PROBLEMS

7.1. If a short-circuited inductor is modeled by an ideal inductance (no resistance) in series with a resistance R equal to its internal resistance, the time-constant for decay of an initial flux $\lambda = \Lambda$ is $\tau = L/R$. Design an inductor with a time-constant L/R of 100 sec. Specify the dimensions and the material involved; for example, you may wish to use

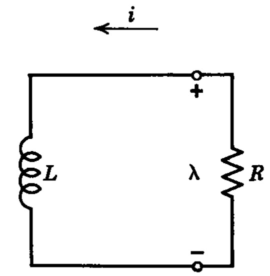

Fig. 7P.1

copper wire, in which case you should specify the size of the wire, the number of turns required, and the size of the winding. You are not allowed to use a superconducting winding.

7.2. This problem is a variation on the situation described in Section 7.1.1. A conducting slab of material with thickness d is positioned between the pole faces of a magnet, as shown in Fig. 7P.2. The switch S has been open for a long time so that the magnetic field in the gap

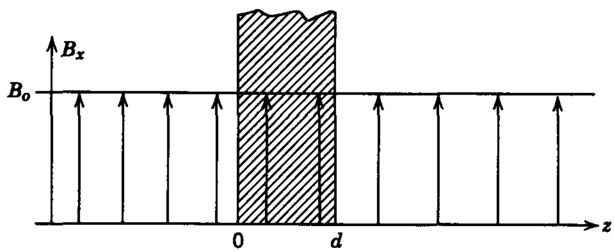

Fig. 7P.2

is as shown. When $t = 0$, the switch is closed and the magnetic field outside the material in the gap collapses.

(a) Compute the flux density distribution as a function of time in the gap.
(b) Find the current distribution from the result of (a).

7.3. A fixed block of material with conductivity σ makes electrical contact with a pair of perfectly conducting parallel plates, as shown in Fig. 7P.3. Dimensions are such that $w \ll L$ and $w \ll D$. A current source I (amperes) is distributed along the edge of the plates at $x = -L$. $I(t) = I_o u_{-1}(t)$, as shown.

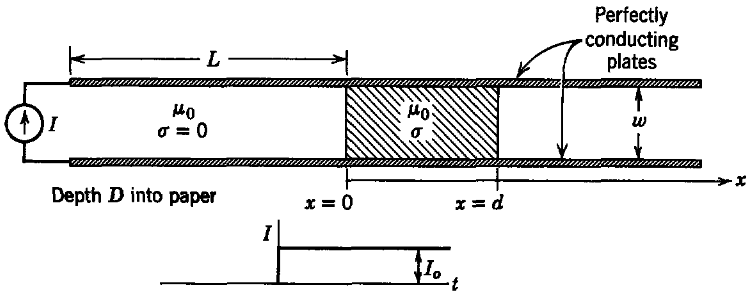

Fig. 7P.3

(a) Sketch and dimension the distribution of current in the block an instant after the step of current is applied.
(b) Sketch and dimension the distribution of current as $t \to \infty$.
(c) In terms of σ and the system dimensions, how long do you *estimate* that it takes the current to reach essentially the distribution of (b)? (This should be an order of magnitude calculation.)

7.4. A pair of perfectly conducting plates is short-circuited through a conducting block, as shown in Fig. 7P.4*a*. The block and plates extend a long distance to the right. A current excitation $i(t) = \text{Re}\,(ie^{j\omega t})$ is applied uniformly to the plates along their left edges.

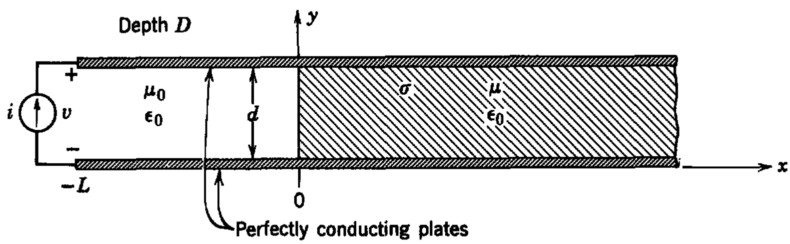

Fig. 7P.4*a*

(a) Find the magnetic flux density in the region between the plates.
(b) Find the current density in the block.
(c) Sketch the results of (a) and (b).
(d) Find the equivalent reactance seen at the current source. Using the equivalent circuit shown in Fig. 7P.4*b*, what are the values of L and R? How do they depend on frequency?

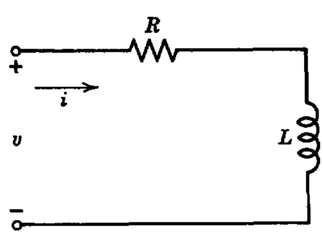

Fig. 7P.4*b*

7.5. A block of material with conductivity σ is bounded by plane-parallel, perfectly conducting electrodes, as shown in Fig. 7P.5. The plates are driven at the left by the current (amperes) $I_0(t) = \text{Re}\,(\hat{I}e^{j\omega t})$, where $\hat{I}$ is a given complex constant. This current is returned through the block of conductor.

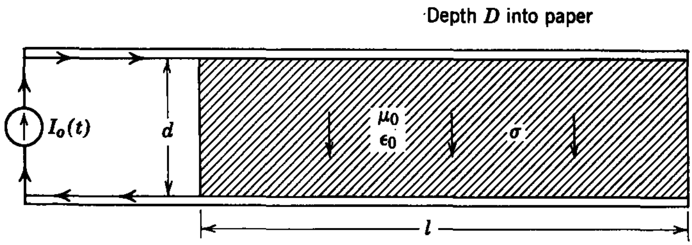

Fig. 7P.5

(a) Determine the distribution of magnetic field and current within the block.
(b) Outline how you would find the time-average magnetic force on the block.

7.6. As an example of electromagnetic phenomena that occur in conductors at rest, consider the system in Fig. 7.1.1 with the constant-current sources and switch replaced by an alternating current source, $i(t) = I \cos \omega t$. Make all of the assumptions of Section 7.1.1 and adopt the coordinate system of Fig. 7.1.2. Interest is confined to the sinusoidal steady-state problem.

(a) Find the magnetic flux density in the conducting slab.
(b) Find the current density in the slab.
(c) Sketch the results of parts (a) and (b).

7.7. A pair of plane-parallel, perfectly conducting plates is driven at the left, as shown in Fig. 7P.7*a*, by a current source $I(t)$, which is a step function. The plates are short-circuited by two conducting slabs spaced a distance l and of thickness Δ. An instant after the current I is established all the current is returned in the left slab; region (1) contains a uniform magnetic field, and region (2) is free of magnetic field. Then, with a characteristic time τ, the field diffuses into region (2).

(a) Use the "equivalent circuit model" shown in Fig. 7P.7*b* to establish this characteristic time.
(b) Physically, why is this characteristic time much longer than the diffusion time based on the thickness Δ of the conducting slabs [(7.1.28) with $d = \Delta$]?

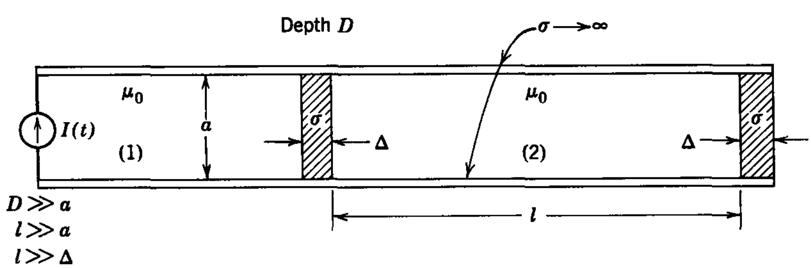

Fig. 7P.7*a*

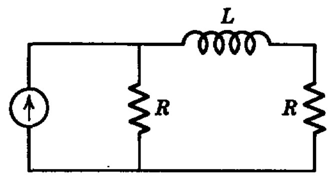

Fig. 7P.7*b*

(c) Discuss how you would solve this problem by using a field description of the diffusion process.

7.8. The cross section of a cylindrical pair of conductors is shown in Fig. 7P.8*a*. The current I is a step function and flows azimuthally in the outer conductor. Hence, when $t = 0$, the magnetic field intensity has the distribution of Fig. 7P.8*b*. The inner conductor acts as a secondary of a transformer and supports an induced current that tends to shield the magnetic

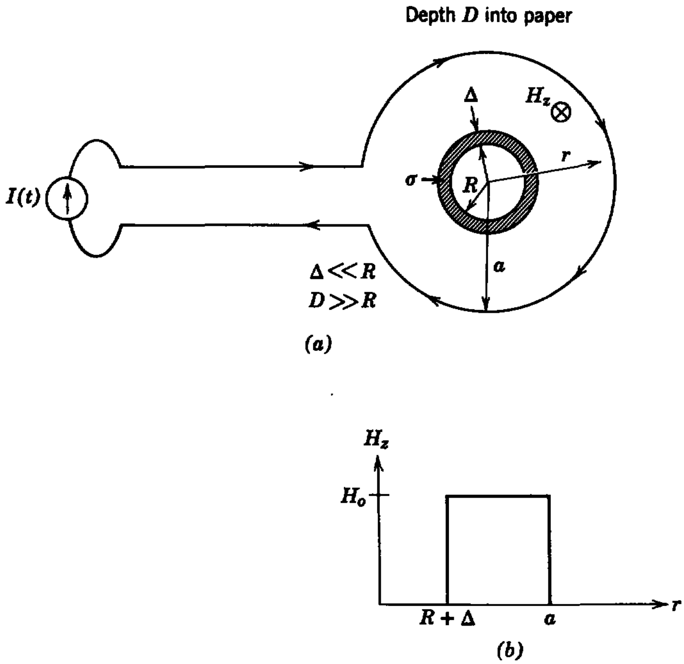

Fig. 7P.8*a*

field induced by I from the center region. Hence inside the center cylinder the magnetic field rises to its final value with some characteristic time. What is this characteristic time in terms of the system dimensions and σ? *Hint.* See Problem 7.7.

7.9. As a possible plasma containment scheme, it has been proposed to create a cylindrical column of plasma within a long solenoid. The plasma is created by use of an arc discharge through a gas. After the plasma is formed the solenoid is excited by rapid discharge of a

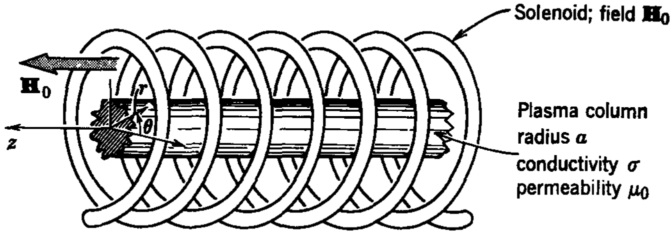

Fig. 7P.9*a*

capacitor bank connected to the terminals. The magnetic field of the solenoid is initially excluded from the interior of the plasma column by surface currents which flow in the theta direction on the plasma; for this reason the device is called a "Theta-pinch machine." Because the plasma has only a finite conductivity, the magnetic field diffuses into the interior. Therefore to design the machine it is necessary to determine the time associated with this field diffusion. In Fig. 7P.9*a* the plasma column is shown in the magnetic field. Neglect end effects by assuming an infinitely long system. Furthermore, assume that the plasma remains stationary during the diffusion process.

(a) Write an equation governing the magnetic flux density **B** inside the plasma.

(b) Because there are only θ currents independent of θ, assume that $\mathbf{B} = \mathbf{i}_z B(r, t)$, where $B(r, t)$ has the form

$$B(r, t) = \mu_0 H_0 - \hat{B}(r)e^{-\alpha t}, \qquad \alpha > 0,$$

which satisfies the condition that $B(r, t) \to \mu_0 H_0$ as $t \to \infty$. Using the results of part (a), write an equation for $\hat{B}(r)$.

(c) The equation obtained in part (b) for $\hat{B}(r)$ is called "Bessel's equation of zeroth order." The general solution to this equation is

$$\hat{B}(r) = C_1 J_0(\sqrt{\mu_0 \sigma \alpha}\, r) + C_2 N_0(\sqrt{\mu_0 \sigma \alpha}\, r),$$

where J_0 is the Bessel function of zeroth order and the first kind and N_0 is the Bessel

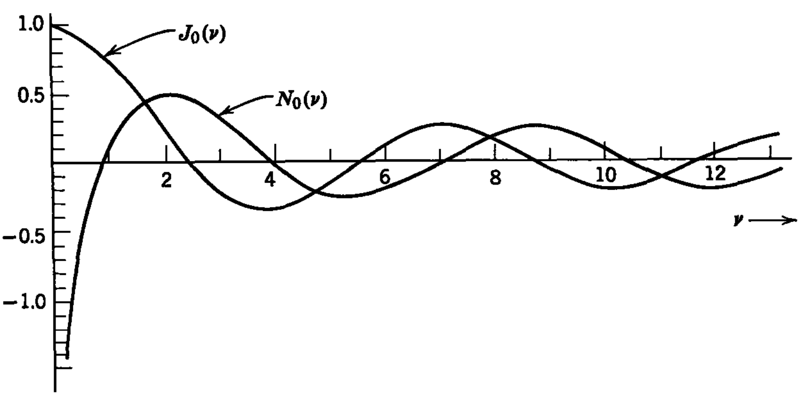

Fig. 7P.9*b*

function of zeroth order and the second kind (sometimes called a "Neumann function"). See Fig. 7P.9*b*.

Using the boundary condition at $r = 0$, argue that $C_2 = 0$, hence that $\hat{B}(r) = C_1 J_0(\sqrt{\mu_0 \sigma \alpha}\, r)$.

(d) Now apply the boundary condition at $r = a$ that $B(r, t) = \mu_0 H_0$ for all $t > 0$ and show that $J_0(\sqrt{\mu_0 \sigma \alpha}\, a) = 0$. This transcendental equation determines the allowed values of α.

Roots of $J_0(\nu_i) = 0$ and the Corresponding Values of $J_1(\nu_i)$

i	ν_i	$J_1(\nu_i)$
1	2.4048	0.5191
2	5.5201	−0.3403
3	8.6537	0.2715
4	11.7915	−0.2325
5	14.9309	0.2065
6	18.0711	−0.1877

The values of α may be obtained from the table as

$$\sqrt{\mu_0 \sigma \alpha}\, a = \nu_i; \qquad \alpha = \frac{1}{\mu_0 \sigma}\left(\frac{\nu_i}{a}\right)^2,$$

where $J_0(\nu_i) = 0$.

(e) On examination of the results, it is evident that $\hat{B}(r) = C_i J_0[\nu_i(r/a)]$ is a solution for any of the allowed values of ν_i. The most general solution for $B(r, t)$ is obtained by using superposition; hence

$$B(r, t) = \mu_0 H_0 - \sum_{i=1}^{\infty} C_i J_0\left(\nu_i \frac{r}{a}\right) \exp\left(-\frac{\nu_i^2 t}{\mu_0 \sigma a^2}\right),$$

and to evaluate the constants C_i the last boundary condition $B(r, t = 0) \equiv 0$ is used. This condition implies that

$$\sum_{i=1}^{\infty} C_i J_0\left(\nu_i \frac{r}{a}\right) = \mu_0 H_0,$$

which is just a series in terms of the functions $J_0[\nu_i(r/a)]$.

Using the integrals

$$\int_0^a r\, J_0\left(\nu_i \frac{r}{a}\right) J_0\left(\nu_j \frac{r}{a}\right) dr = \begin{cases} \dfrac{a^2}{2} J_1^2(\nu_i), & \nu_i = \nu_j, \\ 0, & \nu_i \neq \nu_j, \end{cases}$$

and

$$\int_0^a r\, J_0\left(\nu_i \frac{r}{a}\right) dr = \frac{a^2}{\nu_i} J_1(\nu_i),$$

evaluate C_1, C_2, and C_3.

(f) What is the fundamental time constant for the diffusion? Evaluate this time constant for $a = 5$ cm and $\sigma = 10^4/4\pi(\Omega\text{m})^{-1}$.

7.10. A slab of conducting metal (conductivity σ) moves in the x-direction with the constant velocity U. This slab makes contact with perfectly conducting electrodes which are driven at the left end by a constant current K A/m. This source is distributed along the z-axis but does not get in the way of the moving slab. Assume that there is no fringing and that

$$\mathbf{J} = \mathbf{i}_y J_y(x),$$
$$\mathbf{B} = \mathbf{i}_z B_z(x),$$
$$\mathbf{E} = \mathbf{i}_y E_y(x).$$

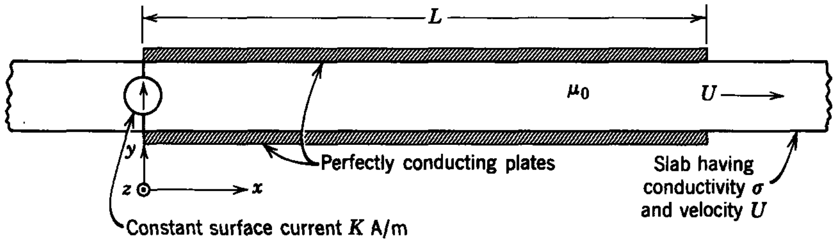

Fig. 7P.10

(a) Write a single differential equation for $B_z(x)$.
(b) Write the boundary conditions satisfied by B_z.
(c) Find an expression for $B_z(x)$ between the perfectly conducting plates. (Reference: see Section 7.1.2a.)

7.11. In the example in Section 7.1.2a we found the flux density and current density given by (7.1.49) and (7.1.50) for the system in Fig. 7.1.9.

(a) Calculate the magnetic force density $\mathbf{J} \times \mathbf{B}$ applied to the moving conductor.
(b) Integrate this force density throughout the volume of the conductor to find the total force that must be supplied by the velocity source. Show that this force is independent of conductivity σ and velocity v.
(c) Evaluate the power supplied by the velocity source and specify how much power goes to J^2/σ losses in the moving conductor and how much power goes into the current source.

7.12. The electric and magnetic fields were found in Section 7.1.2b for a block of conductor moving between shortcircuited, perfectly conducting plates (see Figs 7.0.1 and 7.1.11). In the fixed frame these fields are functions of both z and t. Show that the fields expressed in the fixed frame satisfy the magnetic diffusion equation (7.1.9.).

7.13. A slab of conductor moves in the z-direction with a velocity V, as shown in Figs. 7.1.12 and 7P.13. A perfectly permeable magnetic circuit with the z length L has the same configuration as that in Fig. 7.1.12, except that it has no excitation coils. Instead, an external source is used to constrain $B_y(-L, t)$ to be $B_y(-L, t) = \text{Re}\, B_0 \exp j\omega t$.

(a) What physical arguments would you use to show that the condition at $x = 0$, where the moving slab leaves the region between the pole faces, is described approximately by $B_y(0, t) = 0$?
(b) Find an expression for $B_y(z, t)$, $-L < z < 0$.
(c) What is J_x in the range $-L < z < 0$?
(d) Sketch B_y and J_x in the limit at which $\omega \to 0$. How would you physically arrange the excitation to make these fields a reasonable approximation? (See Section 7.1.2a.)

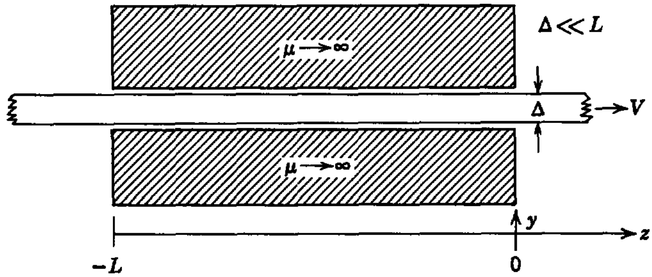

Fig. 7P.13

7.14. You are working on a transportation project and are asked to make an analysis of the following basic method of both levitating and propelling a train. The train rides just above a track which is composed of a slab with conductivity σ. Superconducting coils within the train are arranged to produce a current that can be represented by the current sheet $\mathbf{K}$ (see Fig. 7P.14). This current sheet is backed by a highly permeable magnetic shield ($\mu \to \infty$) which is also attached to the train. (The shield prevents magnetization of the passengers' watches and the attendant possibility that $t \neq t'$.) Here x' is the distance measured with respect to the moving train. Hence, because the train is moving in the

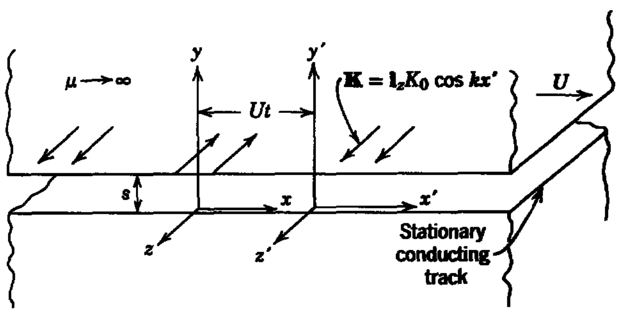

Fig. 7P.14

x-direction with velocity U, $x = Ut + x'$. We wish to compute the time average force per unit area that presumably holds the train a distance s ($ks \ll 1$) above the track.

(a) Express the surface current in the fixed frame $\mathbf{K}(x, t)$.

(b) Assume that the track is infinitely thick in the y-direction (under what conditions is this a good assumption?) and compute the magnetic field and current in the conducting track. Assume that $\partial/\partial z = 0$.

(c) Compute the time average force per unit area (in the x-z plane) that holds up the vehicle.

(d) Compute the force per unit (x-z) area that tends to propel the train in the x-direction. Do you see any basic problems with the proposed scheme of propulsion?

7.15. (a) Compute the fields for the example of Section 7.1.4, taking into account the effect of the spacing s between the moving conductor and the traveling current sheet.

(b) Check to see that the results of Section 7.1.4 are obtained in the limit $ks \ll 1$.
(c) Check to see that currents induced in the moving conductor approach zero as $ks \to \infty$.

7.16. A pair of perfectly conducting electrodes with area A are shown in Fig. 7P.16. Between the plates there is a fluid of depth b, conductivity σ, and permittivity ϵ, bounded from above by an insulating gas. When $t = 0$, there is no charge in the system and the switch S is closed.

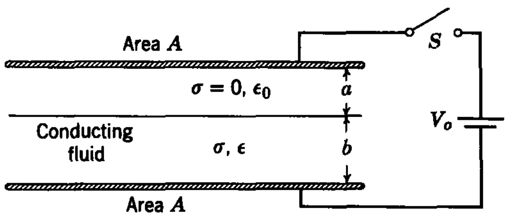

Fig. 7P.16

(a) What are the electric fields in the liquid (E_l) and in the gas (E_g) an instant after $t = 0$?
(b) What are these electric fields as $t \to \infty$?
(c) Find the charge (q) on the fluid-gas interface as a function of time.

7.17. A spherically symmetric system of free charges and conductors is shown in Fig. 7P.17. A sphere of material of conductivity σ_i is embedded at the center of a second material with conductivity σ_o. When $t = 0$, there is a volume charge density $\rho_0(r)$ in the region $r < R_i$, but no charge density in the outer region. There are no surface charges when $t = 0$.

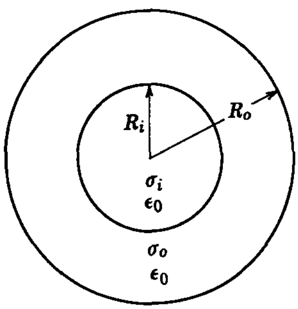

Fig. 7P.17

(a) Find the charge density at points $r < R_i$ in the center sphere and for $R_i < r < R_o$ in the outer region for $t > 0$.
(b) Find the surface charges at $r = R_i$ and $r = R_o$ for $t > 0$.
(c) Sketch the results of (b) for $\sigma_i > \sigma_o$ and $\sigma_i < \sigma_o$.
Hint. You may wish to define the total charge in the inner sphere as

$$q(t) = 4\pi \int_0^{R_i} \rho_0(r) r^2\, dr$$

7.18. Three long, cylindrical, highly conducting shells of radius a, b, and c, respectively, are aligned concentrically, as shown in Fig. 7P.18. The space between the cylinders is filled with a material of permittivity ϵ and conductivity σ. Initially, the cylinders of radii a and b are biased with a battery V_0 and switch S in position (1) as shown. At a time $t = 0$, the switch is connected to terminal (2).

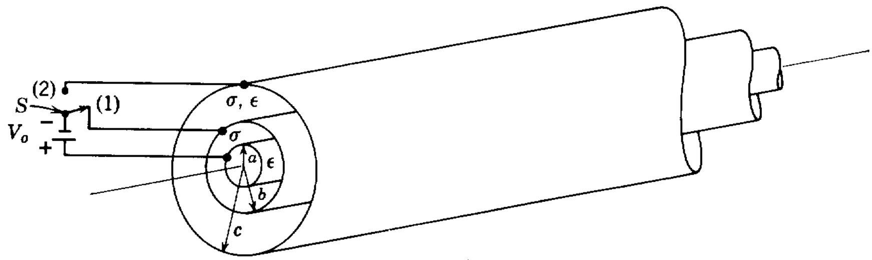

Fig. 7P.18

(a) Find expressions for the electric field intensity **E** in the regions between the cylinders.
(b) Find the expression for the charge per unit length on the cylinder of radius b.
(c) Find the equivalent circuit for a unit length of the cylinders.

7.19. A pair of metal spheres is suspended in a beaker of slightly conducting liquid, as shown in Fig. 7P.19. The lower sphere is fixed to the bottom, whereas the upper one is free to move (say, with deflection x). The fluid has a constant conductivity σ and permittivity ϵ. An external source of potential v is used to establish the charge q on the upper sphere.

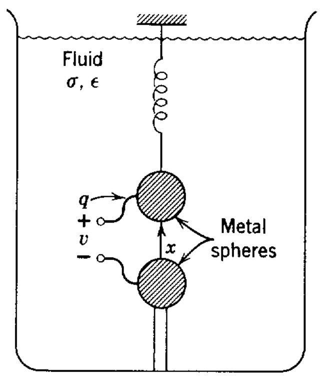

Fig. 7P.19

Then the source of potential is removed, leaving the sphere electrically isolated, except for conduction through the liquid, and free to move. Use the integral laws for an electric field system [(1.1.24) to (1.1.26) and definitions of Table 2.1]* to show that, if the initial charge on the sphere is Q, then as a function of time, the charge q on the sphere is $q = Qe^{-t/\tau}$; $\tau = \epsilon/\sigma$. Your proof should be valid even though the upper sphere is moving [$x = x(t)$].

* Appendix E.

7.20. A pair of parallel slightly conducting plates of conductivity σ and spacing d are insulated from each other, as shown in Fig. 7P.20. When $t = 0$, there is a charge $-Q$ uniformly distributed over the volume of the upper plate and a charge $+Q$ uniformly distributed over the volume of the lower plate. Assume that $d \ll D$:

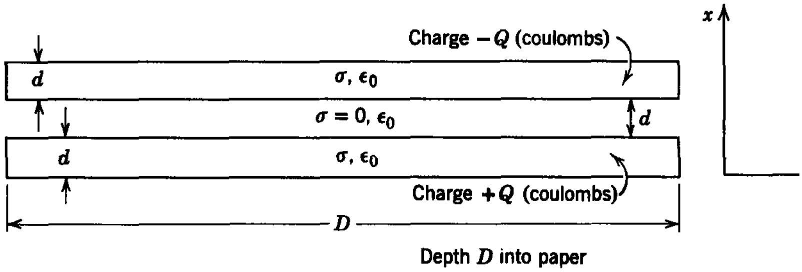

Fig. 7P.20

(a) Make a dimensioned sketch of the electric field E_x as a function of x when $t = 0$ and as $t \to \infty$.
(b) Compute the total force on the lower plate as a function of time.

7.21. Consider the problem of Example 7.2.3 with

$$\epsilon = \epsilon_1 + \frac{\epsilon_2}{l}x,$$

$$\sigma = \sigma_1 + \frac{\sigma_2}{l}x.$$

(a) Find $E_x(x, t)$ and $\rho_f(x, t)$.
(b) Discuss the effect of the nonuniform ϵ.

7.22. A fluid moves with the constant velocity $U\mathbf{i}_x$ between parallel plates, as shown in Fig. 7P.22*a*. The fluid is uncharged ($\rho_f = 0$) for $x < 0$. It has the uniform conductivity σ (which is small) and permittivity ϵ. At $x = 0$ charges are introduced by means of a two-dimensional source midway between the plates. Hence at $x = 0$ the charge has the distribution shown in Fig. 7P.22*b*, where $\hat{\rho}_o$ is a given complex constant and Δ is the initial thickness of the layer of charge that has been injected.

(a) Find the charge distribution across the channel at a downstream position $x = l$.
(b) Describe how you would use the potential induced by this charge on two downstream electrodes to measure the flow velocity U. (Assume that you have control over the frequency. There is more than one way to accomplish this and only a qualitative description is required.)

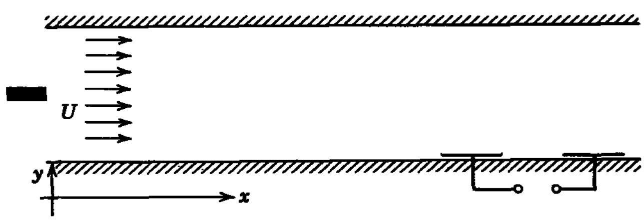

Fig. 7P.22*a*

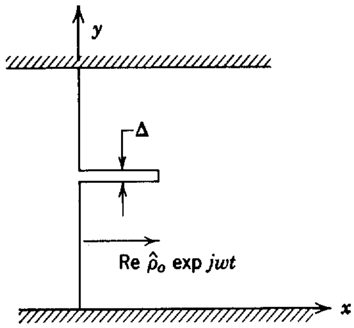

Fig. 7P.22*b*

7.23. A fluid with conductivity σ flows in the x-direction through a pipe with insulating walls and cross-sectional area A. The velocity of the fluid is approximated as constant (U). Ions are injected into the flow at $x = 0$ with a source that maintains the charge density at $x = 0$ equal to ρ_o (a constant).

(a) Find the charge density distribution between the screen electrodes.
(b) Find the distribution of electric field $E_x(x)$ between the electrodes.
(c) What is the voltage V generated across the load resistance R?

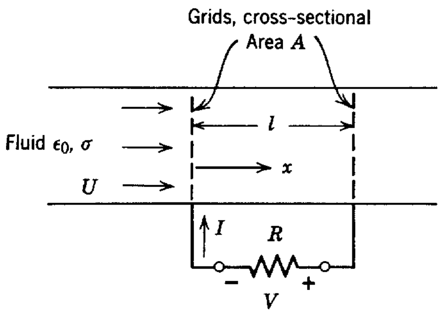

Fig. 7P.23

7.24. When Lord Kelvin invented his electrostatic generator (see Example 7.2.6), he had in mind the generation of dc high voltages. In 1966 William C. Euerle, a student at MIT, elaborated on Lord Kelvin's original idea to make an electrostatic ac generator. This device, with a Y-connected resistive load, is shown in Fig. 7P.24. You can assume that each of the "drippers" is characterized by the same parameters.

(a) Write three equations for the voltages v_1, v_2, and v_3.
(b) Use the dynamic description of the potentials found in (a) to show that it is possible for the system to behave as an oscillator; that is, that any one of the voltages would behave as a sinusoid with an exponentially growing amplitude.
(c) For what values of the parameters will the load limit the oscillations to constant amplitude?
(d) Under the conditions of (c), what is the oscillation frequency?

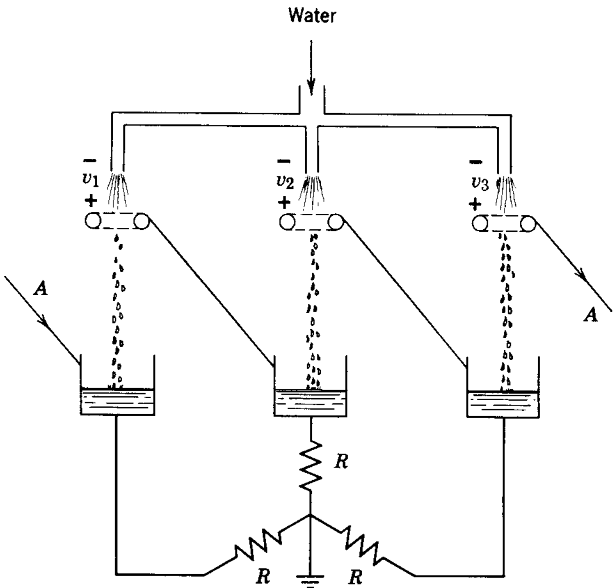

Fig. 7P.24

7.25. Two systems that are proposed to measure the depth d of water in a tank are shown in Fig. 7P.25. In system (a) the depth is indicated by measuring the "inductance" of a loop of copper conductor which encloses a portion of the water; in system (b) the "capacitance" of a pair of copper electrodes is to be used to indicate the depth. You are given that d is on the order of 1 cm., where the properties of the water are about $\sigma = 10^{-2}$ mho/m, $\epsilon = 81 \times \epsilon_0 = 81 \times 8.85 \times 10^{-12}$ F/m, $\mu = \mu_0 = 4\pi \times 10^{-7}$ H/m. Measurements of both the

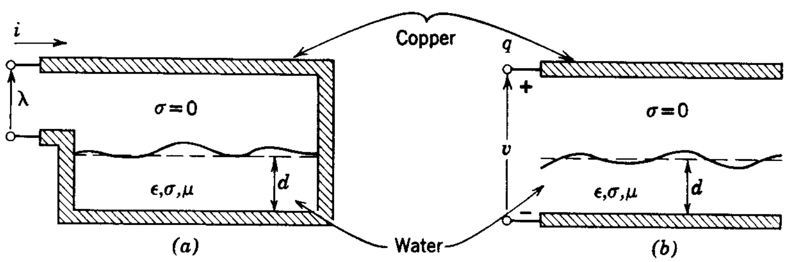

Fig. 7P.25

"capacitance" and "inductance" are to be made with a 100-kc bridge. Which of the two devices would work and why? For what values of σ would both devices be attractive for this application?

7.26. A pair of devices, proposed for measuring the level x of water, is shown in Fig. 7P.26. In case (a) an iron core magnet is driven by a 1-kc signal with the resulting voltage v_o. In case (b) a pair of plates is parallel to the surface and is driven by a voltage source

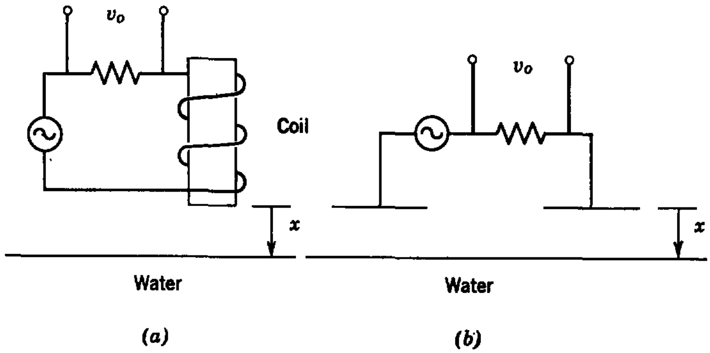

Fig. 7P.26

with a signal v_o across a series resistor. In each case it is desired that v_o be sensitive to the proximity of the water so that the device can be used to indicate the level of the water. Which of the two devices would you use to solve an engineering problem? Give your reasoning.

7.27. Consider the example in Section 7.2.4 but with the additional complication of a perfectly conducting electrode bounding the material (b) from below at $x = -f$.

(a) Find the potential distribution in regions (a) and (b).

(b) Check to see that the results of Section 7.2.4 are obtained if $|fk| \gg 1$.

7.28. A sheet of slightly conducting material moves in the z-direction with the constant velocity U. Above and below the sheet electrodes impose the traveling potential waves shown in Fig. 7P.28. The sheet thickness is small compared with c, so that both sheet surfaces can be considered to be at $x = 0$. The constitutive law for a fixed section of the

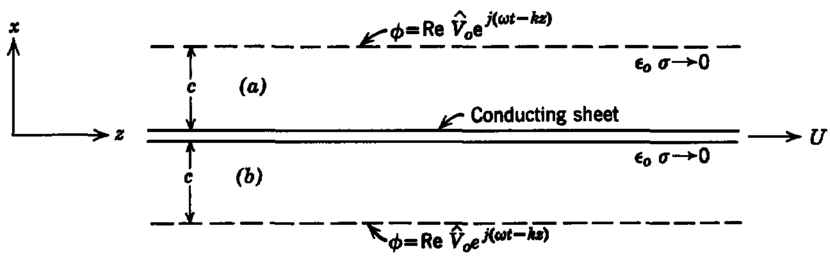

Fig. 7P.28

sheet is $K_f = \sigma_s E_z$, where σ_s is a surface conductivity, K_f is a surface conduction current density, and E_z is the electric field intensity tangential to the sheet.

(a) Write differential equations and boundary conditions in terms of the potentials $\phi(x, z, t)$ above (a) and below (b) the sheet.

(b) Determine the time average force/unit length z and unit width y in the z-direction. At what frequency will it have its largest value?

(c) How would you adjust the traveling potential wave to produce a time average force as $\sigma_s \rightarrow 0$?