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*Electromechanical Dynamics*

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## Chapter 8

# FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

### 8.0 INTRODUCTION

Chapter 7 is restricted to the effects of mechanical motion on magnetic and electric fields. In general, electromechanical interactions involve effects on the mechanical system from the electromagnetic fields as well. These arise from the mechanical forces of electrical origin.

In Chapters 3 through 6 we were concerned with total forces acting on rigid bodies. In systems in which the mechanical medium must be represented by a deformable continuum the details of the force distribution must be known. Hence in continuum electromechanics we are concerned with magnetic or electric force densities, which are, in general, functions of space and time.

Electromagnetic fields are defined by forces composed of two parts: those exerted on free charges by electric fields and those exerted on free currents (moving free charges) by magnetic fields. The relative importance of these forces depends on the type of system being considered. In magnetic field systems, as defined in Section 1.1, the important field excitation is provided by the free current density  $\mathbf{J}_f$ . Hence for magnetic field systems the only important forces arise from the interactions of the free current density  $\mathbf{J}_f$  with magnetic fields. Similarly, the only forces of significance in electric field systems, as defined in Section 1.1, are the interactions of free charge density  $\rho_f$  with electric fields. The validity of these assumptions is checked in particular problems. Following the pattern established in earlier sections, we treat forces in magnetic field and electric field systems separately. Our object is to describe electromagnetic forces mathematically in alternative forms that will prove useful in work with continuum electromechanical systems.

Two other technically important electromagnetic forces are those resulting from the interactions of polarization density  $\mathbf{P}$  with electric fields and magnetization density  $\mathbf{M}$  with magnetic fields. In Chapters 3 to 5 we calculate total forces on polarizable and magnetizable bodies by using an energy method. We extend this method to account for force densities in polarized or magnetized media that are electrically linear, isotropic, and homogeneous. This limitation in our discussion of polarization and magnetization forces is imposed because use of an energy method requires a knowledge of the mechanical and thermodynamic properties of the material.

### 8.1 FORCES IN MAGNETIC-FIELD SYSTEMS

Consider first the force resulting from the interaction of moving free charge (i.e.,  $\mathbf{J}_f$ ) and a magnetic field. The Lorentz force (1.1.28) gives the total magnetic force on a charge  $q$  moving with velocity  $\mathbf{v}$  as

$$\mathbf{f} = q\mathbf{v} \times \mathbf{B}. \quad (8.1.1)$$

The force density  $\mathbf{F}$  (newtons per cubic meter) can be obtained from this expression by writing

$$\mathbf{F} = \lim_{\delta V \rightarrow 0} \frac{\sum_i \mathbf{f}_i}{\delta V} = \lim_{\delta V \rightarrow 0} \frac{\sum_i q_i \mathbf{v}_i \times \mathbf{B}_i}{\delta V}, \quad (8.1.2)$$

where  $\mathbf{f}_i$ ,  $q_i$ , and  $\mathbf{v}_i$  refer to all the particles in  $\delta V$  and  $\mathbf{B}_i$  is the flux density experienced by  $q_i$ . If we can say that all particles within  $\delta V$  experience the same flux density  $\mathbf{B}$ , we can use the definition of free current density (see Section B.1.2)\* to write (8.1.2) as

$$\mathbf{F} = \mathbf{J}_f \times \mathbf{B}. \quad (8.1.3)$$

The general definition of (8.1.2) requires the averaging of products, whereas the result of (8.1.3) is the product of averages. It is not, in general, true for variables  $x$  and  $y$  that

$$[xy]_{\text{av}} = [x]_{\text{av}}[y]_{\text{av}}.$$

The force density expressed by (8.1.3) however, agrees, to a high degree of accuracy, with all experimental results obtained with common conductors. The relation (8.1.3) is valid because the volume  $\delta V$  can be made small enough to enclose a region of essentially constant magnetic flux density, although still including many free charges.

In fact, we could have used (8.1.3) rather than (8.1.1) as the definition of  $\mathbf{B}$ , for the original experiments of Biot and Savart and later Ampère† concerned themselves with relating the force density to the free current density

$$* \mathbf{J}_f = \lim_{\delta V \rightarrow 0} \left[ \left( \sum_i q_i \mathbf{v}_i \right) / \delta V \right]$$

† J. D. Jackson, *Classical Electrodynamics*, Wiley, New York 1962, p. 133.

$\mathbf{J}$ . Some writers start with (8.1.3) as the basic definition of the magnetic force on moving free charge.\* However, the averaging process used to make (8.1.2) and (8.1.3) consistent is then inherent to the definition.

It is important to remember that (8.1.3) represents the average of forces on the *charges*. This is equivalent to the force on a medium if there is some mechanism by which each charge transmits the Lorentz force to the material. For example, in a conductor, the charges can be thought of as particles moving through a viscous material—in which case the force that acts on each charge is transmitted to the medium by the viscous retarding force and (8.1.3) is the force density experienced by the medium.

There are situations in which the charges do not interact individually with the medium. For example, in a polarized medium, *pairs* of charges (dipoles) transmit a force to the medium—each pair being connected through the structure of an atom or molecule. For these cases it is the dipoles rather than the charges that transmit a force to the medium. Then it is appropriate to consider the average of the forces on individual dipoles as equivalent to the force density on the medium. This class of forces is developed in Section 8.5.

The force density given in (8.1.3) is expressed in terms of source and field quantities. It is useful to have the force expressed as a function of field quantities alone because we often solve field problems without calculating the free current density. We find it useful to define the *Maxwell stress tensor* as a function of the field quantities from which the force density can be obtained by space differentiation. The Maxwell stress tensor is particularly useful for finding electromechanical boundary conditions in a concise form. It is useful also for finding the total electromagnetic force on a body.

A tensor has particular properties that are useful in this and the chapters which follow. We therefore devote Section 8.2 to a discussion of the stress tensor, using magnetic field stresses as an example.

We can write (8.1.3) in terms of the magnetic field intensity and in a particularly useful form when the medium has a constant permeability, that is, with the constituent relation†

$$\mathbf{B} = \mu\mathbf{H}. \quad (8.1.4)$$

We can use (8.1.4) and Ampère's law for magnetic field systems (1.1.1)‡ to write (8.1.3) in the form

$$\mathbf{F} = \mu(\nabla \times \mathbf{H}) \times \mathbf{H}. \quad (8.1.5)$$

It is a vector identity that this expression can be written as

$$\mathbf{F} = \mu(\mathbf{H} \cdot \nabla)\mathbf{H} - \frac{\mu}{2} \nabla(\mathbf{H} \cdot \mathbf{H}), \quad (8.1.6)$$

\* See, for example, Jackson, *ibid.*, p. 137.

† Arguments are given in Section 8.5 to show that (8.1.3) is the force density on free currents in the presence of a constant permeability  $\mu$ . For now we assume that this is the case.

‡ Table 1.2, Appendix E.

There are three components to this vector equation, but we usually do not write them out unless specific situations are under consideration. There are manipulations, however, that become easier to perform when the equations are viewed component by component. They can be carried out without dealing with cumbersome expressions by using *index notation*.\*

In what follows we assume a right-hand cartesian coordinate system  $x_1, x_2, x_3$ . The component of a vector in the direction of an axis carries the subscript of that axis. When we write  $F_m$  we mean the  $m$ th component of the vector  $\mathbf{F}$ , where  $m$  can be 1, 2, or 3. The mathematical formalism is illustrated by using the force density of (8.1.6) as an example. When we write the differential operator  $\partial/\partial x_n$ , we mean  $\partial/\partial x_1, \partial/\partial x_2$ , or  $\partial/\partial x_3$ . When the index is repeated in a single term, it implies summation over the three values of the index

$$\frac{\partial H_n}{\partial x_n} = \frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial x_2} + \frac{\partial H_3}{\partial x_3} = \nabla \cdot \mathbf{H}$$

and

$$H_n \frac{\partial}{\partial x_n} = H_1 \frac{\partial}{\partial x_1} + H_2 \frac{\partial}{\partial x_2} + H_3 \frac{\partial}{\partial x_3} = \mathbf{H} \cdot \nabla.$$

This illustrates the *summation convention*. On the other hand,  $\partial H_m/\partial x_n$  represents any one of the nine possible derivatives of components of  $\mathbf{H}$  with respect to coordinates. We define the *Kronecker delta*  $\delta_{mn}$  which has the values

$$\delta_{mn} = \begin{cases} 1, & \text{when } m = n, \\ 0, & \text{when } m \neq n. \end{cases} \quad (8.1.7)$$

The Kronecker delta has the property (remember to sum on an index that appears twice)

$$\delta_{mn} H_n = H_m$$

and

$$\delta_{mn} \frac{\partial}{\partial x_n} = \frac{\partial}{\partial x_m},$$

which can be verified by using the definition (8.1.7).

With these definitions we write the  $m$ th component of (8.1.6) as

$$F_m = \mu H_n \frac{\partial H_m}{\partial x_n} - \frac{\mu}{2} \frac{\partial}{\partial x_m} (H_k H_k). \quad (8.1.8)$$

\* A. J. McConnell, *Applications of the Absolute Differential Calculus*, Blackie, London, 1951, Chapter 1.

We use the property of the Kronecker delta [ $\partial/\partial x_m = \delta_{mn}(\partial/\partial x_n)$ ] and some manipulation to write this expression as

$$F_m = \frac{\partial}{\partial x_n} \left( \mu H_n H_m - \frac{\mu}{2} \delta_{mn} H_k H_k \right) - H_m \frac{\partial \mu H_n}{\partial x_n}. \quad (8.1.9)$$

The last term on the right is

$$H_m (\nabla \cdot \mu \mathbf{H}) = H_m (\nabla \cdot \mathbf{B}) = 0;$$

thus we finally write (8.1.9) in the concise form

$$F_m = \frac{\partial T_{mn}}{\partial x_n}, \quad (8.1.10)$$

where the Maxwell stress tensor  $T_{mn}$  is given by

$$T_{mn} = \mu H_n H_m - \frac{\mu}{2} \delta_{mn} H_k H_k. \quad (8.1.11)$$

If we know the magnetic field intensity  $\mathbf{H}$  in a region of space, we can calculate the components of the stress tensor  $T_{mn}$ . We need only to calculate at most six components because the stress tensor is symmetric:

$$T_{mn} = T_{nm}. \quad (8.1.12)$$

Differentiation of (8.1.11) with respect to the space coordinate according to (8.1.10) gives the force density on the current-carrying matter in that region of space. We should keep in mind that (8.1.10) is simply an alternative way of expressing the  $m$ th component of  $\mathbf{J} \times \mathbf{B}$ . Moreover, we must use the *total*  $\mathbf{H}$  to obtain the correct answer from (8.1.10).

Now suppose we wish to find the  $m$ th component of the total force  $\mathbf{f}$  on material contained within the volume  $V$ . We can find it by performing the volume integration:

$$f_m = \int_V F_m dV = \int_V \frac{\partial T_{mn}}{\partial x_n} dV. \quad (8.1.13)$$

When we define the components of a vector  $\mathbf{A}$  as

$$A_1 = T_{m1}, \quad A_2 = T_{m2}, \quad A_3 = T_{m3}, \quad (8.1.14)$$

we can write (8.1.13) as

$$f_m = \int_V \frac{\partial A_n}{\partial x_n} dV = \int_V (\nabla \cdot \mathbf{A}) dV. \quad (8.1.15)$$

We now use the divergence theorem to change the volume integral to a surface integral,

$$f_m = \oint_S \mathbf{A} \cdot \mathbf{n} da = \oint_S A_n n_n da, \quad (8.1.16)$$

where  $n_n$  is the  $n$ th component of the outward-directed unit vector  $\mathbf{n}$  normal to the surface  $S$  and the surface  $S$  encloses the volume  $V$ . Substitution from (8.1.14) back into this expression yields

$$f_m = \oint_S T_{mn} n_n da. \quad (8.1.17)$$

Hence we can find the total force of magnetic origin on the matter within a volume  $V$  by knowing only the fields along the surface of the volume. This is an important result.

## 8.2 THE STRESS TENSOR

In the preceding section we introduced the Maxwell stress tensor as an ordered array of nine functions of space and time  $T_{mn}(\mathbf{r}, t)$  from which we can calculate magnetic force densities and total forces. The concept of a tensor will be useful to us in later chapters for describing mechanical stresses and deformations in elastic and fluid media. Consequently, we now digress from our study of electromagnetic forces to develop some tensor concepts.

We first consider the tensor representation of stresses with the object of attaching physical significance to the components of a stress tensor. Then mathematical techniques that are used with the stress tensor to find surface stresses (tractions) and volume force densities are introduced. Finally, we introduce some mathematical properties of tensors in general. These properties are introduced in a context in which physical interpretations can be made easily. It is important to remember that tensor analysis is a mathematical formalism that is particularly useful for analyzing a wide variety of physical systems.\*

We have remarked that the Maxwell stress tensor is an ordered array of nine functions of space and time. It is conventional to write this array in matrix form as

$$T_{mn}(\mathbf{r}, t) = \begin{bmatrix} T_{11}(\mathbf{r}, t) & T_{12}(\mathbf{r}, t) & T_{13}(\mathbf{r}, t) \\ T_{21}(\mathbf{r}, t) & T_{22}(\mathbf{r}, t) & T_{23}(\mathbf{r}, t) \\ T_{31}(\mathbf{r}, t) & T_{32}(\mathbf{r}, t) & T_{33}(\mathbf{r}, t) \end{bmatrix}. \quad (8.2.1)$$

The first index marks the row and the second, the column in which the element appears. As indicated by (8.1.11) and (8.1.12), the Maxwell stress tensor is symmetric. In the matrix of (8.2.1) the symmetry is about the diagonal. Although the symmetry property has been established only for the Maxwell stress tensor, we find that all the tensors we use in this book are symmetric.

\* For a more detailed discussion of tensor calculus than we need in this book see, for example, B. Spain, *Tensor Calculus*, Interscience, New York, 1960.

### 8.2.1 Stress and Traction

A physical interpretation of the stress tensor follows from (8.1.17) which relates the total force on matter within the volume  $V$  enclosed by the surface  $S$  to an integral over the surface  $S$ . The integrand  $T_{mn}n_n$  has the dimension of a force per unit area and, in view of the summation convention with a repeated index,  $T_{mn}n_n$  is the  $m$ th component of a vector. The vector whose components are  $T_{mn}n_n$  has special significance and is therefore given the name *traction* and a symbol  $\tau$ . Thus the  $m$ th component of the traction is written as\*

$$\tau_m = T_{mn}n_n = T_{m1}n_1 + T_{m2}n_2 + T_{m3}n_3. \quad (8.2.2)$$

We show subsequently that the traction  $\tau$ , defined by (8.2.2), is actually the vector force per unit area applied to a surface of arbitrary orientation. For the moment, however, we use (8.2.2) to attach some physical significance to the components of the stress tensor.

Assume that the surface integral of (8.1.17) is to be taken over the rectangular volume whose faces are perpendicular to the coordinate axes illustrated in Fig. 8.2.1. We can express (8.1.17) as the sum of six integrals taken over the six plane faces of the volume. As an example, consider the top face, which has the outward directed normal vector.

$$\mathbf{n} = \mathbf{i}_1.$$

The components of this normal vector are

$$n_1 = 1, \quad n_2 = n_3 = 0.$$

Consequently, the three components of the traction on the top surface are

$$\tau_1 = T_{11}, \quad \tau_2 = T_{21}, \quad \tau_3 = T_{31}.$$

These components and the vector  $\tau$  are illustrated in Fig. 8.2.1. Next, consider the bottom face, which has the outward directed normal vector

$$\mathbf{n} = -\mathbf{i}_1.$$

The components of this normal vector are

$$n_1 = -1, \quad n_2 = n_3 = 0.$$

\* Note that the subscript on the traction  $\tau_m$  is the same as the *first* subscript on the stress tensor component  $T_{mn}$ . This choice for the order of subscripts on  $T_{mn}$  is a matter of convention. Although the convention used here is prevalent in the literature, the opposite convention is used. Therefore it is wise to identify the convention used in each case by inspecting equations of the form of (8.1.10) or (8.1.17).



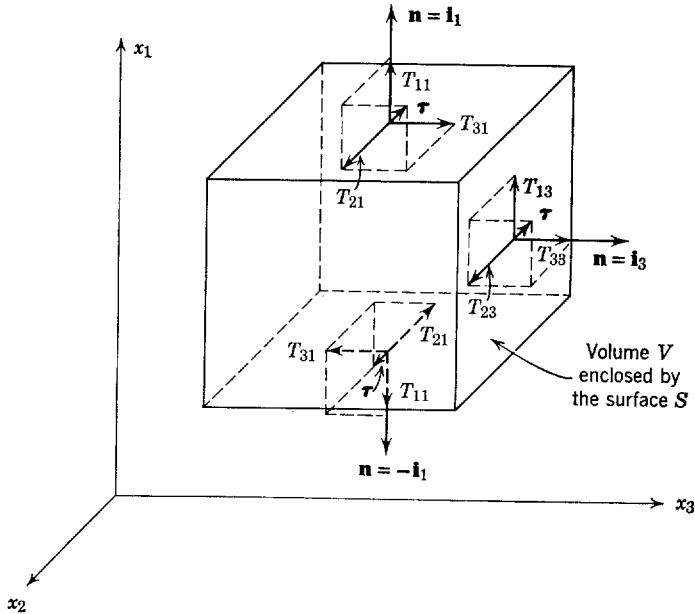


Fig. 8.2.1 A rectangular volume  $V$ , acted on by a stress  $T_{mn}$ .

Thus the three components of the traction on the bottom face are

$$\tau_1 = -T_{11}, \quad \tau_2 = -T_{21}, \quad \tau_3 = -T_{31}.$$

These components and the vector  $\boldsymbol{\tau}$  are illustrated in Fig. 8.2.1.

A similar process can be followed to find the surface traction  $\boldsymbol{\tau}$  on each of the other faces. The vector and its components for the face with outward directed normal vector  $\mathbf{n} = \mathbf{i}_3$  are also shown in Fig. 8.2.1.

We have shown that *the component  $T_{mn}$  of the stress tensor can be physically interpreted as the  $m$ th component of the traction applied to a surface with a normal vector in the  $n$ -direction*. Thus  $T_{23}$  is the  $x_2$ -directed component of the traction applied to a surface whose normal vector is  $\mathbf{i}_3$ .

We use the ideas developed with Fig. 8.2.1 to construct, in component form, the tractions on all six faces of a rectangular volume in Fig. 8.2.2. The faces are perpendicular to the three axes and the position of each face is defined. The corresponding stresses act in opposite directions on opposite faces. Consequently, if each component of the stress tensor is a constant over the whole volume, the stresses exactly oppose one another and no net force is applied to the material inside the volume. The stress tensor must vary with space to produce a net force.

To illustrate this mathematically we assume the dimensions of the volume

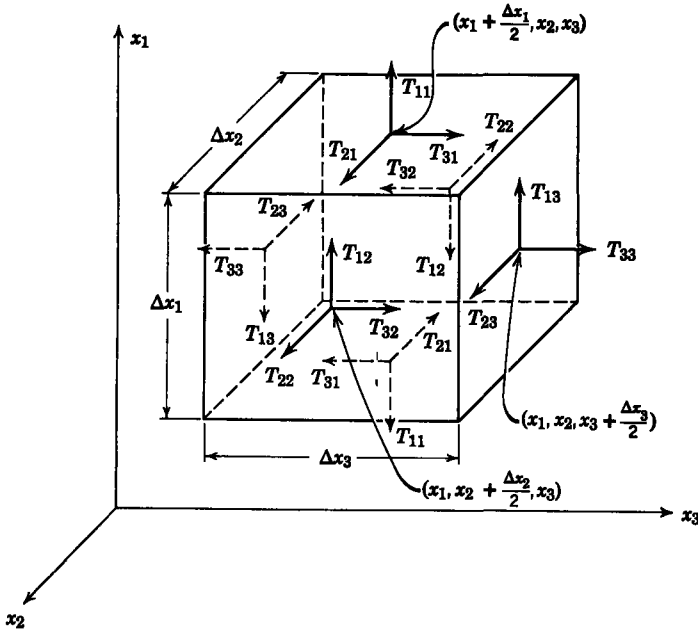


Fig. 8.2.2 Rectangular volume with center at  $(x_1, x_2, x_3)$  showing the surfaces and directions of the stresses  $T_{mn}$ .

to be small enough that components of the stress tensor do not vary appreciably over one face. We use (8.1.17) to evaluate the  $x_1$ -component of the total force applied to the material within the volume as

$$\begin{aligned}
 f_1 = & T_{11} \left( x_1 + \frac{\Delta x_1}{2}, x_2, x_3 \right) \Delta x_2 \Delta x_3 - T_{11} \left( x_1 - \frac{\Delta x_1}{2}, x_2, x_3 \right) \Delta x_2 \Delta x_3 \\
 & + T_{12} \left( x_1, x_2 + \frac{\Delta x_2}{2}, x_3 \right) \Delta x_1 \Delta x_3 - T_{12} \left( x_1, x_2 - \frac{\Delta x_2}{2}, x_3 \right) \Delta x_1 \Delta x_3 \\
 & + T_{13} \left( x_1, x_2, x_3 + \frac{\Delta x_3}{2} \right) \Delta x_1 \Delta x_2 - T_{13} \left( x_1, x_2, x_3 - \frac{\Delta x_3}{2} \right) \Delta x_1 \Delta x_2.
 \end{aligned}
 \tag{8.2.3}$$

Here we have evaluated the components of the stress tensor at the centers of the surfaces on which they act; for example, the stress component  $T_{11}$  acting on the top surface is evaluated at a point having the same  $x_2$ - and  $x_3$ -coordinates as the center of the volume but an  $x_1$  coordinate  $\Delta x_1/2$  above the center.

The dimensions of the volume have already been specified as quite small. In fact, we are interested in the limit as the dimensions go to zero. Consequently, each component of the stress tensor is expanded in a Taylor series

about the value at the volume center with only linear terms in each series retained to write (8.2.3) as

$$\begin{aligned} f_1 = & \left( T_{11} + \frac{\Delta x_1}{2} \frac{\partial T_{11}}{\partial x_1} - T_{11} + \frac{\Delta x_1}{2} \frac{\partial T_{11}}{\partial x_1} \right) \Delta x_2 \Delta x_3 \\ & + \left( T_{12} + \frac{\Delta x_2}{2} \frac{\partial T_{12}}{\partial x_2} - T_{12} + \frac{\Delta x_2}{2} \frac{\partial T_{12}}{\partial x_2} \right) \Delta x_1 \Delta x_3 \\ & + \left( T_{13} + \frac{\Delta x_3}{2} \frac{\partial T_{13}}{\partial x_3} - T_{13} + \frac{\Delta x_3}{2} \frac{\partial T_{13}}{\partial x_3} \right) \Delta x_1 \Delta x_2 \end{aligned}$$

or

$$f_1 = \left( \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} \right) \Delta x_1 \Delta x_2 \Delta x_3. \quad (8.2.4)$$

All terms in this expression are to be evaluated at the center of the volume  $(x_1, x_2, x_3)$ . We have thus verified our physical intuition that space-varying stress tensor components are necessary to obtain a net force.

From (8.2.4) we can obtain the  $x_1$ -component of the force density  $\mathbf{F}$  at the point  $(x_1, x_2, x_3)$  by writing

$$F_1 = \lim_{\Delta x_1, \Delta x_2, \Delta x_3 \rightarrow 0} \frac{f_1}{\Delta x_1 \Delta x_2 \Delta x_3} = \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3}. \quad (8.2.5)$$

The limiting process makes the expansion of (8.2.4) exact. The summation convention is used to write (8.2.5) as

$$F_1 = \frac{\partial T_{1n}}{\partial x_n}. \quad (8.2.6)$$

A similar process for the other two components of the force and force density yields the general result that the  $m$ th component of the force density at a point is

$$F_m = \frac{\partial T_{mn}}{\partial x_n}. \quad (8.2.7)$$

This is the result obtained in (8.1.10), which was derived for magnetic forces. Thus we have made the transition from the integral in (8.1.17) to the derivative in (8.2.7)—the reverse of the process in which we used the divergence theorem to obtain (8.1.17) from (8.1.10).

Although the formalism presented in this section is based on a result derived with magnetic forces, the stress tensor has a more general significance, as we shall see in later chapters; for example, the rectangular volume in Fig. 8.2.2 can be a block of elastic material with mechanical stresses applied to the surfaces. Our derivation and interpretations are still valid with respect to mechanical forces and force densities. For the moment we restrict our

examples to consider only magnetic forces because they are the only ones we have introduced formally.

**Example 8.2.1.** To illustrate some properties of the stress tensor and the mathematical techniques used with it, consider the system illustrated schematically in Fig. 8.2.3. The system consists of a long, cylindrical, nonmagnetic ( $\mu = \mu_0$ ) conductor whose axis coincides with the  $x_3$ -axis. The conductor carries a uniform constant current density

$$\mathbf{J} = \mathbf{i}_3 J. \quad (\text{a})$$

An electromagnet, not shown, produces a uniform magnetic field intensity

$$\mathbf{H}_0 = \mathbf{i}_1 H_0, \quad (\text{b})$$

when  $J = 0$ . The conductor is long enough that we can ignore any variations with  $x_3$ ; thus the problem is two-dimensional.

Because the field problem is linear, we can superimpose the field  $\mathbf{H}_0$  with the field excited by the current density  $\mathbf{J}$ . To calculate the two nonzero components  $H'_1$  and  $H'_2$ , due to  $\mathbf{J}$ , we establish a cylindrical coordinate system as illustrated in Fig. 8.2.4 and use the integral form of Ampère's law to obtain

$$H_\theta = \frac{Jr}{2} \quad \text{for } r < R, \quad (\text{c})$$

$$H_\theta = \frac{JR^2}{2r} \quad \text{for } r > R. \quad (\text{d})$$

The transformation from cylindrical to cartesian coordinates\* is used to find the cartesian components of this field. We then add the externally applied field  $H_0$  to obtain the total field intensity as

$$\left. \begin{aligned} H_1 &= H_0 - \frac{J}{2} x_2 \\ H_2 &= \frac{J}{2} x_1 \end{aligned} \right\} \quad \text{for } x_1^2 + x_2^2 < R^2. \quad (\text{e})$$

$$\left. \begin{aligned} H_1 &= H_0 - \frac{JR^2}{2} \left( \frac{x_2}{x_1^2 + x_2^2} \right) \\ H_2 &= \frac{JR^2}{2} \left( \frac{x_1}{x_1^2 + x_2^2} \right) \end{aligned} \right\} \quad \text{for } x_1^2 + x_2^2 > R^2, \quad (\text{f})$$

The component  $H_3$  is zero; thus we use (8.1.11) and (8.2.1) to write the stress tensor

$$(T_{mn}) = \begin{bmatrix} \frac{\mu_0}{2} (H_1^2 - H_2^2) & \mu_0 H_1 H_2 & 0 \\ \mu_0 H_1 H_2 & \frac{\mu_0}{2} (H_2^2 - H_1^2) & 0 \\ 0 & 0 & -\frac{\mu_0}{2} (H_1^2 + H_2^2) \end{bmatrix}. \quad (\text{g})$$

\* H. B. Phillips, *Analytic Geometry and Calculus*, 2nd ed., Wiley, New York, 1946, p. 206.

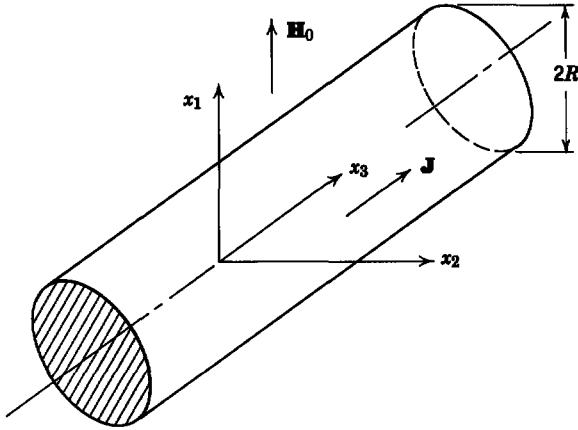


Fig. 8.2.3 A cylindrical conductor carrying uniform current density in the presence of a uniform applied field.

Now (e) or (f) can be used with this expression to find the components of the stress tensor both inside and outside the conductor. First the force density inside the conductor is calculated from (8.1.3):

$$\mathbf{F} = \mathbf{J}_f \times \mathbf{B} = -i_1 J \mu_0 H_2 + i_2 J \mu_0 H_1$$

or

$$\mathbf{F} = -i_1 \frac{\mu_0 J^2 x_1}{2} + i_2 \mu_0 J \left( H_0 - \frac{J x_2}{2} \right). \tag{h}$$

Thus there is a force density term due to the interaction between the current density and the externally applied field and a term due to interaction of the current density with the field it produces.

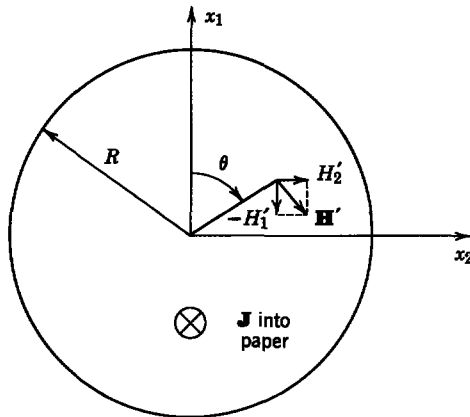


Fig. 8.2.4 Geometry for calculating fields excited by J.

To calculate this same force density from the stress tensor we use (8.1.10) and write for the  $x_2$ -component

$$F_2 = \frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2}. \quad (i)$$

By substitution of (e) into (g) this expression becomes

$$F_2 = \frac{\partial}{\partial x_1} \left[ \frac{\mu_0 J x_1}{2} \left( H_0 - \frac{J x_2}{2} \right) \right] + \frac{\partial}{\partial x_2} \left\{ \frac{\mu_0}{2} \left[ \frac{J^2 x_1^2}{4} - \left( H_0 - \frac{J}{2} x_2 \right)^2 \right] \right\}. \quad (j)$$

Performance of the indicated differentiations yields the  $x_2$ -component of (h), as it should. A similar process can be used to calculate the  $x_1$ -component of the force density and also to show that the  $x_3$ -component of the force density is zero.

It should be evident from a comparison of the effort required to obtain (h) and (j) that the stress tensor is not normally used to calculate force density in a system such as this. We present this example to illustrate the correspondence between the two methods and to illustrate the mathematical processes involved.

It is clear that outside the conductor the force density must be zero because the current density is zero; however, (f) and (g) show that the stress tensor has nonzero components in this region. To show that (8.1.10) yields a zero force density in this region we write the expression for the  $x_2$ -component of (8.1.10) outside the conductor ( $x_1^2 + x_2^2 > R^2$ ):

$$F_2 = \frac{\partial}{\partial x_1} \left\{ \frac{\mu_0 J R^2}{2} \left( \frac{x_1}{x_1^2 + x_2^2} \right) \left[ H_0 - \frac{J R^2}{2} \left( \frac{x_2}{x_1^2 + x_2^2} \right) \right] \right\} + \frac{\partial}{\partial x_2} \left\{ \frac{\mu_0}{2} \left[ \frac{J^2 R^4}{4} \left( \frac{x_1}{x_1^2 + x_2^2} \right)^2 - \left[ H_0 - \frac{J R^2}{2} \left( \frac{x_2}{x_1^2 + x_2^2} \right) \right]^2 \right] \right\}. \quad (k)$$

The indicated differentiation can be carried out to verify that this component of force density is zero. A similar process can be used to show that  $F_1$  and  $F_3$ , calculated from the stress tensor, are zero outside the conductor.

We now turn to the problem of calculating the total magnetic force on a length  $l$  of the conductor. First a volume integration of the force density given by (h) is performed. Because there are no variations of the fields with  $x_3$ , we use as a volume element

$$dV = l dx_1 dx_2$$

and use (h) and the geometry of Fig. 8.2.4 to write

$$\mathbf{f} = \int_{-R}^R \int_{-\sqrt{R^2-x_1^2}}^{\sqrt{R^2-x_1^2}} \left[ -\mathbf{i}_1 \frac{\mu_0 J^2 x_1}{2} + \mathbf{i}_2 \mu_0 J \left( H_0 - \frac{J x_2}{2} \right) \right] l dx_1 dx_2.$$

We integrate this equation with respect to  $x_2$ , evaluate the result at the limits, and obtain

$$\mathbf{f} = \int_{-R}^R [-\mathbf{i}_1 \mu_0 J^2 x_1 \sqrt{R^2 - x_1^2} + \mathbf{i}_2 2 \mu_0 J H_0 \sqrt{R^2 - x_1^2}] l dx_1$$

Evaluation of this integral with the specified limits yields the final result

$$\mathbf{f} = \mathbf{i}_2 J \mu_0 H_0 \pi R^2 l. \quad (l)$$

This is simply the uniform force density due to the externally applied field  $H_0$  multiplied by the volume  $\pi R^2 l$ . That the forces due to the self-field canceled out is a result of the cylindrical symmetry. Thus the force density due to the self-fields tends to deform the conductor but produces no net force that tends to move it.

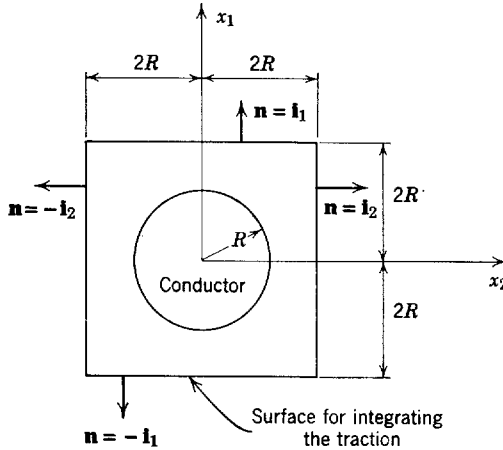


Fig. 8.2.5 Illustrating the surface for integrating the traction.

To use the stress tensor in calculating this same total force we use (8.1.17) with a surface that encloses a length  $l$  of the conductor. To make it quite clear that we can use a surface that is totally outside the body we choose a surface of length  $l$  and of square cross section with sides  $4R$ , as shown in end view in Fig. 8.2.5. Because this surface is completely outside the conductor we must use (f) with (g) to calculate the components of the stress tensor.

None of the quantities varies with  $x_3$ ; consequently, we recognize that the contribution to (8.1.17) from the two ends perpendicular to the  $x_3$ -axis is zero. The contribution from one end is the negative of that from the other end. We calculate only the  $x_2$ -component of the force. A similar process can be carried out for the other two components and (l) indicates that they integrate to zero.

We use the four lateral surfaces whose normal vectors are defined in Fig. 8.2.5 to write (8.1.17) for the  $x_2$ -component as

$$f_2 = \int_{-2R}^{2R} T_{21}(2R, x_2)l dx_2 - \int_{-2R}^{2R} T_{21}(-2R, x_2)l dx_2 + \int_{-2R}^{2R} T_{22}(x_1, 2R)l dx_1 - \int_{-2R}^{2R} T_{22}(x_1, -2R)l dx_1. \quad (m)$$

The stress components in the integrands are given by (g) and can be evaluated in terms of the magnetic field components by using (f). Then integration yields the result

$$f_2 = J\mu_0 H_0 \pi R^2 l.$$

This is the same as (l) which was obtained by integrating the volume force density throughout the conductor.

We have verified in an example that we can obtain the total force on current-carrying material within a volume by integrating the traction over a surface enclosing the volume. It is illuminating to investigate the nature of the tractions involved in this integration. For this purpose we refer to Fig. 8.2.2 in which we interpreted the components of the stress tensor as being the components of the traction. Thus we recognize that the first two integrals in (m) involve the  $x_2$ -component of the traction applied to surfaces whose normal vectors

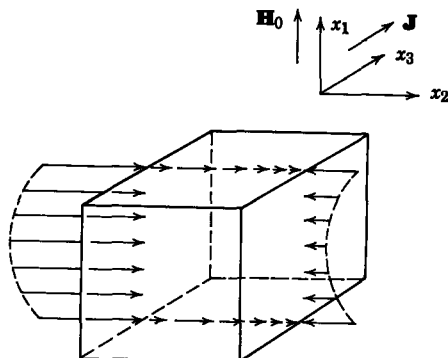


Fig. 8.2.6 Stress distribution.

are in the  $x_1$ -direction. Because these tractions are applied along a surface they are referred to as *shear stresses*. The second two integrals in (m) involve components of the traction that are perpendicular to the surfaces to which they are applied. Such tractions are called *normal stresses*.

If we wish to carry our interpretation a step further and say that there are stresses transmitted through space by the magnetic field as indicated by the Maxwell stress tensor, we can interpret the integrands of (m) as being stresses applied to the four surfaces. We use the integrands to sketch these stresses in Fig. 8.2.6. The shear stresses are equal on top and bottom and are in the direction of the net force. The normal stresses are compressive and there is an excess of stress applied to the left side.

Although the interpretation of the Maxwell stress tensor as representing mechanical stresses transmitted by fields through empty space is often useful it must be employed with understanding; for example, we could add a constant to all components of the stress tensor and not change the results of our calculations of force density and total force. The stress pattern of Fig. 8.2.6, however, would be changed markedly.

In (8.2.2) we defined the  $m$ th component  $\tau_m$  of the traction  $\boldsymbol{\tau}$  as

$$\tau_m = T_{mn}n_n. \quad (8.2.8)$$

The traction was interpreted as the vector force per unit area applied to a surface with components  $n_n$  of the normal vector  $\mathbf{n}$ . The integral force equation (8.1.17) suggests that  $\boldsymbol{\tau}$  represents the force per unit area for a surface of arbitrary orientation. This fact is emphasized by the discussion which follows.

Figure 8.2.7 is a tetrahedron with three of its edges parallel to  $x_1, x_2, x_3$ -axes. One surface of the tetrahedron has a normal vector  $\mathbf{n}$  and supports the traction  $\boldsymbol{\tau}$  (which, in general, is not in the direction of  $\mathbf{n}$ ). Because three of the surfaces have normal vectors that are in the axis directions, the tractions on these surfaces can be written in terms of the components  $T_{mn}$ , whereas the traction on the fourth surface is the unknown  $\boldsymbol{\tau}$ . Although the surface tractions (and in particular  $T_{mn}$ ) depend on the space coordinates, it has been



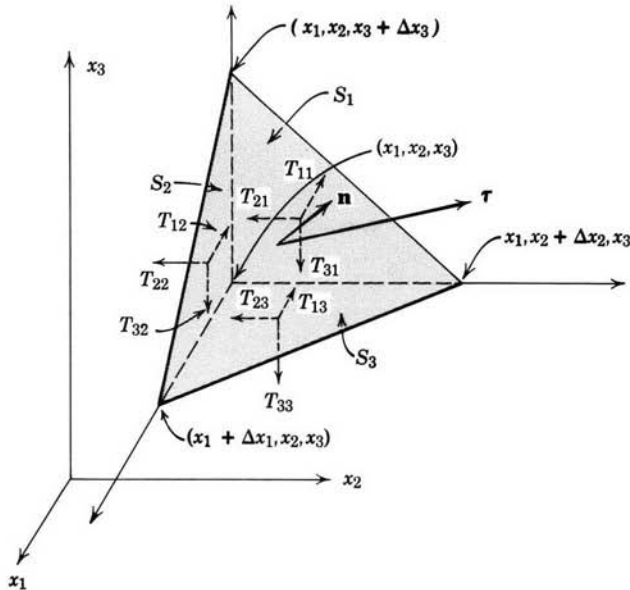


Fig. 8.2.7 The small tetrahedron used to find the surface traction  $\tau$  on a surface with the normal vector  $\mathbf{n}$  in terms of the components of the stress tensor  $T_{mn}$ .

implicitly assumed that  $T_{mn}$  is a continuous function. Hence, as  $\Delta x_1, \Delta x_2, \Delta x_3 \rightarrow 0$ , the traction  $\tau$  must balance the stresses on the negative surfaces. Here we use the fact that the volume forces are proportional to the volume  $\Delta x_1 \Delta x_2 \Delta x_3$ , whereas the surface tractions produce forces proportional to areas, that is,  $\Delta x_1 \Delta x_2$ ,  $\Delta x_2 \Delta x_3$  or  $\Delta x_3 \Delta x_1$ . Hence in the limit in which  $\Delta x_1, \Delta x_2, \Delta x_3 \rightarrow 0$ , the prism of material is not in force equilibrium unless the surface forces balance.

If the surface with the normal  $\mathbf{n}$  has the area  $S$ , the negative surfaces have the areas  $Sn_1, Sn_2, Sn_3$ ,\* respectively, and continuity of the stresses which act in the  $x_1$ -direction gives rise to the equation

$$\tau_1 S \cong T_{11} S n_1 + T_{12} S n_2 + T_{13} S n_3. \quad (8.2.9)$$

In the limit in which the dimensions of the tetrahedron become small (8.2.9) becomes exact. Since the equation can also be written for the other components of the stress, (8.2.8) follows.

\* A proof of this geometric relation can be made by using Gauss's theorem  $\oint_S \mathbf{A} \cdot \mathbf{n} \, da = \int_V (\nabla \cdot \mathbf{A}) \, dV$  with  $\mathbf{A} = \mathbf{i}_1$ . The volume integral vanishes and the surface integral (integrated over the surface of the tetrahedron) becomes  $-S_1 + S n_1 = 0$ , where  $S_1$  is the area of the back surface with the normal  $-\mathbf{i}_1$ . Similar arguments hold using  $\mathbf{A} = \mathbf{i}_2$  and  $\mathbf{A} = \mathbf{i}_3$ .

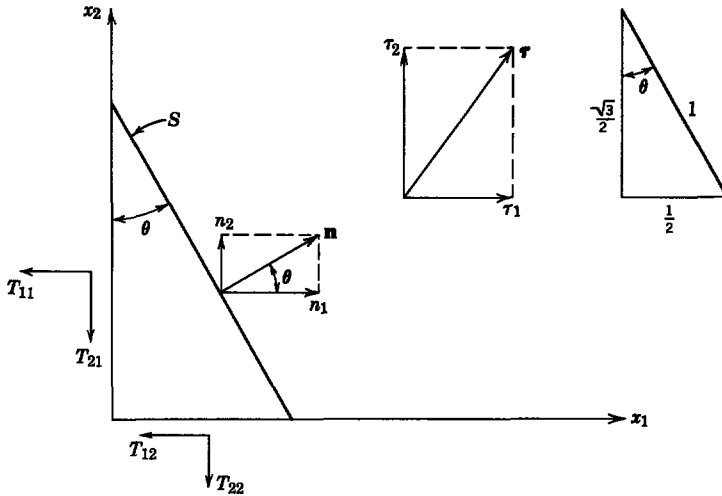


Fig. 8.2.8 Example of surface traction  $\tau$  acting on a particular surface  $S$ .

**Example 8.2.2.** A brief example will help to fix the meaning of (8.2.8). We wish to derive the traction  $\tau$  on the surface  $S$  shown in Fig. 8.2.8, given the stresses  $T_{11}$ ,  $T_{12}$ , etc. It is assumed that  $\mathbf{n}$  lies in the  $x_1$ - $x_2$  plane, so that from the figure the normal vector is

$$\mathbf{n} = \mathbf{i}_1 \frac{\sqrt{3}}{2} + \mathbf{i}_2 \frac{1}{2}. \tag{a}$$

Note that the components of  $\mathbf{n}$  are *not* the unit vectors  $\mathbf{i}_1$ ,  $\mathbf{i}_2$ ,  $\mathbf{i}_3$ . According to (8.2.8), the components of  $\tau$  acting on the surface  $S$  are

$$\begin{aligned} \tau_1 &= T_{11} \frac{\sqrt{3}}{2} + T_{12} \frac{1}{2}, \\ \tau_2 &= T_{21} \frac{\sqrt{3}}{2} + T_{22} \frac{1}{2}, \\ \tau_3 &= 0, \end{aligned} \tag{b}$$

where we have assumed that  $T_{31}$ ,  $T_{32}$ , and  $T_{33}$  are zero or that there are no components of the stress acting in the  $x_3$ -direction. This example should make it clear that all we have done in writing (8.2.8) is to formalize our interpretation of the stress components as forces per unit area acting on surfaces that are perpendicular to the axis directions. The results could be derived from inspection of Fig. 8.2.8 without making use of (8.2.8). Try it!

### 8.2.2 Vector and Tensor Transformations

In our discussion so far we have interpreted the physical properties of the stress tensor in terms of the vector traction  $\tau$  whose components are defined by (8.2.2). We now use the mathematical properties of the vector  $\tau$  to describe some mathematical properties of the stress tensor.

The traction  $\tau$  is a vector. The components of this vector depend on the coordinate system in which  $\tau$  is expressed; for example, the vector might be directed in one of the coordinate directions  $(x_1, x_2, x_3)$ , in which case there would be only one nonzero component of  $\tau$ . In a second coordinate system  $(x'_1, x'_2, x'_3)$ , this same vector might have components in all of the coordinate directions. Analyzing a vector into orthogonal components along the coordinate axes is a familiar process. The components in a cartesian coordinate system  $(x'_1, x'_2, x'_3)$  are related to those in the cartesian coordinate system  $(x_1, x_2, x_3)$  by the three equations

$$\tau'_p = a_{pr}\tau_r, \quad (8.2.10)$$

where  $a_{pr}$  is the cosine of the angle between the  $x'_p$ -axis and the  $x_r$ -axis.

**Example 8.2.3.** Suppose that we wish to use (8.2.10) to compute the components  $\tau'_m$  of the vector  $\tau'$  in the primed coordinate system shown in Fig. 8.2.9, in terms of the known components  $\tau_m$  of  $\tau$  in the unprimed coordinate system. (It should be recognized that the  $x'_1$  axis in this figure is in the direction of the normal in Fig. 8.2.8, so that we can consider this example as an extension of the preceding one.) From the geometry the cosine of the angle between

$$\begin{aligned} x'_1 \text{ and } x_1 &= a_{11} = \frac{\sqrt{3}}{2}, & x'_1 \text{ and } x_2 &= a_{21} = \frac{\sqrt{3}}{2}, \\ x'_1 \text{ and } x_3 &= a_{31} = \frac{1}{2}, & x'_2 \text{ and } x_1 &= a_{12} = -\frac{1}{2}, \\ x'_2 \text{ and } x_2 &= a_{22} = \frac{\sqrt{3}}{2}, & x'_2 \text{ and } x_3 &= a_{32} = 1, \\ x'_3 \text{ and } x_1 &= a_{13} = 0, & x'_3 \text{ and } x_2 &= a_{23} = 0, \\ x'_3 \text{ and } x_3 &= a_{33} = 1, & \text{all others} &= 0. \end{aligned}$$

Hence by definition

$$[a_{mn}] = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

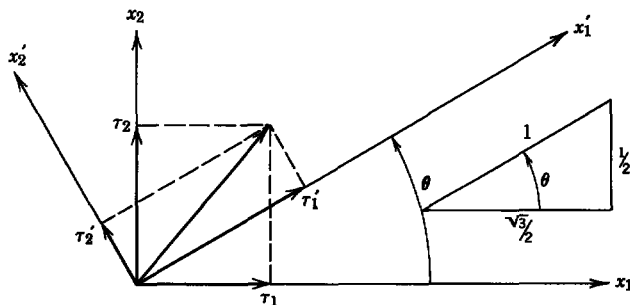


Fig. 8.2.9 Geometrical relationship between the primed and unprimed coordinate systems for Example 8.2.3.

Then (8.2.10) gives

$$\begin{aligned}\tau'_1 &= \frac{\sqrt{3}}{2} \tau_1 + \frac{1}{2} \tau_2, \\ \tau'_2 &= -\frac{1}{2} \tau_1 + \frac{\sqrt{3}}{2} \tau_2.\end{aligned}$$

From this example (8.2.10) should be recognized as a simple statement of vector addition. Again, we could have obtained the result from Fig. 8.2.9 without the formalism of (8.2.10).

Equation 8.2.10 forms the basis for determining how to transform components of the stress tensor from one coordinate system to another.

According to (8.2.2), the components of  $\tau$  are

$$\tau_r = T_{rs} n_s. \quad (8.2.11)$$

Now we consider a particular cartesian coordinate system  $(x'_1, x'_2, x'_3)$  established in such a way that one of the axes (say  $x'_1$ ) has the same direction as  $\mathbf{n}$ . A pictorial representation of the two coordinate systems is given in Fig. 8.2.10. The components  $(n_1, n_2, n_3)$  of the normal vector are the cosines of the angles between the  $(x_1, x_2, x_3)$  axes and the normal direction, which is also the direction of  $x'_1$ . Hence from the definition following (8.2.10)  $(n_1, n_2, n_3) = (a_{11}, a_{12}, a_{13})$  and (8.2.11) can also be written as

$$\tau_r = T_{rs} a_{1s}. \quad (8.2.12)$$

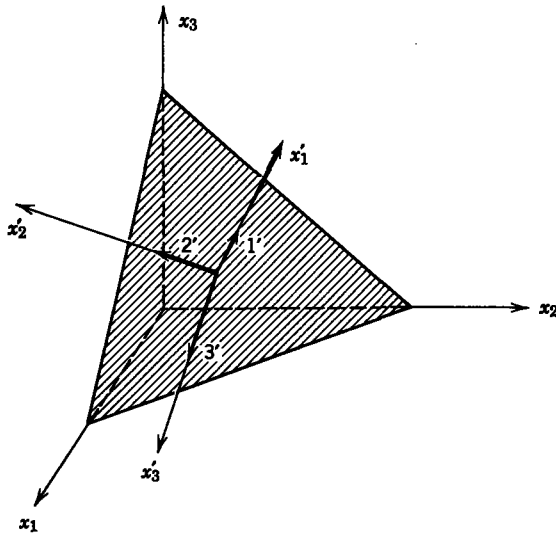


Fig. 8.2.10 Relationship between the primed and unprimed coordinates showing the  $x'_1$ -axis coincident with the normal vector.

Because  $x'_1$  is perpendicular to the surface,  $x'_2$  and  $x'_3$  lie in the surface. We see that  $(\tau'_1, \tau'_2, \tau'_3)$  are just the components of the stress acting on a surface with a normal in the direction of the  $x'_1$ -axis, that is,

$$\tau'_p = T'_{p1}, \quad (8.2.13)$$

but we can also use (8.2.10) to express  $\tau'_p$  as

$$\tau'_p = a_{pr} \tau_r, \quad (8.2.14)$$

which by (8.2.12) gives a relation for  $\tau'_p$  in terms of the stress components in the unprimed coordinates.

$$\tau'_p = a_{pr}(T_{rs} a_{1s}) \quad (8.2.15)$$

Then from 8.2.13

$$T'_{p1} = a_{pr} a_{1s} T_{rs}. \quad (8.2.16)$$

Finally, the designation of the normal direction by the  $x'_1$ -axis is arbitrary, and the preceding arguments could be repeated with 1 replaced by 2 or 1 replaced by 3. Hence we have shown that

$$T'_{pq} = a_{pr} a_{qs} T_{rs}. \quad (8.2.17)$$

This relation provides the rule for finding the components of the stress in the primed coordinates, given the components in the unprimed coordinates. It serves the same purpose in dealing with tensors that (8.2.10) serves in dealing with vectors. In much of the literature a vector or *first-order tensor* is defined as an array of three numbers that transforms according to an equation in the form of (8.2.10). In the same way, a *second-order tensor* is defined as an array of numbers that transforms according to an equation in the form of (8.2.17).\*

**Example 8.2.4.** Suppose we wish to find the stress component  $T'_{11}$  expressed in the primed coordinate system of Fig. 8.2.9 in terms of the components  $T_{mn}$  in the unprimed system. Then (8.2.17) gives

$$T'_{11} = a_{11} a_{11} T_{11} + a_{11} a_{12} T_{12} + a_{11} a_{13} T_{13} + a_{12} a_{11} T_{21} + a_{12} a_{12} T_{22} + a_{12} a_{13} T_{23} \\ + a_{13} a_{11} T_{31} + a_{13} a_{12} T_{32} + a_{13} a_{13} T_{33}$$

or, in particular, from the values of  $a_{mn}$  given in Example 8.2.3,

$$T'_{11} = \frac{\sqrt{3}}{2} \frac{\sqrt{3}}{2} T_{11} + \frac{\sqrt{3}}{2} \left(\frac{1}{2}\right) T_{12} + \left(\frac{1}{2}\right) \frac{\sqrt{3}}{2} T_{21} + \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) T_{22}.$$

A second example provides a useful result.

**Example 8.2.5.** Given the stress components  $T_{mn}$  expressed in a cylindrical coordinate system with the coordinates  $r$ ,  $\theta$ , and  $z$ , what are the components of the stress tensor

\* See, for example, Spain, *op. cit.*, pp. 6-9.

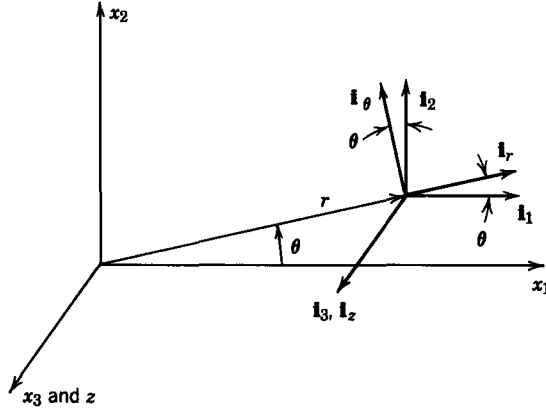


Fig. 8.2.11 Geometrical relationship between cartesian and cylindrical coordinate systems.

expressed in a cartesian coordinate system with axes  $x_1, x_2,$  and  $x_3,$  as illustrated in Fig. 8.2.11.\*

The relationship between the unit vectors is shown in Fig. 8.2.11. The cartesian coordinate system plays the role of the “primed” system. We can see by inspection that the cosine of the angle between

$$\begin{aligned}
 i_1 \text{ and } i_r &= \cos \theta, \\
 i_1 \text{ and } i_\theta &= \cos (\theta + 90^\circ) = -\sin \theta, \\
 i_2 \text{ and } i_r &= \cos (90^\circ - \theta) = \sin \theta, \\
 i_2 \text{ and } i_\theta &= \cos \theta, \\
 i_3 \text{ and } i_z &= 1, \\
 \text{all others} &= 0.
 \end{aligned}$$

Therefore we can write

$$(a_{mn}) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The components of the stress now follow directly by making use of (8.2.17):

$$\begin{aligned}
 T_{11} &= T_{rr} \cos^2 \theta - 2T_{r\theta} \sin \theta \cos \theta + T_{\theta\theta} \sin^2 \theta, \\
 T_{12} &= T_{rr} \sin \theta \cos \theta + T_{r\theta}(\cos^2 \theta - \sin^2 \theta) - T_{\theta\theta} \sin \theta \cos \theta, \\
 T_{13} &= T_{rz} \cos \theta - T_{z\theta} \sin \theta, \\
 T_{22} &= T_{rr} \sin^2 \theta + 2T_{r\theta} \sin \theta \cos \theta + T_{\theta\theta} \cos^2 \theta, \\
 T_{23} &= T_{rz} \sin \theta + T_{z\theta} \cos \theta, \\
 T_{33} &= T_{zz}.
 \end{aligned}$$

\* When the components of a stress tensor are expressed in polar coordinates or any other curvilinear coordinates, care must be exercised in taking space derivatives. This is analogous to taking derivatives of vectors in curvilinear coordinates.

Before we leave the subject of tensor transformations we must make a final important observation. The direction cosines  $a_{mn}$  which transformed the vector in (8.2.10) were defined with the understanding that the components of  $\tau$  were expressed in an *orthogonal* coordinate system. There were therefore implicit trigonometric relations between these direction cosines. If we state them formally, it is possible to extend the concept of a tensor to situations in which the transformations (8.2.10) and (8.2.17) are not geometrical in origin.\* These relations are easily established by means of (8.2.10).

Equation 8.2.10 is the transformation of a vector  $\tau$  from an unprimed to a primed coordinate system. There is, in general, nothing to distinguish the two coordinate systems. We could just as well define a transformation from the primed to the unprimed coordinates by

$$\tau_s = b_{sp}\tau'_p, \quad (8.2.18)$$

where  $b_{sp}$  is the cosine of the angle between the  $x_s$ -axis and the  $x'_p$ -axis. But  $b_{sp}$ , from the definition following (8.2.10), is then also

$$b_{sp} \equiv a_{ps}; \quad (8.2.19)$$

that is, the transformation which reverses the transformation (8.2.10) is

$$\tau_s = a_{ps}\tau'_p. \quad (8.2.20)$$

Now we can establish an important property of the direction cosines  $a_{ps}$  by transforming the vector  $\tau$  to an arbitrary primed coordinate system and then transforming the components  $\tau'_m$  back to the unprimed system in which they must be the same as those we started with. Equation 8.2.10 provides the first transformation, whereas (8.2.20) provides the second; that is, we substitute (8.2.10) into (8.2.20) to obtain

$$\tau_s = a_{ps}a_{pr}\tau_r. \quad (8.2.21)$$

Remember that we are required to sum on both  $p$  and  $r$ ; for example, consider the case in which  $s = 1$ :

$$\begin{aligned} \tau_1 &= (a_{11}a_{11} + a_{21}a_{21} + a_{31}a_{31})\tau_1 \\ &\quad + (a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32})\tau_2 \\ &\quad + (a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33})\tau_3. \end{aligned} \quad (8.2.22)$$

This relation must hold in general. We have not specified either  $a_{ps}$  or  $\tau_m$ . Hence the second two bracketed quantities must vanish and the first must be unity. We can express this fact much more concisely by stating that in general

$$a_{ps}a_{pr} = \delta_{sr} \quad (8.2.23)$$

\* J. C. Slater and N. H. Frank, *Mechanics*, 1st ed., McGraw-Hill, New York, 1947, Appendix V.

[this is the Kronecker delta as defined in (8.1.7)], for then (8.2.21) is reduced to the identity  $\tau_s = \tau_s$ .

### 8.3 FORCES IN ELECTRIC FIELD SYSTEMS

We now consider the forces that develop in electric field systems. The Lorentz force (1.1.28) gives the force on a charge  $q$  in an electric field  $\mathbf{E}$  as

$$\mathbf{f} = q\mathbf{E}. \quad (8.3.1)$$

The force density  $\mathbf{F}$  can be found by averaging (8.3.1) over a small volume:

$$\mathbf{F} = \lim_{\delta V \rightarrow 0} \frac{\sum_i \mathbf{f}_i}{\delta V} = \lim_{\delta V \rightarrow 0} \frac{\sum_i q_i \mathbf{E}_i}{\delta V}, \quad (8.3.2)$$

where  $q_i$  represents all the charges in  $\delta V$ ,  $\mathbf{E}_i$  is the electric field acting on the  $i$ th charge, and  $\mathbf{f}_i$  is the force on the  $i$ th charge. It is found experimentally that free charges are almost never dense enough to make the microscopic field  $\mathbf{E}_i$  seen by a charge appreciably different from the average (macroscopic) field  $\mathbf{E}$ . Consequently, because all charges in the volume  $\delta V$  experience the same electric field  $\mathbf{E}$ , we use the definition  $\rho_f = \lim_{\delta V \rightarrow 0} \sum_i q_i / \delta V$  to write (8.3.2) as

$$\mathbf{F} = \rho_f \mathbf{E}. \quad (8.3.3)$$

Once again, remember that this is the force density on the charges and (as for the magnetic force density (8.1.3)) can be construed as the material force density only if each of the charges transmits its force to the medium.

The constituent relation is

$$\mathbf{D} = \epsilon \mathbf{E}. \quad (8.3.4)$$

For this development we assume that  $\epsilon$  is a constant, but this restriction is relaxed in Section 8.5\*. In this case we write (8.3.3) in terms of the electric field intensity by using Gauss's law (1.1.12)†:

$$\mathbf{F} = (\nabla \cdot \epsilon \mathbf{E}) \mathbf{E}. \quad (8.3.5)$$

We now express (8.3.5) as the space derivative of a stress tensor by recognizing that for electric field systems  $\nabla \times \mathbf{E} = 0$ . Hence (8.3.5) can be written as

$$\mathbf{F} = (\nabla \cdot \epsilon \mathbf{E}) \mathbf{E} + (\nabla \times \mathbf{E}) \times \epsilon \mathbf{E}. \quad (8.3.6)$$

\* Arguments are given in Section 8.5 to show that this is the force density on free charges embedded in a material with a constant permittivity. For now we *assume* that the only effect of a uniform linear dielectric on the free charge force density is to replace  $\epsilon_0 \rightarrow \epsilon$ .

† Table 1.2, Appendix E.



We now use a vector identity on the last term to obtain\*

$$\mathbf{F} = (\nabla \cdot \epsilon \mathbf{E})\mathbf{E} + \epsilon(\mathbf{E} \cdot \nabla)\mathbf{E} - \frac{1}{2}\epsilon \nabla(\mathbf{E} \cdot \mathbf{E}). \quad (8.3.7)$$

Using the index notation introduced in Section 8.1, we combine the first two terms and write the  $m$ th component of this equation as

$$F_m = \frac{\partial}{\partial x_n} (\epsilon E_m E_n) - \frac{\epsilon}{2} \frac{\partial}{\partial x_m} (E_k E_k). \quad (8.3.8)$$

The Kronecker delta is now used to write

$$\frac{\partial}{\partial x_m} = \delta_{mn} \frac{\partial}{\partial x_n}$$

and to put (8.3.8) in the desired form,

$$F_m = \frac{\partial T_{mn}}{\partial x_n}, \quad (8.3.9)$$

where the Maxwell stress tensor  $T_{mn}$  for electric field systems is given by

$$T_{mn} = \epsilon E_m E_n - \frac{\epsilon}{2} \delta_{mn} E_k E_k. \quad (8.3.10)$$

Note that this expression has the same form as (8.1.11) if we replace  $\epsilon$  with  $\mu$  and  $\mathbf{E}$  with  $\mathbf{H}$ . The stress tensor here has all the general properties discussed in Section 8.2.2.

Both electric and magnetic forces are usually included in the Maxwell stress tensor†; however, we have not combined these forces because they usually do not occur in appreciable amounts in the same system. We use the term Maxwell stress tensor to denote that function from which electromagnetic force densities can be obtained by differentiation, as in (8.3.9). In different systems the Maxwell stress tensor represents different functions.

**Example 8.3.1.** To illustrate the use of the different expressions for force density and total force consider the electrostatic problem defined in Fig. 8.3.1.

The system consists of two regions of vacuum separated by a nonpolarizable ( $\epsilon = \epsilon_0$ ) slab of thickness  $\delta$  in the  $x_1$ -direction and of infinite extent in the other two directions. The slab contains a volume charge density

$$\rho_f = \rho_f^0 \left(1 - \frac{x_1}{\delta}\right) \quad (a)$$

for  $0 < x_1 < \delta$ . The electric field in the region

$$x_1 < 0$$

is constrained to be

$$\mathbf{E} = \mathbf{i}_1 E_1^0 + \mathbf{i}_2 E_2^0 + \mathbf{i}_3 E_3^0. \quad (b)$$

\*  $(\nabla \times \mathbf{A}) \times \mathbf{A} = (\mathbf{A} \cdot \nabla)\mathbf{A} - \frac{1}{2}\nabla(\mathbf{A} \cdot \mathbf{A})$ .

† J. A. Stratton, *Electromagnetic Theory*, McGraw-Hill, New York, 1941, pp. 97–103.

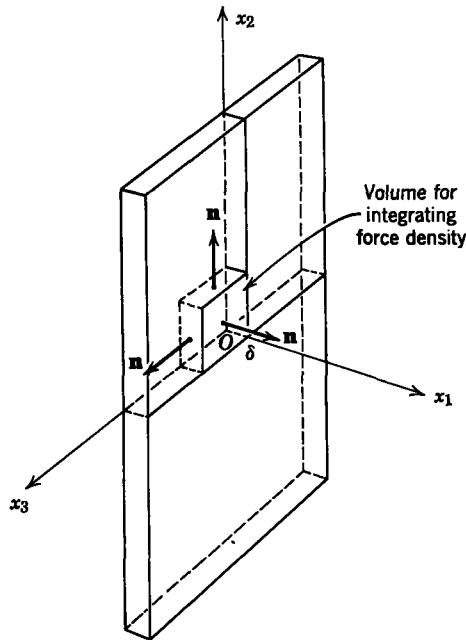


Fig. 8.3.1 Slab of material supporting a volume charge density.

After finding the electric field in the remainder of the system, we wish to compute in two ways the total force per unit area on the slab, first by doing a volume integration of the force density and then by doing a surface integration of the stress tensor.

To find the fields in the system we use the differential equations

$$\nabla \times \mathbf{E} = 0, \quad (\text{c})$$

$$\nabla \cdot \epsilon_0 \mathbf{E} = \rho_f. \quad (\text{d})$$

Because the slab has infinite extent in the  $x_2$ - $x_3$  plane, we can (for purposes of illustration) assume no variation of  $\mathbf{E}$  in the  $x_2$ - and  $x_3$ -directions.

$$\frac{\partial}{\partial x_2} = \frac{\partial}{\partial x_3} = 0.$$

Then (c) shows that everywhere

$$E_2 = E_2^0,$$

$$E_3 = E_3^0.$$

Equations a and d give

$$\frac{\partial E_1}{\partial x_1} = \frac{\rho_f^0}{\epsilon_0} \left( 1 - \frac{x_1}{\delta} \right).$$

Integration of this expression yields

$$E_1 = \frac{\rho_f^0}{\epsilon_0} \left( x_1 - \frac{x_1^2}{2\delta} \right) + C_1.$$

We use the boundary condition on the normal component of  $\mathbf{E}$  at  $x_1 = 0$  to evaluate the constant of integration.

$$C_1 = E_1^0.$$

Thus

$$E_1 = E_1^0 + \frac{\rho_f^0}{\epsilon_0} \left( x_1 - \frac{x_1^2}{2\delta} \right), \quad \text{for } 0 < x_1 < \delta,$$

and use of the boundary condition on the normal component of  $\mathbf{E}$  at  $x_1 = \delta$  yields

$$E_1 = E_1^0 + \frac{\rho_f^0 \delta}{2\epsilon_0}, \quad \text{for } x_1 > \delta.$$

The only region in which free charge exists is for  $0 < x_1 < \delta$ , where we can write

$$\mathbf{F} = \rho_f \mathbf{E} = \rho_f^0 \left( 1 - \frac{x_1}{\delta} \right) \left\{ \mathbf{i}_1 \left[ E_1^0 + \frac{\rho_f^0}{\epsilon_0} \left( x_1 - \frac{x_1^2}{2\delta} \right) \right] + \mathbf{i}_2 E_2^0 + \mathbf{i}_3 E_3^0 \right\}.$$

Taking a volume with unit dimension in the  $x_2$ - and  $x_3$ -directions, we write for the total force per unit area in an  $x_2$ - $x_3$  plane:

$$\mathbf{f} = \int_0^\delta \mathbf{F} dx_1.$$

Performance of the indicated integration yields

$$\mathbf{f} = \mathbf{i}_1 \rho_f^0 \left( \frac{E_1^0 \delta}{2} + \frac{\rho_f^0 \delta^2}{8\epsilon_0} \right) + \mathbf{i}_2 \frac{\rho_f^0 E_2^0 \delta}{2} + \mathbf{i}_3 \frac{\rho_f^0 E_3^0 \delta}{2}.$$

We can obtain this same result by using the stress tensor, which we need only along the surface that encloses the slab.

The surface selected for integrating (8.1.17) is the one shown in Fig. 8.3.1 which has unit area in the  $x_2$ - $x_3$  plane and thickness  $\delta$  in the  $x_1$ -direction. Because the fields are independent of  $x_2$  and  $x_3$ , the contributions from the surfaces perpendicular to  $x_2$  and  $x_3$  add to zero. We need only consider the surfaces of unit area perpendicular to  $x_1$ . Thus we have

$$\begin{aligned} f_1 &= T_{11}(\delta) - T_{11}(0), \\ f_2 &= T_{21}(\delta) - T_{21}(0), \\ f_3 &= T_{31}(\delta) - T_{31}(0). \end{aligned}$$

Using the components  $T_{mn}$  defined with the fields derived earlier, we have [remember that  $E_2(0) = E_2(\delta)$  and  $E_3(0) = E_3(\delta)$ ]

$$\begin{aligned} f_1 &= \frac{\epsilon_0}{2} [E_1^2(\delta) - E_1^2(0)] = \frac{\epsilon_0}{2} \left[ \frac{E_1^0 \rho_f^0 \delta}{\epsilon_0} + \frac{(\rho_f^0)^2 \delta^2}{4\epsilon_0^2} \right], \\ f_2 &= \epsilon_0 [E_1(\delta) E_2(\delta) - E_1(0) E_2(0)] = \epsilon_0 E_2^0 \frac{\rho_f^0 \delta}{2\epsilon_0}, \\ f_3 &= \epsilon_0 [E_1(\delta) E_3(\delta) - E_1(0) E_3(0)] = \epsilon_0 E_3^0 \frac{\rho_f^0 \delta}{2\epsilon_0}. \end{aligned}$$

Thus the result is the same as that obtained by the volume integration. Note that in the surface integration we needed only the fields outside the space occupied by the charge. The fields, of course, are affected by the presence of the charge.

The most significant advantages of a formulation that uses the stress tensor arise because forces on the material within a volume can be determined without knowing the details of the volume force distribution (i.e., the distribution of currents or charges). Moreover, in many problems we are at liberty to choose the surface of integration and this can further simplify the computation. The next example illustrates how the choice of the surface of integration that is most convenient (or makes the integration possible) depends on symmetry and boundary conditions and further shows how the stress tensor can be used to obtain a total force in a situation in which a more direct approach would be difficult if not impossible.

**Example 8.3.2.** A pair of perfectly conducting plates at the potential difference  $V_0$  is shown in Fig. 8.3.2. One of these plates is flat and the other has a step at the middle, as shown. Both plates extend far enough in the  $x_3$ -direction that we can consider the problem as two-dimensional ( $\partial/\partial x_3 = 0$ ). We wish to find the force in the  $x_1$ -direction on a section of length  $l$  (in the  $x_3$ -direction) of the bottom plate, including the effect of the fringing fields. To do this it is assumed that both  $c \gg a$  and  $d \gg b$ , so that the regions of nonuniform electric field near the ends and near the step are separated by regions of essentially uniform electric field intensity.

To carry out the surface integration [the  $x_1$ -component of (8.1.17)]

$$f_1 = \oint_S T_{1n} n_n da \tag{a}$$

we choose the surface shown in Fig. 8.3.2. Surfaces (1), (3) and (5) have the normal vector  $\mathbf{n} = \mathbf{i}_2$ , whereas surfaces (2) and (4) have the normal  $\mathbf{n} = \mp \mathbf{i}_1$ , respectively. Hence we can write (a) as

$$f_1 = \int_{(1)(5)} T_{12} da + \int_{(3)} T_{12} da - \int_{(2)} T_{11} da + \int_{(4)} T_{11} da + \int_{(6)} T_{1n} n_n da. \tag{b}$$

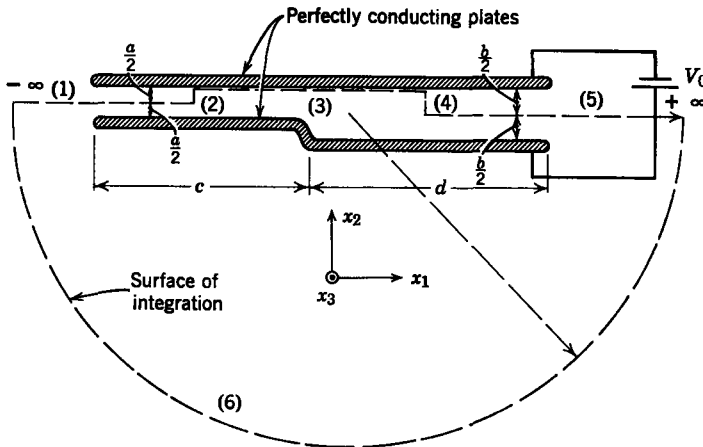


Fig. 8.3.2 Conducting plates at the potential difference  $V_0$ .

The contributions from the surfaces with normals  $\pm i_3$  have been ignored, for they cancel. Because the surface of integration (6) is far from the plates (at infinity), we expect the contribution of the last integral to be zero. We can argue that this is the case by making (6) the surface of a cylinder of radius  $R$ , with the plates at the origin. Far from the plates the electric field distribution is essentially that of a dipole.\* Hence  $|E| \approx 1/R^2$  and  $|T_{1n}| \approx 1/R^4$ . It follows that although the surface area of integration is proportional to  $R^2$  the integral (6) decreases as  $1/R^2$  and vanishes as  $R \rightarrow \infty$ . From (8.3.10)

$$T_{12} = \epsilon E_1 E_2. \quad (c)$$

Surfaces (1) and (5) are half way between the plates where by symmetry  $E_1 = 0$ . Hence the first term in (b) is also zero. Moreover, because  $E_1 = 0$  along the perfectly conducting plate where surface (3) is located, the second integral vanishes also.

From (8.3.10)

$$T_{11} = \frac{1}{2} \epsilon (E_1^2 - E_2^2 - E_3^2). \quad (d)$$

Surfaces (2) and (4) are in regions of uniform electric field intensity. Hence

$$\mathbf{E} = \frac{V_0}{a} \mathbf{i}_2; \quad \text{on surface (2),} \quad (e)$$

$$\mathbf{E} = \frac{V_0}{b} \mathbf{i}_2; \quad \text{on surface (4).}$$

Because these surfaces make the only contribution to the surface integral, (b), (d), and (e) become

$$f_1 = \int_{(2)} \frac{1}{2} \epsilon \left( \frac{V_0}{a} \right)^2 da - \int_{(4)} \frac{1}{2} \epsilon \left( \frac{V_0}{b} \right)^2 da. \quad (f)$$

The stresses are constant over the surfaces of integration and therefore the integral is performed by multiplication of the appropriate areas:

$$f_1 = \frac{1}{2} \epsilon \left( \frac{V_0}{a} \right)^2 \left( \frac{la}{2} \right) - \frac{1}{2} \epsilon \left( \frac{V_0}{b} \right)^2 \left( \frac{lb}{2} \right) \quad (g)$$

or

$$f_1 = \frac{\epsilon V_0^2 l}{4} \left( \frac{1}{a} - \frac{1}{b} \right). \quad (h)$$

The electric force on the lower plate (for  $a < b$ , as shown in Fig. 8.3.2) tends to pull in the  $x_1$ -direction. If we had closed the surface above the top plate, the signs of the normal vectors involved would have been reversed to give an equal and opposite force on the top plate.

#### 8.4 THE SURFACE FORCE DENSITY

Magnetic and electric fields in many situations are found by modeling the current or charge distributions by surface currents or surface charges. In these systems surface forces of electrical origin must be considered; for

\* R. M. Fano, L. J. Chu, and R. B. Adler, *Electromagnetic Fields, Energy, and Forces*, Wiley, New York, 1960, p. 92.

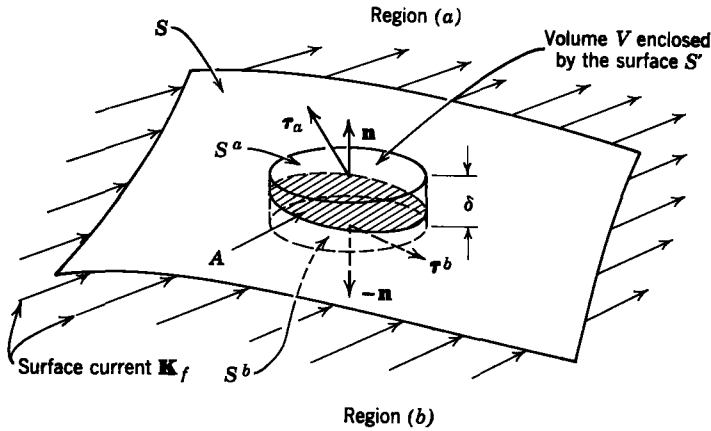


Fig. 8.4.1 Small thin volume  $V$  which encloses a section  $A$  of a surface  $S$  supporting the surface current  $\mathbf{K}_f$ .

example, if the surface  $S$  supports a surface current density  $\mathbf{K}_f$ , as shown in Fig. 8.4.1, and is immersed in a magnetic field  $\mathbf{H}$ , we expect a surface force similar in form to (8.1.3) with  $\mathbf{K}_f$  playing the role of  $\mathbf{J}_f$ .

The surface force density  $\mathbf{T}$  (newtons per square meter) is defined in terms of the force  $\mathbf{f}$  on the material within the small thin volume shown in Fig. 8.4.1. It is the force per unit area  $A$  on the surface  $S$  intersected by the volume  $V$  in the limit in which first the thickness  $\delta$  and then the area  $A$  become small. The stress tensor provides a convenient means of evaluating  $\mathbf{T}$ , for the total force can be written as a surface integral (8.1.17). In the limit in which  $\delta \rightarrow 0$  the contribution to this integral along the sides (of height  $\delta$ ) of the volume  $V$  becomes vanishingly small and the only contributions come from the tractions acting on the surfaces  $S^a$  and  $S^b$ . In the limit  $S^a \rightarrow S^b \rightarrow A$  we have

$$T_m = \tau_m^a + \tau_m^b, \tag{8.4.1}$$

where  $\tau^a$  and  $\tau^b$  are the tractions acting on the surfaces  $S^a$  and  $S^b$ , respectively.

If we define  $\mathbf{n}$  as the unit vector *normal to the surface  $S$*  and directed from region (b) to region (a), the surface tractions can be evaluated by using (8.2.8):

$$T_m = (T_{mn}^a - T_{mn}^b)n_n. \tag{8.4.2}$$

Remember that in (8.2.8)  $\mathbf{n}$  is the unit vector normal to a surface that *encloses* the volume of integration in Fig. 8.4.1. Over the top surface the normal vector is  $\mathbf{n}$ , but over the bottom surface it is  $-\mathbf{n}$ . Hence the minus sign in (8.4.2).

The Maxwell stresses  $T_{mn}$  are functions of either the magnetic or electric fields. Therefore (8.4.2) is a convenient expression for the surface force density on either a surface current or a surface charge.

### 8.4.1 Magnetic Surface Forces

As already pointed out, the magnetic surface force should be equivalent to the cross product of the surface current with a magnetic field. In this cross product, however, do we use the value of  $\mathbf{H}$  from region (a) or from region (b)? In fact the average value of  $\mathbf{H}$  should be used and the force per unit area  $\mathbf{T}$  acting on a surface current  $\mathbf{K}_f$  is

$$\mathbf{T} = \mu \mathbf{K}_f \times \frac{(\mathbf{H}^a + \mathbf{H}^b)}{2}. \quad (8.4.3)$$

We can prove that this relation is, in fact, valid by showing that it is equivalent to (8.4.2).

The surface current density from (6.2.14)\* is

$$\mathbf{K}_f = \mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b) \quad (8.4.4)$$

and (8.4.3) becomes

$$\mathbf{T} = \mu [\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b)] \times \frac{(\mathbf{H}^a + \mathbf{H}^b)}{2}. \quad (8.4.5)$$

We now use a vector identity† to rewrite this expression in component form as

$$T_m = \mu (H_m^a - H_m^b) \mathbf{n} \cdot \frac{(\mathbf{H}^a + \mathbf{H}^b)}{2} - \frac{\mu n_m}{2} (\mathbf{H}^a \cdot \mathbf{H}^a - \mathbf{H}^b \cdot \mathbf{H}^b). \quad (8.4.6)$$

The first term of this equation can be simplified by using (6.2.7)\*  $\mathbf{n} \cdot \mu \mathbf{H}^a = \mathbf{n} \cdot \mu \mathbf{H}^b$ , whereas we replace  $n_m$  with  $n_n \delta_{mn}$  in the second term.

$$T_m = (\mu H_m^a H_n^a - \frac{1}{2} \delta_{mn} \mu H_k^a H_k^a) n_n - (\mu H_m^b H_n^b - \frac{1}{2} \delta_{mn} \mu H_k^b H_k^b) n_n. \quad (8.4.7)$$

Our expression is now identical with (8.4.2), if we note that the magnetic stress is given by (8.1.11). We can alternatively write the surface force in terms of the fields alone (stresses) by using (8.4.2) or in terms of surface currents and an average magnetic field (8.4.3).

### 8.4.2 Electric Surface Forces

The surface force in an electric field system can be expressed as the product of the surface charge density  $\sigma_f$  and the average electric field intensity.

$$\mathbf{T} = \sigma_f \frac{(\mathbf{E}^a + \mathbf{E}^b)}{2}. \quad (8.4.8)$$

\* Table 6.1, Appendix E.

†  $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{A}(\mathbf{C} \cdot \mathbf{B})$ .

**8.1 Electromagnetic Force Densities, Stress Tensors, and Surface Force Densities for Quasi-Static Magnetic and Electric Field Systems\***

Description	Force Density $\mathbf{F}$	Stress Tensor $T_{mn}$ $F_m = \frac{\partial T_{mn}}{\partial x_n}$ (8.1.10)	Surface Force Density $T_m = [T_{mn}]n_n$
Force on media carrying free current density $\mathbf{J}_f$ , $\mu$ constant	$\mathbf{J}_f \times \mathbf{B}$ (8.1.3)	$T_{mn} = \mu H_m H_n - \delta_{mn} \frac{1}{2} \mu H_k H_k$ (8.1.11)	$\mathbf{T} = \mathbf{K}_f \times \mathbf{l}$ $\mathbf{K}_f = \mathbf{n} \times [\mathbf{J}]$ (8.4.3)
Force on media supporting free charge density $\rho_f$ , $\epsilon$ constant	$\rho_f \mathbf{E}$ (8.3.3)	$T_{mn} = \epsilon E_m E_n - \delta_{mn} \frac{1}{2} \epsilon E_k E_k$ (8.3.10)	$\mathbf{T} = \sigma_f \langle \mathbf{E} \rangle$ $\sigma_f = \mathbf{n} \cdot [\epsilon \mathbf{E}]$ (8.4.8)
Force on free current plus magnetization force in which $\mathbf{B} = \mu \mathbf{H}$ both before and after media are deformed	$\mathbf{J}_f \times \mathbf{B} - \frac{1}{2} \mathbf{H} \cdot \mathbf{H} \nabla \mu$ $+ \frac{1}{2} \nabla \left( \mathbf{H} \cdot \mathbf{H} \rho \frac{\partial \mu}{\partial \rho} \right)$ (8.5.38)	$T_{mn} = \mu H_m H_n$ $- \frac{1}{2} \delta_{mn} \left( \mu - \rho \frac{\partial \mu}{\partial \rho} \right) H_k H_k$ (8.5.41)	
Force on free charge plus polarization force in which $\mathbf{D} = \epsilon \mathbf{E}$ both before and after media are deformed	$\rho_f \mathbf{E} - \frac{1}{2} \mathbf{E} \cdot \mathbf{E} \nabla \epsilon$ $+ \frac{1}{2} \nabla \left( \mathbf{E} \cdot \mathbf{E} \rho \frac{\partial \epsilon}{\partial \rho} \right)$ (8.5.45)	$T_{mn} = \epsilon E_m E_n$ $- \frac{1}{2} \delta_{mn} \left( \epsilon - \rho \frac{\partial \epsilon}{\partial \rho} \right) E_k E_k$ (8.5.46)	

$$\equiv \frac{\mathbf{A}^a + \mathbf{A}^b}{2}$$

$$\equiv \mathbf{A}^a - \mathbf{A}^b$$



This result expresses the surface force in a form that is similar to that of the force density (8.3.3). We can show that this equation is correct by demonstrating that it is equivalent to (8.4.2). First, we write (8.4.8) in terms of the electric fields, using Gauss's law to express  $\sigma_f$  (6.2.33)\*

$$\mathbf{T} = \epsilon \mathbf{n} \cdot (\mathbf{E}^a - \mathbf{E}^b) \frac{(\mathbf{E}^a + \mathbf{E}^b)}{2}. \quad (8.4.9)$$

By use of a vector identity,† this becomes

$$\mathbf{T} = \epsilon \mathbf{n} (\mathbf{E}^a - \mathbf{E}^b) \cdot \frac{(\mathbf{E}^a + \mathbf{E}^b)}{2} - \epsilon (\mathbf{E}^a - \mathbf{E}^b) \times \left[ \mathbf{n} \times \frac{(\mathbf{E}^a + \mathbf{E}^b)}{2} \right]. \quad (8.4.10)$$

This looks like the long way to go about it, but in this form the expression can be factored by using the condition (6.2.31)\*,  $\mathbf{n} \times \mathbf{E}^a = \mathbf{n} \times \mathbf{E}^b$ .

$$\mathbf{T} = \frac{1}{2} \epsilon \mathbf{n} (\mathbf{E}^a \cdot \mathbf{E}^a - \mathbf{E}^b \cdot \mathbf{E}^b) - \epsilon [\mathbf{E}^a \times (\mathbf{n} \times \mathbf{E}^a) - \mathbf{E}^b \times (\mathbf{n} \times \mathbf{E}^b)]. \quad (8.4.11)$$

If we now use this same vector identity again,‡

$$\mathbf{T} = \epsilon [\mathbf{E}^a (\mathbf{n} \cdot \mathbf{E}^a) - \mathbf{E}^b (\mathbf{n} \cdot \mathbf{E}^b)] - \frac{1}{2} \epsilon \mathbf{n} (\mathbf{E}^a \cdot \mathbf{E}^a - \mathbf{E}^b \cdot \mathbf{E}^b), \quad (8.4.12)$$

and this equation is equivalent to the traction in terms of the stress (8.4.2), as can be seen by writing the  $m$ th component of  $\mathbf{T}$  from (8.4.12)

$$T_m = [(\epsilon E_m^a E_n^a - \frac{1}{2} \delta_{mn} \epsilon E_k^a E_k^a) - (\epsilon E_m^b E_n^b - \frac{1}{2} \delta_{mn} \epsilon E_k^b E_k^b)] n_n \quad (8.4.13)$$

and using (8.3.10). Surface forces and their corresponding stresses are summarized in Table 8.1.

**Example 8.4.1.** The three plane parallel electrodes of Fig. 8.4.2. provide an example of a force on a surface charge. The plates are assumed to be perfectly conducting, with the outer plates connected together. If we ignore the fringing fields, we have

$$\begin{aligned} \mathbf{E}^a &= \frac{v}{d-x} \mathbf{i}_1, \\ \mathbf{E}^b &= \frac{-v}{x} \mathbf{i}_1, \end{aligned} \quad (a)$$

for the fields between the plates.

We now use several methods to compute the force acting on the middle plate.

First, we use the stress, as given in (8.4.2). The force of electrical origin on the middle plate in the  $\mathbf{i}_1$ -direction is

$$f^e = AT_1 = A(T_{11}^a - T_{11}^b), \quad (b)$$

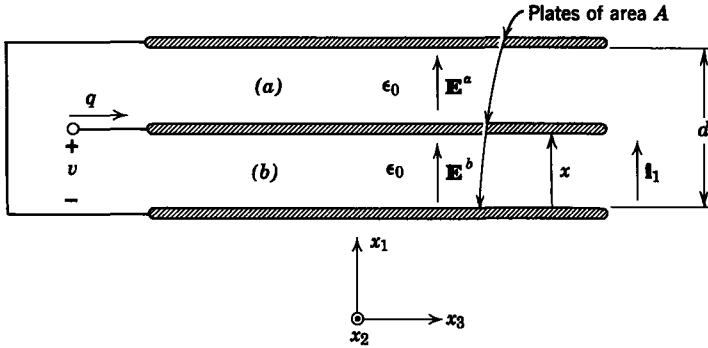
which in view of (a) becomes

$$f^e = \frac{A\epsilon_0}{2} [(E_1^a)^2 - (E_1^b)^2] = \frac{A\epsilon_0}{2} \left[ \frac{v^2}{(d-x)^2} - \frac{v^2}{x^2} \right]. \quad (c)$$

\* Table 6.1, Appendix E.

†  $(\mathbf{C} \cdot \mathbf{A})\mathbf{B} = \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) - \mathbf{A} \times (\mathbf{C} \times \mathbf{B})$ .

‡  $\mathbf{A} \times (\mathbf{B} \times \mathbf{A}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{A}) - \mathbf{A}(\mathbf{A} \cdot \mathbf{B})$ .

Fig. 8.4.2 Plane-parallel electrodes with Area  $A$ .

This same force can be calculated from (8.4.8). We first compute the surface charge density as

$$\sigma_f = \epsilon_0(\mathbf{E}^a - \mathbf{E}^b) \cdot \mathbf{i}_1 = \epsilon_0 \left( \frac{v}{d-x} + \frac{v}{x} \right) \quad (d)$$

and then use (8.4.8)

$$f^e = AT_1 = \frac{A\epsilon_0}{2} \left( \frac{v}{d-x} + \frac{v}{x} \right) \left( \frac{v}{d-x} - \frac{v}{x} \right) = \frac{A\epsilon_0}{2} \left[ \frac{v^2}{(d-x)^2} - \frac{v^2}{x^2} \right]. \quad (e)$$

This is the same expression as in (c).

Finally, we use the energy method introduced in Chapter 3\* to find the force on the middle plate by noting that the system has one mechanical terminal pair ( $f^e, x$ ). The capacitance of the electrical terminal pair ( $v, q$ ) (Fig. 8.4.2) is

$$C = \frac{A\epsilon_0}{x} + \frac{A\epsilon_0}{d-x}. \quad (f)$$

Hence the stored coenergy (which is the same as the energy, since the system is electrically linear) is

$$W'(v, x) = \frac{1}{2} C v^2 = \frac{1}{2} A \epsilon_0 \left( \frac{1}{x} + \frac{1}{d-x} \right) v^2, \quad (g)$$

and we have

$$f^e = \frac{\partial W'}{\partial x} = \frac{1}{2} A \epsilon_0 \left[ \frac{1}{(d-x)^2} - \frac{1}{x^2} \right] v^2, \quad (h)$$

where we have used (h) of Table 3.1; of course, this result is also the same as given by (c).

## 8.5 THE MAGNETIZATION AND POLARIZATION FORCE DENSITIES

So far in this chapter the discussion has been limited to electric and magnetic forces on media that support free charges and free currents. In Chapter 3 examples often involve forces on magnetized or polarized media. In these examples the electric or magnetic fields are excited by means of free charges

\* See Table 3.1, Appendix E.

or free currents. However, in many cases, the media subjected to the forces of electric origin do not themselves support free charges or free currents. Such forces, which are found by means of the energy method, must be attributed to the magnetization or polarization of the media.

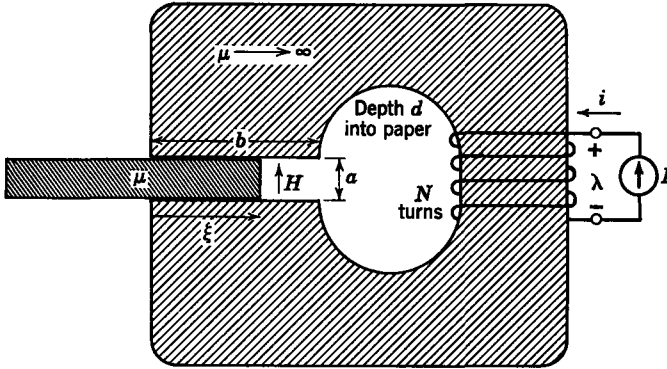
When an atom or molecule of a substance is subjected to an external electric or magnetic field, the physical microscopic structure is distorted. Although the medium may be electrically neutral on a macroscopic scale, on an atomic scale it is composed in part of charged particles. It is the reaction of these charged particles to the Lorentz force that gives rise to the distortion of the microscopic structure. On a macroscopic level these effects are observed as a magnetization or polarization of the medium. For a wide range of substances it is possible to characterize the magnetization or polarization by simple constitutive laws, such as those introduced in Section 1.1.1; for example, certain isotropic materials can be characterized by a linear relation between the magnetic flux density  $\mathbf{B}$  and the field intensity  $\mathbf{H}$ ,  $\mathbf{B} = \mu\mathbf{H}$ . Similarly, for many isotropic dielectrics,  $\mathbf{D} = \epsilon\mathbf{E}$ .

In Chapter 3 we found forces of electrical origin by first establishing the electrical terminal relations for the system, then computing the electrical energy (or coenergy) stored in the system and finally using the energy function and the principle of conservation of energy to find the force of electric origin. In problems involving magnetization or polarization the first step in this procedure is made possible by knowing the appropriate electrical constitutive law.

In this section we wish to derive the force density by using the energy approach introduced in Chapter 3. Hence the derivation begins with the constitutive laws. Because these laws hold only for particular classes of material, the resulting force expressions are also restricted in validity. In particular, we consider media that are isotropic both before and after the magnetic or electric field is applied. Liquids and gases are most clearly in this category, some types of interaction with solids can be so modeled. Attention is given first to magnetization forces. The derivation is then easily revised to account for polarization forces.

### 8.5.1 Examples with One Degree of Freedom

Two simple examples help to establish the nature of the magnetization force density and show how its derivation relates to the energy method of Chapter 3. Figure 8.5.1 shows a slab of magnetizable material that is free to slide between the pole faces of a magnetic yoke. The force of electric origin tends to make the slab move into the region between the pole faces. This problem involves the rigid body motion of the material. By contrast, a second example (in Fig. 8.5.2) involves a medium that has an interface at  $\xi$

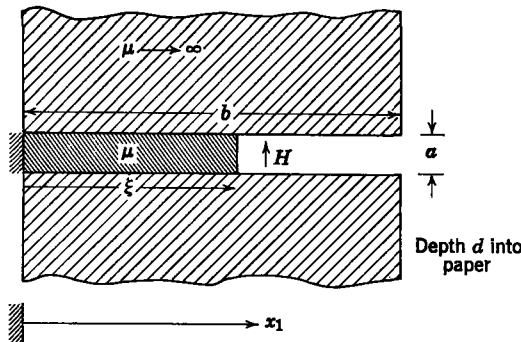


**Fig. 8.5.1** A magnetic field intensity  $H$  is produced in the gap of the magnetic yoke. As a result, a magnetization force tends to pull the slab of magnetic material into the region between the pole faces.

but is otherwise surrounded by rigid walls. Hence a deflection of the interface must lead to a change in the volume occupied by the material. For the present purposes we assume that the material can deform only in the  $x_1$ -direction.

These two examples have been selected for discussion because they characterize situations in which the magnetization force density is commonly operative. In the first case the force arises because the region occupied by the magnetic field includes a magnetically inhomogeneous material (the air and the magnetic solid). In the example of Fig. 8.5.2 there is an additional contribution to the force, caused by the change in volume of the material. This contribution is called the magnetostriction force.

The force in these examples can be computed by using the energy method of Chapter 3, since in each case there is only one degree of freedom. We first



**Fig. 8.5.2** The gap in the magnetic yoke of Fig. 8.5.1, with an experiment that demonstrates the magnetostrictive force. The surface at  $\xi$  is free to move. Because the material is otherwise surrounded by rigid walls, the motions of the surface must involve a compression or expansion of the medium.

review the energy approach by finding the force in each case associated with the displacement  $\xi$ . Then in the next section the same technique is extended to find the continuum force density and we return to these examples to illustrate its significance.

In each of the examples conservation of energy in the electromechanical coupling requires that

$$\lambda \delta i = \delta W' - f \delta \xi, \quad (8.5.1)$$

where  $W'$  is the total coenergy, as defined in Chapter 3\*, and  $f$  is the total force of electrical origin associated with the displacement  $\xi$ . The symbol  $\delta$  is used to indicate incremental changes in the independent variables  $i$  and  $\xi$ . It has the same significance as  $d$  in Chapter 3 and is introduced to avoid confusion with integration symbols such as  $da$  and  $dV$ , which indicate surface and volume elements.

We can establish the coenergy  $W'$  by integrating (8.5.1) in such a way that it is not necessary to know  $f$ . First we integrate on  $\xi$  (put the system together mechanically) with  $i = 0$ , but because the force of electrical origin is then zero, this integration makes no contribution. Here, of course, we preclude the possibility that the material is initially magnetized. The remaining integration takes the familiar form

$$W' = \int \lambda \delta i. \quad (8.5.2)$$

If the magnetic material is electrically linear,  $\lambda$  and  $i$  are related by the inductance and (8.5.2) yields

$$W' = \frac{1}{2} i^2 L(\xi). \quad (8.5.3)$$

Hence we have established the function  $W'$  from information about the electrical system, essentially the  $\lambda-i$  relation.

We now hold the independent electrical variable  $i$  fixed (say by means of a constant current source). Then the left-hand side of (8.5.1) makes no contribution to the energy balance and this equation becomes,

$$\left( \frac{1}{2} i^2 \frac{\partial L}{\partial \xi} - f \right) \delta \xi = 0. \quad (8.5.4)$$

Here it is important to recognize that in this context  $\xi$  is an independent variable. Incremental displacements  $\delta \xi$  are arbitrary. It therefore follows from (8.5.4) that the quantity in parentheses is zero.

$$f = \frac{1}{2} i^2 \frac{\partial L}{\partial \xi}. \quad (8.5.5)$$

In the next section we use this procedure to find the continuum force density at each point in the movable medium. Before embarking on that

\* See Tables 2.1 and 3.1, Appendix E.

development we consider the specific examples shown in Figs. 8.5.1. and 8.5.2.

In the example of Fig. 8.5.1 the yoke is assumed to be perfectly permeable, hence in the gap  $H = Ni/a$ . In addition,  $B = \mu H$  in the movable slab. It follows that the inductance  $L$  is

$$L = \frac{N^2 d}{a} [b\mu_0 + \xi(\mu - \mu_0)]. \quad (8.5.6)$$

Then from (8.5.5) the force is

$$f = (da)\frac{1}{2}H^2(\mu - \mu_0). \quad (8.5.7)$$

In deriving this expression we have assumed that  $\mu$  in the movable slab is independent of the displacement  $\xi$ . This is reasonable as long as the material moves as a rigid body. In the example in Fig. 8.5.2 a displacement of the interface at  $\xi$  clearly is accompanied by a change in the density of the material. We expect that there is an associated change in the permeability which can be expressed as

$$\mu = \mu(\rho), \quad (8.5.8)$$

where  $\rho$  is the density (mass per unit volume) of the material. It is clear that  $\rho$  is in turn a function of  $\xi$ , for conservation of mass requires that

$$\rho \xi ad = \text{total mass of material} = \text{constant}. \quad (8.5.9)$$

Differentiation of this expression with respect to  $\xi$  gives

$$\frac{\partial \rho}{\partial \xi} = -\frac{\rho}{\xi}, \quad (8.5.10)$$

which shows how changes in density arise from motions of the surface at  $\xi$ .

We now use (8.5.6) to find the total force, including the dependence of  $\mu$  on  $\xi$  (through the density  $\rho$ ).

$$f = da\frac{1}{2}H^2 \left[ (\mu - \mu_0) + \xi \frac{\partial \mu}{\partial \rho} \frac{\partial \rho}{\partial \xi} \right]. \quad (8.5.11)$$

From (8.5.10) this force can also be written as

$$f = da\frac{1}{2}H^2 \left[ (\mu - \mu_0) - \rho \frac{\partial \mu}{\partial \rho} \right]. \quad (8.5.12)$$

The compressibility of the material gives rise to an additional term, as can be seen by comparing (8.5.7) and (8.5.12). This magnetostrictive force is significant when material deformations that lead to changes in the density are important.

We have considered these special cases to make it clear that the basic thermodynamic techniques introduced in Chapter 3 provide the fundamental means by which magnetization and polarization forces can be derived. The derivations of the next two sections are something new in our development only because the objective is a force density rather than a finite number of total forces. One way to consider the continuum situation is shown schematically in Fig. 8.5.3. A magnet is excited by a current  $i$  and a magnetizable material is subjected to the resulting magnetic field. Now, if we divide the material into small volume elements, deformations can be described by simply indicating the displacement  $\xi^i$  of each element.

There are three degrees of freedom for each volume element and therefore the  $i$ th displacement must be represented by three terminal pairs, which are summarized by vector terminal variables  $\xi^i$  and  $\mathbf{f}^i$ . Say that the medium has been divided into  $m$  regions. Then conservation of energy for the electromechanical coupling requires that

$$i \delta \lambda = \delta W + \sum_{i=1}^m \mathbf{f}^i \cdot \delta \xi^i; \quad (8.5.13)$$

that is, an increment of energy  $i \delta \lambda$  introduced through the electrical terminals either increases the magnetic energy stored by the amount  $\delta W$  or does work  $\mathbf{f}^i \cdot \delta \xi^i$  on one or more of the elements of volume.

In terms of the coenergy (as discussed in Section 3.1.2b)\* this statement of conservation of energy becomes

$$\lambda \delta i = \delta W' - \sum_{i=1}^m \mathbf{f}^i \cdot \delta \xi^i. \quad (8.5.14)$$

Now, if we used a large number of elements  $m$ , the force  $\mathbf{f}^i$  divided by the volume of the  $i$ th element would constitute the force density acting in the neighborhood of the  $i$ th element. Hence it should be clear that in principle we can find the force density by using this familiar energy method. Rather than using the summation, we take the limit in which  $m \rightarrow \infty$  at the outset and represent the summation by an integration.

One significant point can be made without further mathematical developments. In Section 8.1 we claim (without proof) that the force density  $\mathbf{J}_f \times \mathbf{B}$  on free currents remains valid even if the currents are immersed in a material

\* See Table 3.1, Appendix E.

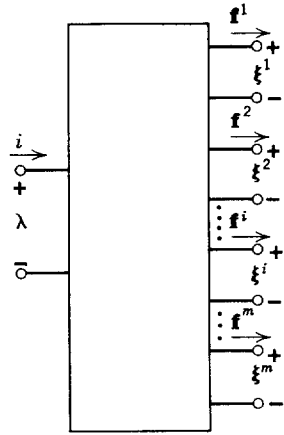


Fig. 8.5.3 Schematic representation of electromechanical coupling in which deformations of a continuum are represented by  $m$  vector displacements.

with a uniform and constant  $\mu$ . We can use the energy method of finding forces to see that this must be true.

Suppose for a moment that we had a system that did not involve magnetization and used the energy method to find the force density  $\mathbf{F} = \mathbf{J}_f \times \mathbf{B}$ . Now, we do not need to carry out this formalism because we already know the answer. The point is this. Consider the same system, but with a uniform permeability  $\mu$ . The energy function  $W'$ , from which the force density is found, is computed from an electrical terminal relation that in turn is found by using Maxwell's equations. The only change in the laws governing the fields is that  $\mu_0 \rightarrow \mu$ . Hence the only change in the energy function  $W'$  is that  $\mu_0 \rightarrow \mu$ , and the force density remains the same as in the form without the uniform  $\mu: \mathbf{J}_f \times \mathbf{B}$ . Of course,  $\mathbf{B}$  is computed by using Maxwell's equations with  $\mu_0 \rightarrow \mu$ .

Note that this conclusion is consistent with the derivation of the Maxwell stress tensor given in Section 8.1. In what follows we concern ourselves with finding the magnetization force on material in which  $\mathbf{J}_f = 0$ .

### 8.5.2 The Magnetization Force Density

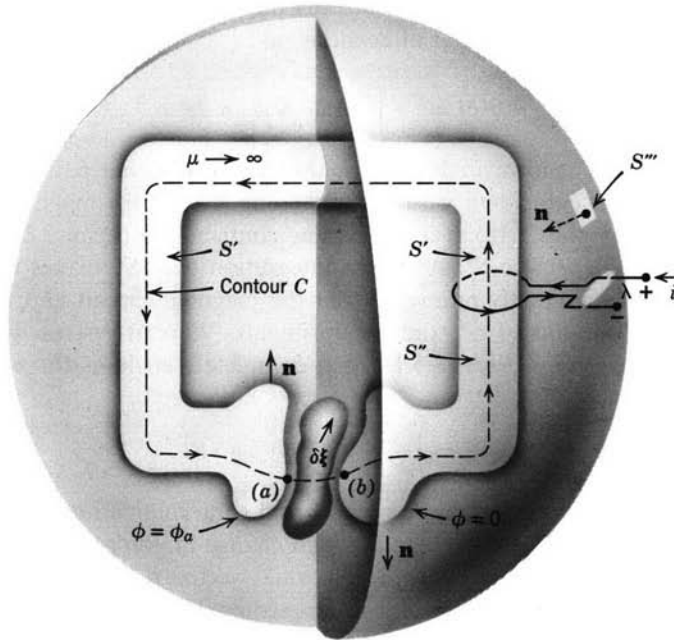
The force density on magnetizable material can be found by following the same procedure outlined in Section 8.5.1. For this purpose we consider the experiment shown in Fig. 8.5.4. A perfectly permeable magnetic yoke is excited by the current  $i$ . Our experiment is carried out in the region between the rigid pole faces, where the magnetic field is concentrated (just as it was in Fig. 8.5.1). In this region a deformable magnetic material has a displacement from the coordinate position  $\mathbf{r}$  given by  $\delta \boldsymbol{\xi}(\mathbf{r})$ .

We define the force per unit volume  $\mathbf{F}(\mathbf{r})$  as acting on the material at  $\mathbf{r} + \delta \boldsymbol{\xi}$ . This makes it possible to write (8.5.14) as

$$\lambda \delta i = \int_V \delta w' dV - \int_V \mathbf{F} \cdot \delta \boldsymbol{\xi} dV. \quad (8.5.15)$$

The function  $w'$  is defined as the *coenergy density* and can be integrated over the volume  $V$  to find the total coenergy  $W'$ . Although the specific geometry of the magnetic circuit is superfluous, it does help to fix attention on a physically reasonable system. For convenience we have included only a single one-turn electrical excitation in the system, with the magnetic circuit arranged to concentrate the magnetic field in the volume  $V$ . It is convenient to define this volume  $V$  as being enclosed by three surfaces  $S'$ ,  $S''$ , and  $S'''$ , shown in Fig. 8.5.4. The surface  $S'$  is bounded by the current path for  $i$  and encloses the magnetic circuit to the left. (It covers the pole face to the left like a sock on a foot with the current path  $i$  as the garter.) The surface  $S''$  plays a similar role for the remaining section of magnetic circuit to the right (the other foot). Finally, the surface  $S'''$  encloses the entire system, with a slit





**Fig. 8.5.4** Perfectly permeable magnetic circuit excited by the current  $i$ . A deformable magnetic material with permeability  $\mu(\mathbf{r})$  is subjected to the concentration of magnetic field intensity between the pole faces.

left open for the electrical terminals. These three surfaces, taken together, enclose the volume  $V$  occupied by the deformable magnetizable material. Note that the normal vector  $\mathbf{n}$  is directed *into* the volume  $V$ .

We require a field representation of the electrical input  $\lambda \delta i$  for the purpose of writing all the terms in (8.5.15) as volume integrals. There is no free current  $\mathbf{J}_f$  in the volume  $V$ , hence  $\nabla \times \mathbf{H} = 0$ . It is therefore convenient to write  $\mathbf{H} = -\nabla\phi$ . We are free to define  $\phi$  as zero on the right-hand pole face in Fig. 8.5.4; hence since  $\mathbf{H} = 0$  inside the magnetic yoke

$$i = \oint \mathbf{H} \cdot d\mathbf{l} = -\oint \nabla\phi \cdot d\mathbf{l} = -\phi_b + \phi_a = \phi_a, \quad (8.5.16)$$

where  $a$  is any point on the left pole face, as shown in Fig. 8.5.4. Remember that by definition

$$\lambda = \int_{S'} \mathbf{B} \cdot \mathbf{n} \, da = - \int_{S'} \mu \nabla\phi \cdot \mathbf{n} \, da. \quad (8.5.17)$$

The surface  $S'$  used to compute the flux  $\lambda$  is bounded by the conductor carrying the current  $i$ . In Fig. 8.5.4 this surface is defined so that it is coincident with the surface of the perfectly permeable magnetic circuit. Since  $\phi_a$  is

the potential evaluated on the surface  $S'$ , these last two equations are used to write the incremental input of coenergy as

$$\lambda \delta i = - \int_{S'} \mu \delta \phi \nabla \phi \cdot \mathbf{n} \, da. \quad (8.5.18)$$

The surface  $S''$  shown in Fig. 8.5.4 also has the current path for  $i$  as its periphery but is coincident with the remaining part of the magnetic circuit. The integration of (8.5.18) over  $S''$  gives no contribution because  $\phi$  is defined as zero over the surface  $S''$ . A similar integration over  $S'''$  makes no contribution, for  $S'''$  is greatly removed from the magnetic circuit and  $\mathbf{n} \cdot \mathbf{B} = 0$  in the neighborhood of  $S'''$  and the terminals. We can just as well use a surface of integration  $S$  in (8.5.18) that completely encloses the volume  $V$ . It follows that

$$\lambda \delta i = \int_V \nabla \cdot [(\mu \nabla \phi)(\delta \phi)] \, dV. \quad (8.5.19)$$

Here the surface integral has been converted to a volume integral by using Gauss's theorem. Note that there is a sign change in going from the surface integral to the volume integral. The normal vector  $\mathbf{n}$  in Fig. 8.5.4 points into the volume  $V$  rather than outward as required in the usual statement of Gauss's theorem.

Because  $\nabla \cdot \mathbf{B} = -\nabla \cdot \mu \nabla \phi = 0$ , we can use an identity\* to convert (8.5.19) to

$$\lambda \delta i = \int_V \frac{1}{2} \mu \delta (\nabla \phi)^2 \, dV. \quad (8.5.20)$$

It is now possible to write all the terms of (8.5.15) as volume integrals and express conservation of energy as

$$\int_V \frac{1}{2} \mu \delta (\nabla \phi)^2 \, dV = \int_V \delta w' \, dV - \int_V \mathbf{F} \cdot \delta \boldsymbol{\xi} \, dV. \quad (8.5.21)$$

We now put the system together, first mechanically and then electrically to find  $w'$ . As in Section 8.5.1, the last term in (8.5.21) makes no contribution to the coenergy stored during this process. We must remember that because the material is deformable the permeability at a given point is a function of deflection (e.g., the permeability at a given point in the gap of the magnetic circuit shown in Fig. 8.5.1 could be  $\mu$  or  $\mu_0$ , depending on the position of the slab); that is,  $\mu = \mu(\boldsymbol{\xi})$ . Once the system is assembled mechanically, however,  $\mu$  is constant and the remaining integration of (8.5.21) becomes

$$\int_V \delta [\frac{1}{2} \mu (\nabla \phi)^2] \, dV = \int_V \delta w' \, dV. \quad (8.5.22)$$

\*  $\nabla \cdot \mathbf{A} \psi = \psi \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla \psi$ .

Rather than carrying out the integration, it serves our purpose to recognize that if we integrate the quantity

$$\delta w' = \delta \left[ \frac{1}{2} \mu (\nabla \phi)^2 \right] \quad (8.5.23)$$

over the volume  $V$ , the incremental change in total coenergy will have been computed. This completes the first step in finding the force density  $\mathbf{F}$  in that the coenergy has been found from the electrical properties of the material. Note that we have assumed that  $\mathbf{B} = \mu \mathbf{H}$  both before and after the material is deformed.

As in the preceding section, the next step uses the coenergy density to determine the force density. This is done by constraining the current  $i$  to be a constant so that the left-hand side of (8.5.15), and hence (8.5.21), is zero. From (8.5.16) this means that  $\phi$  is held constant on  $S'$ . Changes in coenergy now occur because of changes  $\delta \boldsymbol{\xi}$  in the material displacement.

A few manipulations on  $\delta w'$  make the remaining terms assume a familiar form. From (8.5.23)

$$\delta w' = \frac{1}{2} (\nabla \phi)^2 \delta \mu + \frac{1}{2} \mu \delta (\nabla \phi)^2. \quad (8.5.24)$$

The integral over the volume of the second term in this equation vanishes, as can be seen by first using an identity to write it as

$$\frac{1}{2} \mu \delta (\nabla \phi)^2 = \mu \nabla \phi \cdot \nabla (\delta \phi) = \nabla \cdot (\delta \phi \mu \nabla \phi) - \delta \phi \nabla \cdot (\mu \nabla \phi). \quad (8.5.25)$$

Because  $\nabla \cdot \mathbf{B} = 0$ , the last term is zero, whereas the integral over the volume of the remaining term can be transformed by Gauss's theorem to an integral over  $S$ , where  $\phi$  is constant, hence  $\delta \phi = 0$ .

Because the last term in (8.5.24) makes no contribution, the conservation of energy equation (8.5.15) becomes (remember,  $\mathbf{H} = -\nabla \phi$ )

$$\int_V \left( \frac{1}{2} \mathbf{H} \cdot \mathbf{H} \delta \mu - \mathbf{F} \cdot \delta \boldsymbol{\xi} \right) dV = 0. \quad (8.5.26)$$

This equation is the generalization of (8.5.4). Note that the permeability plays the same role as the inductance in determining the dependence of the coenergy on the displacement of the material. To determine the force density we must relate the permeability  $\mu$  to the material displacement. (This is analogous to finding the inductance  $L$  of (8.5.6) as a function of  $\xi$ .)

There are two ways in which the permeability at a point  $\mathbf{r}$  can change. Either the material is initially inhomogeneous, in which case a displacement can transport material of different permeability into the region of  $\mathbf{r}$ , or the density of the material can change with a resulting change in permeability.

Consider first the effect of inhomogeneities. After the displacement  $\delta \boldsymbol{\xi}$ , the permeability at  $\mathbf{r}$  is that of the material that was at  $\mathbf{r} - \delta \boldsymbol{\xi}$  before the displacement. Hence

$$\delta \mu = \lim_{\delta \boldsymbol{\xi} \rightarrow 0} [\mu(\mathbf{r} - \delta \boldsymbol{\xi}) - \mu(\mathbf{r})]. \quad (8.5.27)$$

Taylor's expansion makes it possible to write this as

$$\delta\mu = \lim_{\delta\xi \rightarrow 0} \left[ \mu(\mathbf{r}) - \delta\xi_i \frac{\partial\mu}{\partial x_i}(\mathbf{r}) + \cdots - \mu(\mathbf{r}) \right]. \tag{8.5.28}$$

In the limit

$$\delta\mu = -\delta\xi \cdot \nabla\mu. \tag{8.5.29}$$

If this is the only mechanism by which the permeability can change, (8.5.26) becomes

$$\int_V \left( -\frac{1}{2} \mathbf{H} \cdot \mathbf{H} \nabla\mu - \mathbf{F} \right) \cdot \delta\xi \, dV = 0. \tag{8.5.30}$$

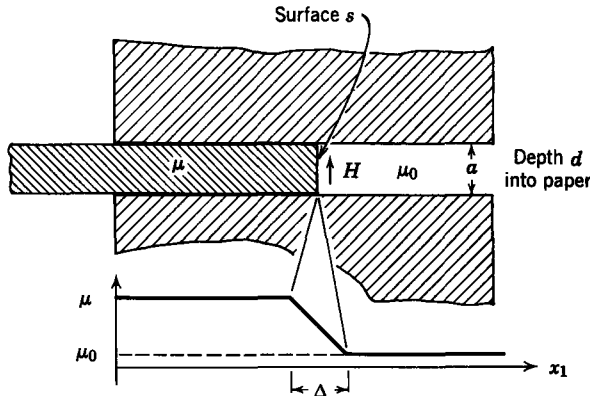
It is now crucial to recognize that the displacement  $\delta\xi$  is arbitrary, in that  $\xi$  is an independent variable in the same sense as in the analogous lumped parameter derivation [see (8.5.4)]. Hence to satisfy (8.5.30) the quantity in parentheses must vanish.

$$\mathbf{F} = -\frac{1}{2} \mathbf{H} \cdot \mathbf{H} \nabla\mu. \tag{8.5.31}$$

This contribution to the magnetization force density results because of inhomogeneities in the magnetic material. An example involving a force of this type, considered in Section 8.5.1 (Fig. 8.5.1), serves to illustrate the significance of (8.5.31).

**Example 8.5.1.** The slab of magnetic material and adjacent pole faces for the problem of Fig. 8.5.1 is shown in Fig. 8.5.5. Here the distribution of  $\mu$  in the gap is plotted as a function of  $x_1$  with the transition from  $\mu$  to  $\mu_0$  at the surface ( $s$ ) expanded over a thickness  $\Delta$ . The magnetic field intensity  $H$  is uniform throughout the gap. The gradient of  $\mu$  in the  $x_1$ -direction is zero in the bulk of the slab but has the value

$$\nabla\mu = \frac{\mu_0 - \mu}{\Delta} \mathbf{i}_1 \tag{a)}$$



**Fig. 8.5.5** Magnetic slab of Fig. 8.5.1 free to slide in the  $x_1$ -direction. Near the surface  $s$  the permeability undergoes a rapid change. This region is shown (expanded) to have a thickness  $\Delta$  over which the permeability varies linearly from  $\mu$  to  $\mu_0$ .

in the expanded region of the surface ( $s$ ). Hence the force per unit volume acting near the surface  $s$  is constant and given by

$$F_1 = -\frac{1}{2}H^2 \frac{(\mu_0 - \mu)}{\Delta} \mathbf{i}_1. \quad (b)$$

The total force is the integral of the force density over the volume of the slab. Because  $\nabla\mu$  is constant over the volume  $\Delta ad$  and zero elsewhere, this integration reduces to

$$f_1 = -\frac{1}{2}H^2 \frac{(\mu_0 - \mu)}{\Delta} (\Delta ad) \mathbf{i}_1. \quad (c)$$

Note that this result is in agreement with (8.5.7). We may view the force density given by (8.5.31) as the generalization of (8.5.5). The force on the slab does not depend on the thickness  $\Delta$ , as can be seen from (c). We could have used a distribution of  $\mu(x_1)$  other than that shown in Fig. 8.5.5 to arrive at the same answer. Certainly the answer holds in the limit in which  $\Delta \rightarrow 0$ . This point is easily seen if the force density is represented in terms of a stress tensor, a point to which we return in the next section.

There remains the task of computing the force density that results from changes in the density of the material. As we saw in Section 8.5.1, compression of the material leads to a magnetostriction force. There it was accounted for by including the effect of changes in density on the inductance  $L$ . Here it is incorporated as it leads to a change in  $\mu$ ; that is, in addition to the change in  $\mu$  given by (8.5.29), there is a change

$$\delta\mu = \frac{\partial\mu}{\partial\rho} \delta\rho. \quad (8.5.32)$$

The decrease in density  $-\delta\rho$  is proportional to the density  $\rho$  and the increase in the volume occupied by the material  $\nabla \cdot \delta\xi$ . Hence\*

$$-\delta\rho = \rho\nabla \cdot \delta\xi. \quad (8.5.33)$$

Now, if we combine these last two equations, the first term in (8.5.26) can be written as

$$\int_V \frac{1}{2} \mathbf{H} \cdot \mathbf{H} \delta\mu \, dV = \int_V -\frac{1}{2} \mathbf{H} \cdot \mathbf{H} \frac{\partial\mu}{\partial\rho} \rho \nabla \cdot \delta\xi \, dV. \quad (8.5.34)$$

In order to find the force density, we must write the integrand of this expression in the form  $(\quad) \cdot \delta\xi$ . With this end in mind, we use an identity† to write

\* A statement that the decrease in mass within the volume  $V_e$  is equal to the mass transported out of the volume through the surface  $S_e$  is given by  $-\int_{V_e} \delta\rho \, dV = \oint_{S_e} \rho \delta\xi \cdot \mathbf{n} \, da$ .

Gauss's theorem converts the surface integral to a volume integral. To first order in  $\delta\rho$  and  $\delta\xi$ ,  $\nabla \cdot \rho \delta\xi = \rho \nabla \cdot \delta\xi$  and (8.5.33) follows.

†  $\psi \nabla \cdot \mathbf{A} = \nabla \cdot \psi \mathbf{A} - \mathbf{A} \cdot \nabla \psi$ .

(8.5.34) as

$$\int_V \frac{1}{2} \mathbf{H} \cdot \mathbf{H} \delta\mu \, dV = - \int_V \nabla \cdot \left( \frac{1}{2} \mathbf{H} \cdot \mathbf{H} \frac{\partial\mu}{\partial\rho} \rho \delta\xi \right) dV \\ + \int_V \nabla \left( \frac{1}{2} \mathbf{H} \cdot \mathbf{H} \frac{\partial\mu}{\partial\rho} \rho \right) \cdot \delta\xi \, dV. \quad (8.5.35)$$

The first integral is in a form in which by Gauss's theorem it can be written as an integral over the surface  $S$  of the volume  $V$ . On the surface  $S$  either the fields  $\mathbf{H}$  are zero (the surface is outside the field region in Fig. 8.5.4) or  $\delta\xi \cdot \mathbf{n}$  is zero (because the surface is adjacent to the rigid pole faces). Hence the first term in (8.5.35) makes no contribution and the last term is in the desired form.

It is now possible to write (8.5.26) with the effects of inhomogeneity and changes in density included. There is a contribution to  $\delta\mu$  from (8.5.29) (due to inhomogeneity) and from (8.5.32) which has already been incorporated into (8.5.35):

$$\int_V \left[ -\frac{1}{2} \mathbf{H} \cdot \mathbf{H} \nabla\mu + \nabla \left( \frac{1}{2} \mathbf{H} \cdot \mathbf{H} \frac{\partial\mu}{\partial\rho} \rho \right) - \mathbf{F} \right] \cdot \delta\xi \, dV = 0. \quad (8.5.36)$$

As before, we use the arbitrary nature of  $\delta\xi$  to conclude that the force density is

$$\mathbf{F} = -\frac{1}{2} \mathbf{H} \cdot \mathbf{H} \nabla\mu + \nabla \left( \frac{1}{2} \mathbf{H} \cdot \mathbf{H} \frac{\partial\mu}{\partial\rho} \rho \right). \quad (8.5.37)$$

Of course, the first term is the same as that given by (8.5.31). The second term is added to account for forces that accompany (or cause) changes in the density of the material and is referred to as the *magnetostriction force density*.

### 8.5.3 The Stress Tensor

It is often convenient to express the force density in terms of a stress tensor. This is done in this section, with both forces on free currents and magnetization forces included. Thus the appropriate force density is the superposition of (8.1.3) and (8.5.37):

$$\mathbf{F} = \mathbf{J}_f \times \mu \mathbf{H} - \frac{1}{2} \mathbf{H} \cdot \mathbf{H} \nabla\mu + \nabla \left( \frac{1}{2} \mathbf{H} \cdot \mathbf{H} \frac{\partial\mu}{\partial\rho} \rho \right). \quad (8.5.38)$$

The  $m$ th component of this equation can be written [using (8.1.8) to express  $\mathbf{J}_f \times \mathbf{B}$ ] as

$$F_m = \mu H_n \frac{\partial H_m}{\partial x_n} - \frac{\mu}{2} \frac{\partial}{\partial x_m} H_k H_k - \frac{1}{2} H_k H_k \frac{\partial\mu}{\partial x_m} + \frac{\partial}{\partial x_m} \left( \frac{1}{2} H_k H_k \frac{\partial\mu}{\partial\rho} \rho \right). \quad (8.5.39)$$

Because  $\partial\mu H_n/\partial x_n = 0$ , the first term in this equation can be written as  $\partial\mu H_n H_m/\partial x_n$ . The second and third terms combine. Then, by introducing the Kronecker delta  $\delta_{mn}$  (8.1.7), (8.5.39) can be written as

$$F_m = \frac{\partial T_{mn}}{\partial x_n}, \quad (8.5.40)$$

where

$$T_{mn} = \mu H_n H_m - \frac{1}{2} \delta_{mn} H_k H_k \left( \mu - \frac{\partial\mu}{\partial\rho} \rho \right). \quad (8.5.41)$$

Note that except for the magnetostriction term the stress tensor takes the same form as it did in Section 8.1, in which only the force on the free current  $\mathbf{J}_f$  is considered. This similarity is deceptive unless it is remembered that the magnetic field intensity  $\mathbf{H}$  [in (8.5.41)] is not the same with and without the current density  $\mathbf{J}_f$ . Moreover,  $\mu$  in (8.5.41) is a function of position rather than a constant, as it was in (8.1.11).

**Example 8.5.2.** The problem shown in Fig. 8.5.2 serves as an illustration for the application of the stress tensor. In this example the slab of magnetic material is free to slide in the  $x_1$ -direction but is constrained at  $x_1 = 0$  so that the left end of the slab is fixed. The magnetic field intensity  $\mathbf{H}$  is uniform throughout the slab and adjacent region of free space. It is therefore apparent from (8.5.38) that the force density in the  $x_1$ -direction is present only at the left and right extremes of the slab in which the permeability  $\mu$  and magnetostriction constant  $\rho$   $\partial\mu/\partial\rho$  undergo rapid variations. Deformations are independent of forces at the left end because it is fixed. At the right end there is a surface force that can be found by using the stress tensor. From (8.4.2) the force per unit area acting on the right end of the slab is

$$T_1 = T_{11}^a - T_{11}^b, \quad (a)$$

where (a) and (b) indicate the regions to the right and left of the surface. From (8.5.41) (a) becomes

$$T_1 = -\frac{1}{2}\mu_0 H^2 + \frac{1}{2} \left( \mu - \frac{\partial\mu}{\partial\rho} \rho \right) H^2. \quad (b)$$

Here we have taken the  $\partial\mu/\partial\rho$  as zero in the free-space region (a). The total force on the end of the slab is (b) multiplied by the area  $ad$ , and this result agrees with that found in Section 8.5.1 (8.5.12) by using a model with a single degree of freedom.

Force densities, stress tensors, and surface force densities in magnetic field systems are summarized in Table 8.1. Note that the superposition of the free current force density and the magnetization force density leads to the same stress tensor as for the magnetization force density alone.

### 8.5.4 Polarization Force Density and Stress Tensor

So far we have limited our discussion to forces induced in magnetic materials by magnetic fields. Polarization forces, induced in dielectric materials

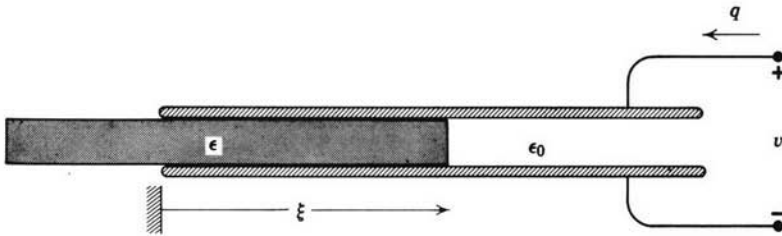


Fig. 8.5.6 A voltage  $v$  applied to perfectly conducting electrodes leads to a polarization force that tends to draw the block of dielectric into the region between the plates.

subjected to electric fields, are similar to magnetization forces. The lumped-parameter example shown in Fig. 8.5.6 is analogous to that shown in Fig. 8.5.1. A potential between the perfectly conducting electrodes induces a polarization force in the dielectric material which tends to draw it into the region between the plates. We could find this force by again using the energy methods introduced in Chapter 3\* and writing a conservation of energy equation analogous to (8.5.1).

$$q \delta v = \delta W' - f \delta \xi. \quad (8.5.42)$$

Now,  $W'$  is the electric coenergy,  $q$  is the charge on the upper electrode, and  $v$  is the potential of the upper electrode with the potential of the lower electrode defined as zero.

We are interested here in finding the polarization force density and so generalize (8.5.42) to write a conservation of energy expression analogous to (8.5.15) (the physical system is shown in Fig. 8.5.7).

$$q \delta v = \int_V \delta w' dV - \int_V \mathbf{F} \cdot \delta \boldsymbol{\xi} dV. \quad (8.5.43)$$

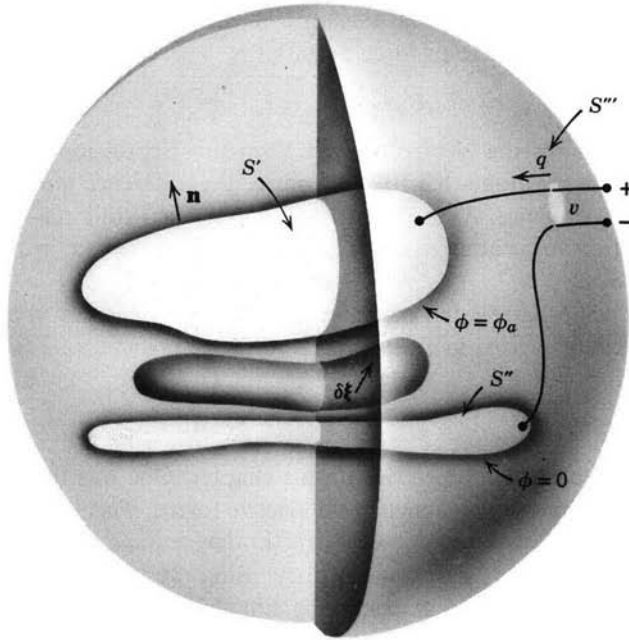
If we define the potential  $\phi (\mathbf{E} = -\nabla \phi)$  as being zero on the lower electrode, then  $v = \phi$  evaluated on the upper electrode, whereas the total charge  $q$  is the integral of the surface charge  $-\epsilon \mathbf{n} \cdot \nabla \phi$  over the surface of the upper electrode. Hence (8.5.43) becomes

$$-\int_{S'} \epsilon \nabla \phi \cdot \mathbf{n} \delta \phi da = \int_V \delta w' dV - \int_V \mathbf{F} \cdot \delta \boldsymbol{\xi} dV. \quad (8.5.44)$$

As in Section 8.5.2, the surface of integration  $S'$  can be extended to enclose a volume  $V$  that includes all the deformable material with no further contribution to the integral. This is true because  $\phi$  is zero on the lower electrode which is enclosed by a surface  $S''$ . Moreover, there is no contribution to an integration over a surface  $S'''$  that encloses the entire system, for this surface

\* Table 3.1, Appendix E.





**Fig. 8.5.7** A pair of perfectly conducting rigid electrodes imposes an electric field  $\mathbf{E} = -\nabla\phi$  on a polarizable deformable dielectric medium. The surface  $S'$  encloses the upper electrode which is at the potential  $\phi = v$ .

is either far from the system or arranged (in the neighborhood of the terminals) so that  $\mathbf{n} \cdot \mathbf{D} = 0$ . The surface  $S = S' + S'' + S'''$  completely encloses the volume  $V$ , and we can make use of Gauss's theorem to convert the left-hand side of (8.5.44) to a volume integral over  $V$ .

As a consequence of these manipulations, the left-hand side of (8.5.44) takes the same form as (8.5.19) with  $\mu \rightarrow \epsilon$ . Of course,  $\phi$  now has the physical significance of being the potential for the electric field rather than for the magnetic field. All of the mathematical steps following (8.5.19) are valid, however, and we are led to a polarization force density with the same form as (8.5.37), with  $\mathbf{H} \rightarrow \mathbf{E}$  and  $\mu \rightarrow \epsilon$ . If we superimpose on this force density, the force density on free charges  $\rho_f$ , the force density is

$$\mathbf{F} = \rho_f \mathbf{E} - \frac{1}{2} \mathbf{E} \cdot \mathbf{E} \nabla \epsilon + \nabla \left( \frac{1}{2} \mathbf{E} \cdot \mathbf{E} \frac{\partial \epsilon}{\partial \rho} \rho \right). \quad (8.5.45)$$

The first term is the free charge force density, the second is due to inhomogeneities in the dielectric, and the last results from changes in the material density. This last term is called the *electrostriction force density*.

Manipulations of (8.5.45) that incorporate the irrotational nature of the electric field intensity show that the stress tensor representation of combined free charge and polarization force densities is

$$T_{mn} = \epsilon E_m E_n - \frac{1}{2} \delta_{mn} E_k E_k [\epsilon - (\partial\epsilon/\partial\rho)\rho]. \quad (8.5.46)$$

Note that without the electrostriction term this expression is as obtained for the free charge alone (8.3.10). Of course, the difference now is that  $\epsilon$  can be a function of space. At the same time the electric field intensity that must be used in (8.5.46) is affected by the presence of free charge in the material, for  $\nabla \cdot \epsilon \mathbf{E} = \rho_f$ .

Force densities, stress tensors, and surface force densities in electric field systems are summarized in Table 8.1.

## 8.6 DISCUSSION

There have been two objectives in this chapter. One was the development of a field description of magnetic and electric forces. This led to the concept of a stress tensor, which was convenient in determining total forces from a knowledge of the fields over a surface enclosing the volume of interest. The stress tensor is also useful in describing singular force distributions such as surface forces. The stress tensor, as developed here, is of interest as a basic mathematical representation. As illustrated in the chapters that follow, it can be used to represent a variety of physical quantities.

Our second objective has been to develop a picture of the distribution of forces due to magnetization and polarization. This was done while illustrating the important fact that the energy methods which form the theme of Chapter 3 are of equal significance in formulating a continuum description of electro-mechanical interactions.

From our derivations and discussion it should be clear that attention has been confined to a simple class of materials but that similar techniques can be used to determine force densities in more complicated media; for example, extensions of the energy method should allow us to find the force density in materials that are electrically nonlinear. Certainly the energy methods of Chapter 3 are not confined to electrically linear systems. Most solids do not exhibit the simple linear isotropic constitutive laws used here. Nonetheless, energy methods can be employed to find the force density in such situations,\* although the formalisms used may be somewhat different from those used here.†

\* J. A. Stratton, *Electromagnetic Theory*, McGraw-Hill, New York, 1941, p. 140.

† P. Penfield and H. Haus, *Electrodynamics of Moving Media*, M.I.T. Press, Cambridge, Mass., 1967; W. F. Brown, Jr., "Theory of Magnetoelastic Effects in Ferromagnetism," *J. Appl. Phys.*, **36**, 994 (1965).