Appendix G

SUMMARY OF PARTS I AND II AND USEFUL THEOREMS

IDENTITIES

$$\mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{B} \times \mathbf{C},$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

$$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi,$$

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B},$$

$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B},$$

$$\nabla(\phi\psi) = \phi\,\nabla\psi + \psi\,\nabla\phi,$$

$$\nabla \cdot (\psi\mathbf{A}) = \mathbf{A} \cdot \nabla\psi + \psi\nabla \cdot \mathbf{A},$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B},$$

$$\nabla \cdot \nabla\phi = \nabla^2\phi,$$

$$\nabla \cdot \nabla \times \mathbf{A} = 0,$$

$$\nabla \times \nabla\phi = 0,$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A},$$

$$(\nabla \times \mathbf{A}) \times \mathbf{A} = (\mathbf{A} \cdot \nabla)\mathbf{A} - \tfrac{1}{2}\nabla(\mathbf{A} \cdot \mathbf{A}),$$

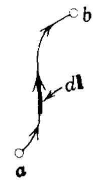
$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$$

$$\nabla \times (\phi\mathbf{A}) = \nabla\phi \times \mathbf{A} + \phi\nabla \times \mathbf{A},$$

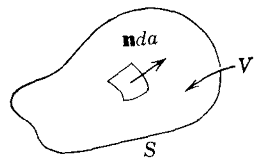
$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}.$$

THEOREMS

$$\int_a^b \boldsymbol{\nabla}\phi \cdot d\mathbf{l} = \phi_b - \phi_a.$$



Divergence theorem $\displaystyle\oint_S \mathbf{A} \cdot \mathbf{n}\, da = \int_V \boldsymbol{\nabla} \cdot \mathbf{A}\, dV$



Stokes's theorem $\displaystyle\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathbf{n}\, da$

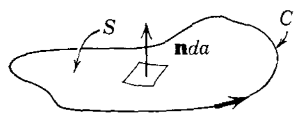
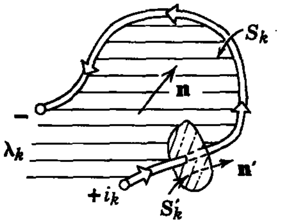
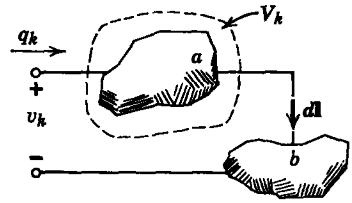


Table 1.2 Summary of Quasi-Static Electromagnetic Equations

	Differential Equations		Integral Equations	
Magnetic field system	$\nabla \times \mathbf{H} = \mathbf{J}_f$	(1.1.1)	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J}_f \cdot \mathbf{n}\, da$	(1.1.20)
	$\nabla \cdot \mathbf{B} = 0$	(1.1.2)	$\oint_S \mathbf{B} \cdot \mathbf{n}\, da = 0$	(1.1.21)
	$\nabla \cdot \mathbf{J}_f = 0$	(1.1.3)	$\oint_S \mathbf{J}_f \cdot \mathbf{n}\, da = 0$	(1.1.22)
	$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	(1.1.5)	$\oint_C \mathbf{E}' \cdot d\mathbf{l} = -\dfrac{d}{dt}\int_S \mathbf{B} \cdot \mathbf{n}\, da$ where $\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}$	(1.1.23)
Electric field system	$\nabla \times \mathbf{E} = 0$	(1.1.11)	$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$	(1.1.24)
	$\nabla \cdot \mathbf{D} = \rho_f$	(1.1.12)	$\oint_S \mathbf{D} \cdot \mathbf{n}\, da = \int_V \rho_f\, dV$	(1.1.25)
	$\nabla \cdot \mathbf{J}_f = -\dfrac{\partial \rho_f}{\partial t}$	(1.1.14)	$\oint_S \mathbf{J}_f' \cdot \mathbf{n}\, da = -\dfrac{d}{dt}\int_V \rho_f\, dV$	(1.1.26)
	$\nabla \times \mathbf{H} = \mathbf{J}_f + \dfrac{\partial \mathbf{D}}{\partial t}$	(1.1.15)	$\oint_C \mathbf{H}' \cdot d\mathbf{l} = \int_S \mathbf{J}_f' \cdot \mathbf{n}\, da + \dfrac{d}{dt}\int_S \mathbf{D} \cdot \mathbf{n}\, da$ where $\mathbf{J}_f' = \mathbf{J}_f - \rho_f \mathbf{v}$ $\mathbf{H}' = \mathbf{H} - \mathbf{v} \times \mathbf{D}$	(1.1.27)

Table 2.1 Summary of Terminal Variables and Terminal Relations

Magnetic field system	Electric field system
	

Definition of Terminal Variables

Magnetic field system	Electric field system
Flux	Charge
$\lambda_k = \int_{S_k} \mathbf{B} \cdot \mathbf{n}\, da$	$q_k = \int_{V_k} \rho_f \, dV$
Current	Voltage
$i_k = \int_{S_k'} \mathbf{J}_f \cdot \mathbf{n}'\, da$	$v_k = \int_a^b \mathbf{E} \cdot d\mathbf{l}$

Terminal Conditions

Magnetic field system	Electric field system
$v_k = \dfrac{d\lambda_k}{dt}$	$i_k = \dfrac{dq_k}{dt}$
$\lambda_k = \lambda_k(i_1 \cdots i_N;$ geometry$)$	$q_k = q_k(v_1 \cdots v_N;$ geometry$)$
$i_k = i_k(\lambda_1 \cdots \lambda_N;$ geometry$)$	$v_k = v_k(q_1 \cdots q_N;$ geometry$)$

Table 3.1 Energy Relations for an Electromechanical Coupling Network with N Electrical and M Mechanical Terminal Pairs*

Magnetic Field Systems		Electric Field Systems	
Conservation of Energy			
$dW_m = \sum_{j=1}^{N} i_j \, d\lambda_j - \sum_{j=1}^{M} f_j^e \, dx_j$	(a)	$dW_e = \sum_{j=1}^{N} v_j \, dq_j - \sum_{j=1}^{M} f_j^e \, dx_j$	(b)
$dW'_m = \sum_{j=1}^{N} \lambda_j \, di_j + \sum_{j=1}^{M} f_j^e \, dx_j$	(c)	$dW'_e = \sum_{j=1}^{N} q_j \, dv_j + \sum_{j=1}^{M} f_j^e \, dx_j$	(d)
Forces of Electric Origin, $j = 1, \ldots, M$			
$f_j^e = -\dfrac{\partial W_m(\lambda_1, \ldots, \lambda_N; x_1, \ldots, x_M)}{\partial x_j}$	(e)	$f_j^e = -\dfrac{\partial W_e(q_1, \ldots, q_N; x_1, \ldots, x_M)}{\partial x_j}$	(f)
$f_j^e = \dfrac{\partial W'_m(i_1, \ldots, i_N; x_1, \ldots, x_M)}{\partial x_j}$	(g)	$f_j^e = \dfrac{\partial W'_e(v_1, \ldots, v_N; x_1, \ldots, x_M)}{\partial x_j}$	(h)
Relation of Energy to Coenergy			
$W_m + W'_m = \sum_{j=1}^{N} \lambda_j i_j$	(i)	$W_e + W'_e = \sum_{j=1}^{N} v_j q_j$	(j)
Energy and Coenergy from Electrical Terminal Relations			
$W_m = \sum_{j=1}^{N} \int_0^{\lambda_j} i_j(\lambda_1, \ldots, \lambda_{j-1}, \lambda'_j, 0, \ldots, 0; x_1, \ldots, x_M) \, d\lambda'_j$	(k)	$W_e = \sum_{j=1}^{N} \int_0^{q_j} v_j(q_1, \ldots, q_{j-1}, q'_j, 0, \ldots, 0; x_1, \ldots, x_M) \, dq'_j$	(l)
$W'_m = \sum_{j=1}^{N} \int_0^{i_j} \lambda_j(i_1, \ldots, i_{j-1}, i'_j, 0, \ldots, 0; x_1, \ldots, x_M) \, di'_j$	(m)	$W'_e = \sum_{j=1}^{N} \int_0^{v_j} q_j(v_1, \ldots, v_{j-1}, v'_j, 0, \ldots, 0; x_1, \ldots, x_M) \, dv'_j$	(n)

* The mechanical variables f_j and x_j can be regarded as the jth force and displacement or the jth torque T_j and angular displacement θ_j.

Table 6.1 Differential Equations, Transformations, and Boundary Conditions for Quasi-static Electromagnetic Systems with Moving Media

	Differential Equations		Transformations		Boundary Conditions	
Magnetic field systems	$\nabla \times \mathbf{H} = \mathbf{J}_f$	(1.1.1)	$\mathbf{H}' = \mathbf{H}$	(6.1.35)	$\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b) = \mathbf{K}_f$	(6.2.14)
	$\nabla \cdot \mathbf{B} = 0$	(1.1.2)	$\mathbf{B}' = \mathbf{B}$	(6.1.37)	$\mathbf{n} \cdot (\mathbf{B}^a - \mathbf{B}^b) = 0$	(6.2.7)
	$\nabla \cdot \mathbf{J}_f = 0$	(1.1.3)	$\mathbf{J}_f' = \mathbf{J}_f$	(6.1.36)	$\mathbf{n} \cdot (\mathbf{J}_f^{\,a} - \mathbf{J}_f^{\,b}) + \nabla_\Sigma \cdot \mathbf{K}_f = 0$	(6.2.9)
	$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	(1.1.5)	$\mathbf{E}' = \mathbf{E} + \mathbf{v}^r \times \mathbf{B}$	(6.1.38)	$\mathbf{n} \times (\mathbf{E}^a - \mathbf{E}^b) = v_n(\mathbf{B}^a - \mathbf{B}^b)$	(6.2.22)
	$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$	(1.1.4)	$\mathbf{M}' = \mathbf{M}$	(6.1.39)		
Electric field systems	$\nabla \times \mathbf{E} = 0$	(1.1.11)	$\mathbf{E}' = \mathbf{E}$	(6.1.54)	$\mathbf{n} \times (\mathbf{E}^a - \mathbf{E}^b) = 0$	(6.2.31)
	$\nabla \cdot \mathbf{D} = \rho_f$	(1.1.12)	$\mathbf{D}' = \mathbf{D}$	(6.1.55)	$\mathbf{n} \cdot (\mathbf{D}^a - \mathbf{D}^b) = \sigma_f$	(6.2.33)
			$\rho_f' = \rho_f$	(6.1.56)		
	$\nabla \cdot \mathbf{J}_f = -\dfrac{\partial \rho_f}{\partial t}$	(1.1.14)	$\mathbf{J}_f' = \mathbf{J}_f - \rho_f \mathbf{v}^r$	(6.1.58)	$\mathbf{n} \cdot (\mathbf{J}_f^{\,a} - \mathbf{J}_f^{\,b}) + \nabla_\Sigma \cdot \mathbf{K}_f = v_n(\rho_f^{\,a} - \rho_f^{\,b}) - \dfrac{\partial \sigma_f}{\partial t}$	(6.2.36)
	$\nabla \times \mathbf{H} = \mathbf{J}_f + \dfrac{\partial \mathbf{D}}{\partial t}$	(1.1.15)	$\mathbf{H}' = \mathbf{H} - \mathbf{v}^r \times \mathbf{D}$	(6.1.57)	$\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b) = \mathbf{K}_f + v_n \mathbf{n} \times [\mathbf{n} \times (\mathbf{D}^a - \mathbf{D}^b)]$	(6.2.38)
	$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$	(1.1.13)	$\mathbf{P}' = \mathbf{P}$	(6.1.59)		

From Chapter 8; The Stress Tensor and Related Tensor Concepts

In what follows we assume a right-hand cartesian coordinate system x_1, x_2, x_3. The component of a vector in the direction of an axis carries the subscript of that axis. When we write F_m we mean the mth component of the vector $\mathbf{F}$, where m can be 1, 2, or 3. When the index is repeated in a single term, it implies summation over the three values of the index

$$\frac{\partial H_n}{\partial x_n} = \frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial x_2} + \frac{\partial H_3}{\partial x_3} = \nabla \cdot \mathbf{H}$$

and

$$H_n \frac{\partial}{\partial x_n} = H_1 \frac{\partial}{\partial x_1} + H_2 \frac{\partial}{\partial x_2} + H_3 \frac{\partial}{\partial x_3} = \mathbf{H} \cdot \nabla.$$

This illustrates the *summation convention.* On the other hand, $\partial H_m/\partial x_n$ represents any one of the nine possible derivatives of components of $\mathbf{H}$ with respect to coordinates. We define the *Kronecker delta* δ_{mn} which has the values

$$\delta_{mn} = \begin{cases} 1, \text{ when } m = n, \\ 0, \text{ when } m \neq n. \end{cases} \tag{8.1.7}$$

The component T_{mn} of the stress tensor can be physically interpreted as the mth component of the traction (force per unit area) applied to a surface with a normal vector in the n-direction.

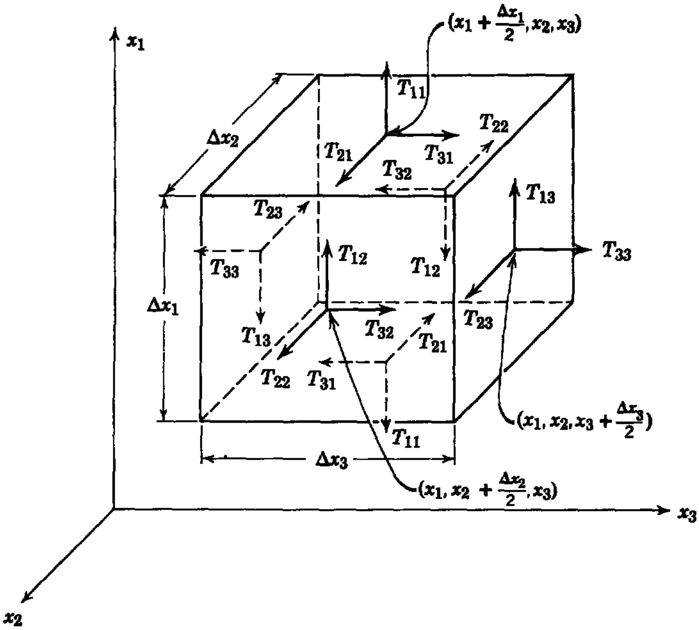


Fig. 8.2.2 Rectangular volume with center at (x_1, x_2, x_3) showing the surfaces and directions of the stresses T_{mn}.

The x_1-component of the total force applied to the material within the volume of Fig. 8.2.2 is

$$f_1 = T_{11}\left(x_1 + \frac{\Delta x_1}{2}, x_2, x_3\right)\Delta x_2 \Delta x_3 - T_{11}\left(x_1 - \frac{\Delta x_1}{2}, x_2, x_3\right)\Delta x_2 \Delta x_3$$
$$+ T_{12}\left(x_1, x_2 + \frac{\Delta x_2}{2}, x_3\right)\Delta x_1 \Delta x_3 - T_{12}\left(x_1, x_2 - \frac{\Delta x_2}{2}, x_3\right)\Delta x_1 \Delta x_3$$
$$+ T_{13}\left(x_1, x_2, x_3 + \frac{\Delta x_3}{2}\right)\Delta x_1 \Delta x_2 - T_{13}\left(x_1, x_2, x_3 - \frac{\Delta x_3}{2}\right)\Delta x_1 \Delta x_2. \tag{8.2.3}$$

Here we have evaluated the components of the stress tensor at the centers of the surfaces on which they act; for example, the stress component T_{11} acting on the top surface is evaluated at a point having the same x_2- and x_3-coordinates as the center of the volume but an x_1 coordinate $\Delta x_1/2$ above the center.

The dimensions of the volume have already been specified as quite small. In fact, we are interested in the limit as the dimensions go to zero. Consequently, each component of the stress tensor is expanded in a Taylor series about the value at the volume center with only linear terms in each series retained to write (8.2.3) as

$$f_1 = \left(T_{11} + \frac{\Delta x_1}{2}\frac{\partial T_{11}}{\partial x_1} - T_{11} + \frac{\Delta x_1}{2}\frac{\partial T_{11}}{\partial x_1}\right)\Delta x_2 \Delta x_3$$
$$+ \left(T_{12} + \frac{\Delta x_2}{2}\frac{\partial T_{12}}{\partial x_2} - T_{12} + \frac{\Delta x_2}{2}\frac{\partial T_{12}}{\partial x_2}\right)\Delta x_1 \Delta x_3$$
$$+ \left(T_{13} + \frac{\Delta x_3}{2}\frac{\partial T_{13}}{\partial x_3} - T_{13} + \frac{\Delta x_3}{2}\frac{\partial T_{13}}{\partial x_3}\right)\Delta x_1 \Delta x_2$$

or

$$f_1 = \left(\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3}\right)\Delta x_1 \Delta x_2 \Delta x_3. \tag{8.2.4}$$

All terms in this expression are to be evaluated at the center of the volume (x_1, x_2, x_3). We have thus verified our physical intuition that space-varying stress tensor components are necessary to obtain a net force.

From (8.2.4) we can obtain the x_1-component of the force density **F** at the point (x_1, x_2, x_3) by writing

$$F_1 = \lim_{\Delta x_1, \Delta x_2, \Delta x_3 \to 0} \frac{f_1}{\Delta x_1 \Delta x_2 \Delta x_3} = \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3}. \tag{8.2.5}$$

The limiting process makes the expansion of (8.2.4) exact. The summation convention is used to write (8.2.5) as

$$F_1 = \frac{\partial T_{1n}}{\partial x_n}. \tag{8.2.6}$$

A similar process for the other two components of the force and force density yields the general result that the *m*th component of the force density at a point is

$$F_m = \frac{\partial T_{mn}}{\partial x_n}. \tag{8.2.7}$$

Now suppose we wish to find the *m*th component of the total force **f** on material contained within the volume V. We can find it by performing the volume integration:

$$f_m = \int_V F_m \, dV = \int_V \frac{\partial T_{mn}}{\partial x_n} dV. \tag{8.1.13}$$

When we define the components of a vector **A** as

$$A_1 = T_{m1}, \qquad A_2 = T_{m2}, \qquad A_3 = T_{m3}, \tag{8.1.14}$$

we can write (8.1.13) as

$$f_m = \int_V \frac{\partial A_n}{\partial x_n} dV = \int_V (\nabla \cdot \mathbf{A}) \, dV. \tag{8.1.15}$$

We now use the divergence theorem to change the volume integral to a surface integral,

$$f_m = \oint_S \mathbf{A} \cdot \mathbf{n} \, da = \oint_S A_n n_n \, da, \tag{8.1.16}$$

where n_n is the *n*th component of the outward-directed unit vector **n** normal to the surface S and the surface S encloses the volume V. Substitution from (8.1.14) back into this expression yields

$$f_m = \oint_S T_{mn} n_n \, da. \tag{8.1.17}$$

where $T_{mn}n_n$ is the *m*th component of the surface traction $\boldsymbol{\tau}$.

The traction $\boldsymbol{\tau}$ is a vector. The components of this vector depend on the coordinate system in which $\boldsymbol{\tau}$ is expressed; for example, the vector might be directed in one of the coordinate directions (x_1, x_2, x_3), in which case there would be only one nonzero component of $\boldsymbol{\tau}$. In a second coordinate system (x_1', x_2', x_3'), this same vector might have components in all of the coordinate directions. Analyzing a vector into orthogonal components along the coordinate axes is a familiar process. The components in a cartesian coordinate system (x_1', x_2', x_3') are related to those in the cartesian coordinate system (x_1, x_2, x_3) by the three equations

$$\tau_p' = a_{pr}\tau_r, \tag{8.2.10}$$

where a_{pr} *is the cosine of the angle between the* x_p'*-axis and the* x_r*-axis.*

Similarly, the components of the stress tensor transform according to the equation

$$T'_{pq} = a_{pr}a_{qs}T_{rs}. \tag{8.2.17}$$

This relation provides the rule for finding the components of the stress in the primed coordinates, given the components in the unprimed coordinates. It serves the same purpose in dealing with tensors that (8.2.10) serves in dealing with vectors.

Equation 8.2.10 is the transformation of a vector $\boldsymbol{\tau}$ from an unprimed to a primed coordinate system. There is, in general, nothing to distinguish the two coordinate systems. We could just as well define a transformation from the primed to the unprimed coordinates by

$$\tau_s = b_{sp}\tau'_p, \tag{8.2.18}$$

where b_{sp} is the cosine of the angle between the x_s-axis and the x'_p-axis. But b_{sp}, from the definition following (8.2.10), is then also

$$b_{sp} \equiv a_{ps}; \tag{8.2.19}$$

that is, the transformation which reverses the transformation (8.2.10) is

$$\tau_s = a_{ps}\tau'_p. \tag{8.2.20}$$

Now we can establish an important property of the direction cosines a_{ps} by transforming the vector $\boldsymbol{\tau}$ to an arbitrary primed coordinate system and then transforming the components τ'_m back to the unprimed system in which they must be the same as those we started with. Equation 8.2.10 provides the first transformation, whereas (8.2.20) provides the second; that is, we substitute (8.2.10) into (8.2.20) to obtain

$$\tau_s = a_{ps}a_{pr}\tau_r. \tag{8.2.21}$$

Remember that we are required to sum on both p and r; for example, consider the case in which $s = 1$:

$$\begin{aligned} \tau_1 = {} & (a_{11}a_{11} + a_{21}a_{21} + a_{31}a_{31})\tau_1 \\ & + (a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32})\tau_2 \\ & + (a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33})\tau_3. \end{aligned} \tag{8.2.22}$$

This relation must hold in general. We have not specified either a_{ps} or τ_m. Hence the second two bracketed quantities must vanish and the first must be unity. We can express this fact much more concisely by stating that in general

$$a_{ps}a_{pr} = \delta_{sr} \tag{8.2.23}$$

Table 8.1 Electromagnetic Force Densities, Stress Tensors, and Surface Force Densities for Quasi-static Magnetic and Electric Field Systems*

Description	Force Density $\mathbf{F}$	Stress Tensor T_{mn} $F_m = \dfrac{\partial T_{mn}}{\partial x_n}$ (8.1.10)	Surface Force Density* $T_m = [T_{mn}]n_n$ (8.4.2)
Force on media carrying free current density $\mathbf{J}_f$, μ constant	$\mathbf{J}_f \times \mathbf{B}$ (8.1.3)	$T_{mn} = \mu H_m H_n - \delta_{mn}\frac{1}{2}\mu H_k H_k$ (8.1.11)	$\mathbf{T} = \mathbf{K}_f \times \mu\langle\mathbf{H}\rangle$ $\mathbf{K}_f = \mathbf{n} \times [\mathbf{H}]$ (8.4.3)
Force on media supporting free charge density ρ_f, ϵ constant	$\rho_f \mathbf{E}$ (8.3.3)	$T_{mn} = \epsilon E_m E_n - \delta_{mn}\frac{1}{2}\epsilon E_k E_k$ (8.3.10)	$\mathbf{T} = \sigma_f\langle\mathbf{E}\rangle$ $\sigma_f = \mathbf{n} \cdot [\epsilon \mathbf{E}]$ (8.4.8)
Force on free current plus magnetization force in which $\mathbf{B} = \mu\mathbf{H}$ both before and after media are deformed	$\mathbf{J}_f \times \mathbf{B} - \frac{1}{2}\mathbf{H}\cdot\mathbf{H}\,\nabla\mu + \frac{1}{2}\nabla\left(\mathbf{H}\cdot\mathbf{H}\rho\dfrac{\partial\mu}{\partial\rho}\right)$ (8.5.38)	$T_{mn} = \mu H_m H_n - \frac{1}{2}\delta_{mn}\left(\mu - \rho\dfrac{\partial\mu}{\partial\rho}\right)H_k H_k$ (8.5.41)	
Force on free charge plus polarization force in which $\mathbf{D} = \epsilon\mathbf{E}$ both before and after media are deformed	$\rho_f\mathbf{E} - \frac{1}{2}\mathbf{E}\cdot\mathbf{E}\,\nabla\epsilon + \frac{1}{2}\nabla\left(\mathbf{E}\cdot\mathbf{E}\rho\dfrac{\partial\epsilon}{\partial\rho}\right)$ (8.5.45)	$T_{mn} = \epsilon E_m E_n - \frac{1}{2}\delta_{mn}\left(\epsilon - \rho\dfrac{\partial\epsilon}{\partial\rho}\right)E_k E_k$ (8.5.46)	

$$* \langle\mathbf{A}\rangle \equiv \frac{\mathbf{A}^a + \mathbf{A}^b}{2}$$

$$[\mathbf{A}] \equiv \mathbf{A}^a - \mathbf{A}^b$$

Table 9.1 Modulus of Elasticity E and Density ρ for Representative Materials*

Material	E-units of 10^{11} N/m^2	ρ-units of 10^3 kg/m^3	v_p-units† of m/sec
Aluminum (pure and alloy)	0.68–0.79	2.66–2.89	5100
Brass (60–70% Cu, 40–30% Zn)	1.0–1.1	8.36–8.51	3500
Copper	1.17–1.24	8.95–8.98	3700
Iron, cast (2.7–3.6% C)	0.89–1.45	6.96–7.35	4000
Steel (carbon and low alloy)	1.93–2.20	7.73–7.87	5100
Stainless steel (18% Cr, 8% Ni)	1.93–2.06	7.65–7.93	5100
Titanium (pure and alloy)	1.06–1.14	4.52	4900
Glass	0.49–0.79	2.38–3.88	4500
Methyl methacrylate	0.024–0.034	1.16	1600
Polyethylene	1.38–3.8 × 10^{-3}	0.915	530
Rubber	0.79–4.1 × 10^{-5}	0.99–1.245	46

* See S. H. Crandall, and N. C. Dahl, *An Introduction to the Mechanics of Solids*, McGraw-Hill, New York, 1959, for a list of references for these constants and a list of these constants in English units.
† Computed from average values of E and ρ.

Table 9.2 Summary of One-Dimensional Mechanical Continua Introduced in Chapter 9

Thin Elastic Rod
$\rho \frac{\partial^2 \delta}{\partial t^2} = E \frac{\partial^2 \delta}{\partial x^2} + F_x$ $T = E \frac{\partial \delta}{\partial x}$ δ—longitudinal (x) displacement T—normal stress ρ—mass density E—modulus of elasticity F_x—longitudinal body force density
Wire or "String"
$m \frac{\partial^2 \xi}{\partial t^2} = f \frac{\partial^2 \xi}{\partial x^2} + S_z$ ξ—transverse displacement m—mass/unit length f—tension (constant force) S_z—transverse force/unit length
Membrane
$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S\left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2}\right) + T_z$ ξ—transverse displacement σ_m—surface mass density S—tension in y- and z-directions (constant force per unit length) T_z—z-directed force per unit area

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