = T_o + T_e(t) \qquad (a)$$

If $T_e(t)$ is thought of as a driving term, the response time of the mechanical circuit is

$$\tau = \frac{J_r \omega_o}{T_o} = 0.0785 \text{ sec} \qquad (b)$$

In Probs. 6.15 to 6.16 we have already calculated the armature circuit time constant to be

$$\tau = \frac{L_a}{R_a + R_L} \simeq 2.5 \times 10^{-3} \text{ sec} \qquad (c)$$

We conclude that the rise time of the armature circuit may be neglected, this is equivalent to ignoring the armature inductance. The circuit equation for the armature is then

$$(R_a + R_L) i_L = G\omega i_f \qquad (d)$$

Then

$$T_e = -G i_f i_L = \frac{-(G i_f)^2 \omega}{R_a + R_L} \qquad (e)$$

Plugging into (a)

$$J_r \frac{d\omega}{dt} + K\omega = T_o \qquad (f)$$

Here

$$K = \left(\frac{T_o}{\omega_o} + \frac{(G i_f)^2}{R_a + R_L}\right); \; i_f = \frac{V_f}{R_f} \qquad (g)$$

Using the initial condition that $\omega(0) = \omega_o$

$$\omega(t) = \frac{T_o}{K} + \left(\omega_o - \frac{T_o}{K}\right) e^{-(K/J_r)t} \qquad t \geq 0 \qquad (h)$$

From which we can calculate the net torque on the shaft as

$$T = J_r \frac{d\omega}{dt} = (T_o - K\omega_o) e^{-(K/J_r)t} \; u_{-1}(t) \qquad (i)$$

and the armature current $i_L(t)$

$$i_L(t) = \left(\frac{G i_f}{R_a + R_L}\right) \omega(t) \qquad t \geq 0 \qquad (j)$$

PROBLEM 6.17 (Continued)

From the given data

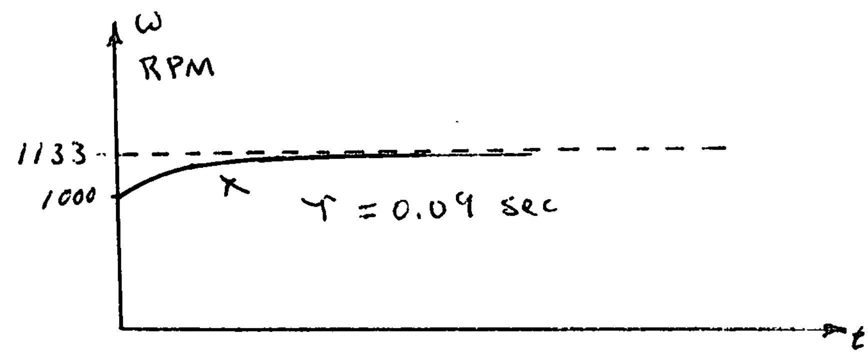
$$\omega_{final} = \frac{T_o}{K} = 119.0 \text{ rad/sec} = 1133 \text{ RPM} \tag{k}$$

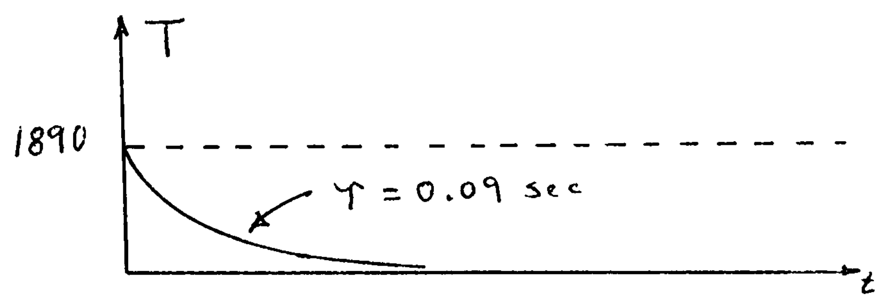
$$T_{max} = (T_o - K\omega_o) \simeq 1890 \text{ newton-m} \tag{l}$$

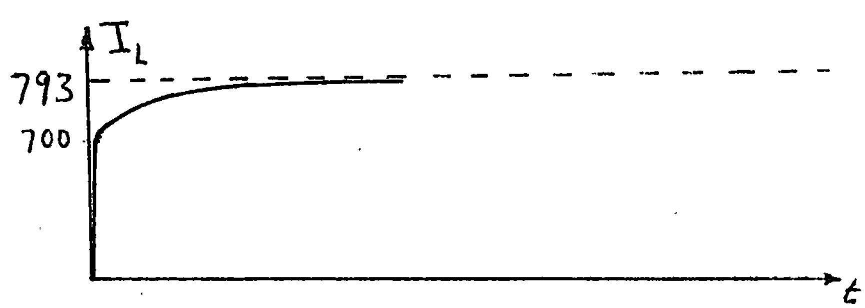
$$i_{L_{min}} = \frac{Gi_f}{R_a+R_L}\,\omega_o \simeq 700 \text{ amps} \tag{m}$$

$$i_{L_{max}} = \left(\frac{Gi_f}{R_a+R_L}\right)\omega_{final} \simeq 793 \text{ amps} \tag{n}$$

$$K = 134.5 \text{ newton-meters},\ \tau = J_r/K \simeq 0.09 \text{ sec} \tag{o}$$







PROBLEM 6.18

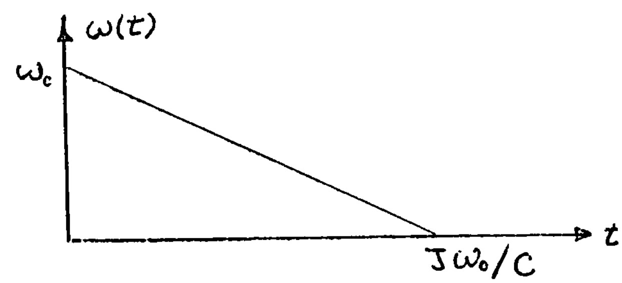
Part a

Let the coulomb torque be C, then the equation of motion is

$$J \frac{d\omega}{dt} + C = 0 \qquad (a)$$

Since $\omega(0) = \omega_o$

$$\omega(t) = \omega_o(1 - \frac{C}{J\omega_o} t) \qquad 0 \le t \le (J/C)\,\omega_o \qquad (b)$$

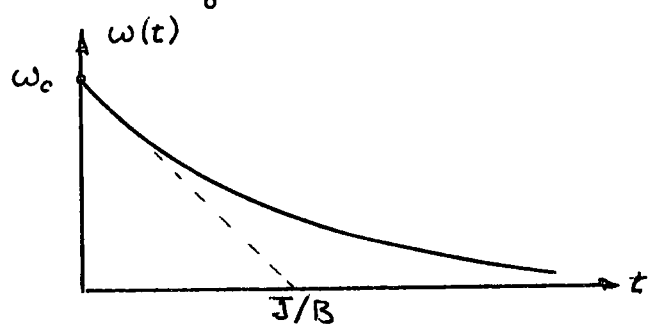


Part b

Now the equation of motion is

$$J \frac{d\omega}{dt} + B\omega = 0 \qquad (c)$$

$$\omega(t) = \omega_o\, e^{-(B/J)t} \qquad (d)$$



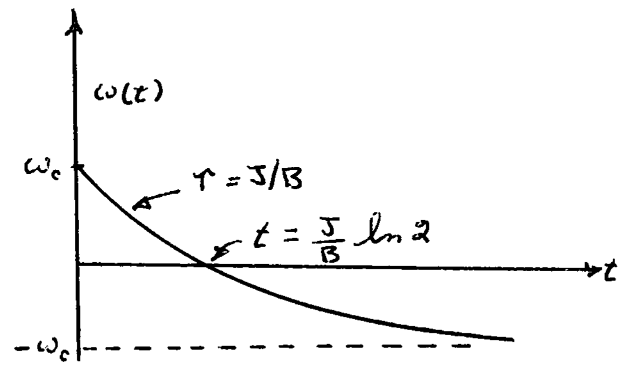
Part c

Let $C = B\omega_o$, the equation of motion is now

$$J \frac{d\omega}{dt} + B\omega = -B\omega_o \qquad (e)$$

$$\left\{\omega(t) = -\omega_o + 2\omega_o e^{-\frac{B}{J}t} \qquad 0 < t \le \frac{J}{B} \ln 2\right\} \qquad (f)$$

PROBLEM 6.18 (Continued)



PROBLEM 6.19

Part a

The armature circuit equation is

$$R_a i_L + L_a \frac{di_L}{dt} = G\omega i_f - V_a\, u_{-1}(t) \qquad \text{(a)}$$

Differentiating

$$L_a \frac{d^2 i_L}{dt^2} + R_a \frac{di_L}{dt} = G i_f \frac{d\omega}{dt} - V_a u_o(t) \qquad \text{(b)}$$

The mechanical equation of motion is

$$J_r \frac{d\omega}{dt} = - G i_L i_f \qquad \text{(c)}$$

Thus, (b) becomes

$$L_a \frac{d^2 i_L}{dt^2} + R_a \frac{di_L}{dt} + \frac{(G i_f)^2}{J_r} i_L = - V_a u_o(t) \qquad \text{(d)}$$

Initial conditions are

$$i_L(0^+) = 0, \quad \frac{di_L}{dt}(0^+) = - \frac{V_a}{L_a} \qquad \text{(e)}$$

and it follows from (d) that

$$i_L(t) = \left(- \frac{V_a}{L_a \beta} e^{-\alpha t} \sin \beta t\right) u_{-1}(t) \qquad \text{(f)}$$

where

$$\alpha = \frac{R_a}{2L_a} = 7.5/\text{sec} \qquad \text{(g)}$$

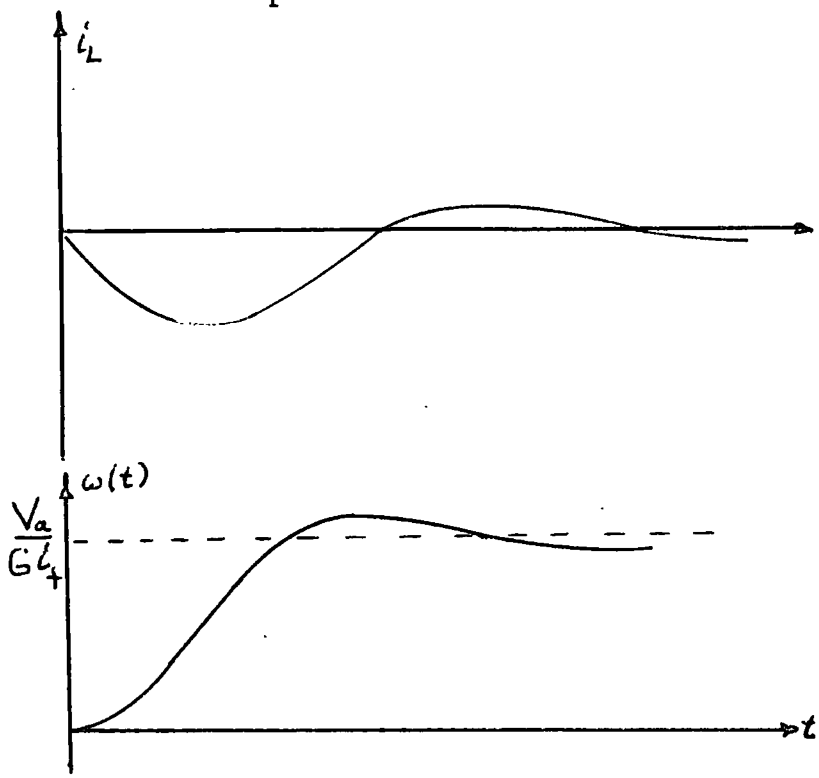
$$\beta = \sqrt{\frac{(G i_f)^2}{J_r L_a} - \left(\frac{R_a}{2L_a}\right)^2} \simeq 19.9 \text{ rad/sec} \qquad \text{(h)}$$

PROBLEM 6.19 (Continued)

$$\frac{V_a}{L_a\beta} \simeq 1160 \text{ amps} \tag{i}$$

$$\omega(t) = -\frac{V_a}{Gi_f}\left[\frac{\alpha}{\beta} e^{-\alpha t}\sin \beta t + (e^{-\alpha t}\cos \beta t - 1)\right]$$

$$\frac{V_a}{Gi_f} = 153.3 \text{ rad/sec} \tag{k}$$



Part b

Now we replace R_a by R_a+R_L in part (a). Because of the additional damping

$$i_L(t) = -\frac{V_a}{2L_a\gamma}\left(e^{-(\alpha-\gamma)t} - e^{-(\alpha+\gamma)t}\right)u_{-1}(t) \tag{l}$$

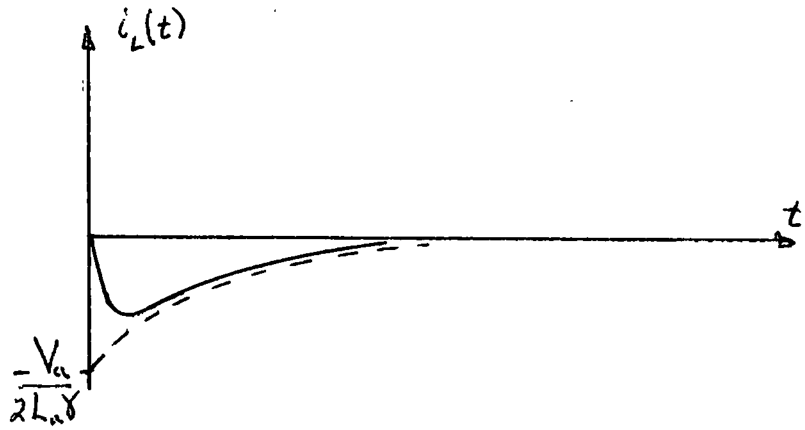
where

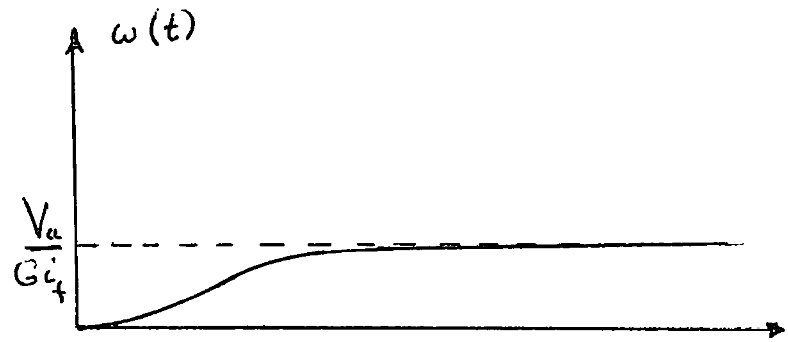
$$\alpha = \frac{R_a + R_L}{2L_a} = 75/\text{sec} \tag{m}$$

$$\gamma = \sqrt{\left(\frac{R_a + R_L}{2L_a}\right)^2 - \frac{(Gi_f)^2}{J_r L_a}} = 10.6/\text{sec.} \tag{n}$$

PROBLEM 6.19 (Continued)

$$\omega(t) = \frac{V_a}{2L_a\gamma}\frac{Gi_f}{J_r}\left[\frac{1}{(\gamma-\alpha)}e^{-(\alpha-\gamma)t} + \frac{1}{\gamma+\alpha}e^{-(\alpha+\gamma)t} + \frac{2\gamma}{\alpha^2-\gamma^2}\right] \quad \text{(o)}$$





PROBLEM 6.20

Part a

The armature circuit equation is

$$v_a = R_a i_a + GI_f\omega \quad \text{(a)}$$

The equation of motion is

$$J\frac{d\omega}{dt} = GI_f i_a \quad \text{(b)}$$

Which may be integrated to yield

$$\omega(t) = \frac{G}{J}\int_{-\infty}^{t} i_a(t) \quad \text{(c)}$$

PROBLEM 6.20 (Continued)

Combining (c) with (a)

$$v_a = R_a i_a + \frac{(GI_f)^2}{J_r} \int_{-\infty}^{t} i_a(t) \tag{d}$$

We recognize that

$$C = \frac{J_r}{(GI_f)^2} \tag{e}$$

Part b

$$C = \frac{J_r}{(GI_f)^2} = \frac{(0.5)}{(1.5)^2(1)} = 0.22 \text{ farads}$$

PROBLEM 6.21

According to (6.4.30) the torque of electromagnetic origin is

$$T^e = Gi_f i_a$$

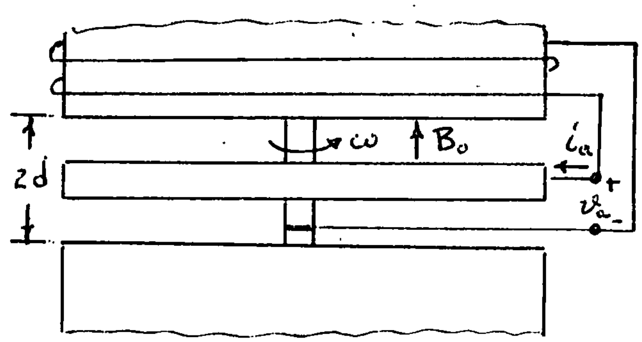
For operation on a-c, maximum torque is produced when i_f and i_a are in phase, a situation assured for all loading conditions by a series connection of field and armature. Parallel operation, on the other hand, will yield a phase relation between i_f and i_a that varies with loading. This gives reduced performance unless phase connecting means are employed. This is so troublesome and expensive that the series connection is used almost exclusively.

PROBLEM 6.22

From (6.4.50) et. seq. the homopolar machine, viewed from the disk terminals in the steady state, has the volt ampere relation

$$v_a = R_a i_a + G\omega i_f$$

$$R_a = \frac{\ln(b/a)}{2\pi\sigma d}$$



For definition of v_a and i_a shown to the right and with the interconnection with the coil shown in Fig. 6P.22

$$B_o = \frac{\mu_o N i_a}{2d}$$

Then from (6.4.52)

$$G\omega i_f = \frac{\omega B_o}{2}(b^2-a^2) = \frac{\omega \mu_o N i_a}{4d}(b^2-a^2)$$

PROBLEM 6.22 (Continued)

Substitution of this into the voltage equation yields for steady state (because the coil resistance is zero).

$$0 = R_a i_a + \frac{\omega\mu_o N i_a}{4d}(b^2-a^2)$$

for self-excitation with $i_a \neq 0$

$$\frac{\omega\mu_o N}{4d}(b^2-a^2) = -R_a$$

Because all terms on the left are positive except for ω, we specify $\omega < 0$ (it rotates in the direction opposite to that shown). With this provision the number of turns must be

$$N = \frac{4dR_a}{|\omega|\mu_o(b^2-a^2)} = \frac{4d\ln(b/a)}{2\pi\sigma d|\omega|\mu_o(b^2-a^2)}$$

$$N = \frac{2\ln(b/a)}{\pi\sigma\mu_o|\omega|(b^2-a^2)}$$

PROBLEM 6.23

Part a

Denoting the left disk and magnet as 1 and the right one as 2, the flux densities defined as positive upward are

$$B_1 = -\frac{\mu_o N}{\ell}(i_1-i_2)$$

$$B_2 = -\frac{\mu_o N}{\ell}(i_1+i_2)$$

Adding up voltage drops around the loop carrying current i_1 we have:

$$-N\pi a^2\frac{dB_2}{dt} - N\pi a^2\frac{dB_1}{dt} + i_1R_L + i_1R_a \underset{-}{+} \frac{\Omega B_1}{2}(b^2-a^2) = 0$$

where $R_a = \dfrac{\ln(\overset{a/b}{\cancel{b/a}})}{2\pi\sigma h}$

Part b

Substitution of the expression for B_1 and B_2 into this voltage expression and simplification yield

$$L\frac{di_1}{dt} + i_1(R_L+R_a) - G\Omega i_1 + G\Omega i_2 = 0$$

PROBLEM 6.23 (Continued)

where

$$L = \frac{2\mu_o N^2 \pi a^2}{\ell}$$

$$G = \frac{-\mu_o N(b^2-a^2)}{2\ell}$$

The equation for the circuit carrying current i_2 can be written similarly as

$$L\frac{di_2}{dt} + i_2(R_L+R_a) - G\Omega i_2 - G\Omega i_1 = 0$$

These are linear differential equations with constant coefficients, hence, assume

$$i_1 = I_1 e^{st}; \quad i_2 = I_2 e^{st}$$

Then

$$[Ls + R_L+R_a-G\Omega]I_1 + G\Omega I_2 = 0$$

$$[Ls + R_L+R_a-G\Omega]I_2 - G\Omega I_1 = 0$$

Elimination of I_1 yields

$$\left[\frac{[Ls + R_L+R_a-G\Omega]^2}{G\Omega} + G\Omega\right] I_2 = 0$$

If $I_2 \neq 0$ as it must be if we are to supply current to the load resistances, then

$$[Ls + R_L+R_a-G\Omega]^2 + (G\Omega)^2 = 0$$

For steady-state sinusoidal operation s must be purely imaginary. This requires

$$R_L + R_a - G\Omega = 0$$

a||b

or

$$G = \frac{-\mu_o N(b^2-a^2)}{2\ell} = \frac{R_L + \frac{\ln\cancel{(b/a)}}{2\pi\sigma h}}{\Omega}$$

This is the condition required.

Part c

When the condition of (b) is satisfied

$$s = \pm j\omega = \pm j\frac{G\Omega}{L}$$

$$\omega = \frac{-\mu_o N(b^2-a^2)\ell\Omega}{2\ell\mu_o N^2\pi a^2} = \frac{-(\frac{b^2}{a^2}-1)\Omega}{2\cdot 2\pi N}$$

PROBLEM 6.23 (Continued)

Thus the system will operate in the sinusoidal steady-state with amplitudes determined by initial conditions. With the condition of part (b) satisfied the voltage equations show that

$$I_1 = jI_2$$

and the currents form a balanced two-phase set.