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SOLUTIONS TO CHAPTER 9

9.1 MAGNETIZATION DENSITY

9.2 LAWS AND CONTINUITY CONDITIONS WITH MAGNETIZATION

9.2.1

$$\mathbf{M} = M_o \cos\beta x(\mathbf{i_x} + \mathbf{i_y})$$

The volume charge density

$$\rho_m = -\nabla \cdot \mu_o \mathbf{M} = \mu_o M_o \sin\beta x$$

$$\sigma_m = \mathbf{n} \cdot \mu_o(\mathbf{M}^a - \mathbf{M}^b)$$

and thus there is positive surface charge density on top

$$\sigma_m = \mu_o M_o \cos\beta x \qquad y = d$$

and a charge density of opposite sign at the bottom, $y = -d$.

9.2.2 (a) The magnetization is uniform, with the orientation shown in Fig. P9.2.1. Thus, it is solenoidal and the right hand side of (9.2.2) is zero and therefore equal to the left hand side, which is zero because $\mathbf{H} = 0$. Certainly a zero H field is irrotational, so Ampère's law is also satisfied. Associated with $\mathbf{M}$ inside is a magnetic surface charge density. However, this is cancelled by a surface charge density of opposite sign induced in the infinitely permeable wall so as to prevent there being an $\mathbf{H}$ outside the cylinder.

(b) In view of the direction defined as positive for the wire, the flux linked by the coil is

$$\lambda = \mathbf{B} \cdot \mathbf{i_y} 2Rd = \mu_o M_y 2Rd = \mu_o 2RdM_o \cos\gamma \tag{1}$$

Thus, with the terminus of the right wire defined as the + terminal and $\gamma = \Omega t$, the voltage is

$$v = \frac{d\lambda}{dt} = -\mu_o 2RdM_o\Omega \sin\Omega t \tag{2}$$

9.2.3 (a) From Ampère's law

$$\oint_C \mathbf{H} \cdot d\mathbf{s} = \int_S \mathbf{J} \cdot d\mathbf{a}$$

we find

$$\oint \mathbf{H} \cdot d\mathbf{s} = 0$$

because there is no $\mathbf{J}$ present. This means that $\mathbf{H} = -\nabla\Psi$ and Ψ is a scalar potential that satisfies Laplace's equations, since $\mathbf{H}$ is divergence-free. The only possible solution to this problem, subject to $\Psi = \text{const}$ at $y = 0$ and $y = a$, is $\Psi = \text{const}$; and hence $\mathbf{H} = 0$.

(b) Since

$$\mathbf{B} = \mu_o(\mathbf{H} + \mathbf{M}) \tag{1}$$

we have

$$\mathbf{B} = \mathbf{i_y}\mu_o M_o \cos\beta(x - Ut) \tag{2}$$

The flux linked by the turn is

$$\begin{aligned}
\lambda &= \mu_o l \int_{x=-d}^{x=d} M_o \cos\beta(x - Ut)dx \\
&= \mu_o l d M_o \left\{ \frac{\sin(\beta d - \beta U t)}{\beta d} + \frac{\sin(\beta d + \beta U t)}{\beta d} \right\} \\
&= \mu_o l d M_o \left\{ \frac{\sin\beta d \cos\beta U t - \cos\beta d \sin\beta U t}{\beta d} \right. \\
&\quad \left. + \frac{\sin\beta d \cos\beta U t + \cos\beta d \sin\beta U t}{\beta d} \right\} \\
&= 2\mu_o l d M_o \frac{\sin\beta d}{\beta d} \cos\beta U t
\end{aligned}$$

The voltage is

$$v = \frac{d\lambda}{dt} = -2\beta U \mu_o l d M_o \frac{\sin\beta d}{\beta d} \sin\beta U t$$

9.3 PERMANENT MAGNETIZATION

9.3.1 The given answer is the result of using (4.5.24) twice. First, the result is written with the identification of variables

$$\frac{\sigma_o}{\epsilon_o} \to \frac{\mu_o M_o}{\mu_o}; \quad x_1 = a, x_2 = -a, y \to y - b \tag{1}$$

representing the upper magnetic surface charge. Second, representing the potential of the lower magnetic surface charge,

$$\frac{\sigma_o}{\mu_o} \to -M_o; \quad x_1 = a, x_2 = -a, y \to y + b \tag{2}$$

The sum of these two results is the given answer.

9.3.2 In the upper half-space, where there is the given magnetization density, the magnetic charge density is

$$\rho_m = -\nabla \cdot \mu_o \mathbf{M} = \mu_o M_o \alpha \cos \beta x e^{-\alpha y} \tag{1}$$

while at the interface there is the surface magnetic charge density

$$\sigma_m = -\mu_o M_x(y = 0) = -\mu_o M_o \cos \beta x \tag{2}$$

In the upper region, a particular solution is needed to balance the source term, (1) introduced into the magnetic potential Poisson's equation

$$\nabla^2 \Psi_p = -M_o \alpha \cos \beta x e^{-\alpha y} \tag{3}$$

given the constant coefficient nature of the Laplacian on the left, it is natural to look for a product solution having the same x and y dependence as what is on the right. Thus, if

$$\Psi_p = F \cos \beta x e^{-\alpha y} \tag{4}$$

then (3) requires that

$$F[-\beta^2 + \alpha^2] = -M_o \alpha \Rightarrow F = M_o \alpha / (\beta^2 - \alpha^2) \tag{5}$$

Thus, to satisfy the boundary conditions at $y = 0$

$$\Psi^a = \Psi^b; \quad -\mu_o \frac{\partial \Psi^a}{\partial y} + \mu_o \frac{\partial \Psi^b}{\partial y} = -\mu_o M_o \cos \beta x \tag{6}$$

we take the solution in the upper region to be a superposition of (5) and a suitable solution to Laplace's equation that goes to zero at $y \to \infty$ and has the same x dependence.

$$\Psi^a = \left[A e^{-\beta y} + \frac{M_o \alpha}{(\beta^2 - \alpha^2)} e^{-\alpha y}\right] \cos \beta x \tag{7}$$

Similarly, in the lower region where there is no source,

$$\Psi^b = C e^{\beta y} \cos \beta x \tag{8}$$

Substitution of these solutions into the two boundary conditions of (6) gives

$$A = \frac{M_o}{2(\alpha - \beta)} \tag{9}$$

$$C = -\frac{M_o}{2(\alpha + \beta)} \tag{10}$$

and hence the given solution.

9.3.3 We have

$$\nabla \cdot \mu_o \mathbf{H} = -\mu_o \nabla^2 \Psi = -\nabla \cdot \mu_o \mathbf{M} = -\mu_o \beta M_o \cos \beta x \exp \alpha y$$

This is Poisson's equation for Ψ with the particular solution:

$$\Psi_p = \frac{\beta M_o}{\alpha^2 - \beta^2} \cos \beta x \exp \alpha y$$

The homogeneous solution has to take care of the fact that at $y = 0$ the magnetic charge density stops. We have the following solutions of Laplace's equation

$$\Psi_h = \begin{cases} A \cos \beta x e^{-\beta y} & y > 0 \\ B \cos \beta x e^{\beta y} & y < 0 \end{cases}$$

There is no magnetic surface charge density. At the boundary, Ψ and $\partial \Psi / \partial y$ must be continuous

$$-\frac{\beta M_o}{\alpha^2 - \beta^2} + B = A$$

and

$$\frac{\alpha \beta M_o}{\alpha^2 - \beta^2} + \beta B = -\beta A$$

Solving, we find

$$B = -\frac{M_o}{2(\alpha - \beta)} \left(1 + \frac{\alpha}{\beta}\right)$$

and

$$A = -\frac{M_o}{2(\alpha + \beta)} \left(1 - \frac{\alpha}{\beta}\right)$$

9.3.4 The magnetic volume charge density is

$$\begin{aligned} \rho_m &= -\nabla \cdot \mu_o \mathbf{M} = -\mu_o \frac{1}{r} \frac{\partial}{\partial r}(r M_r) - \mu_o \frac{1}{r} \frac{\partial}{\partial \phi} M_\phi \\ &= -\mu_o \frac{M_o}{r} p(r/R)^{p-1} \cos p(\phi - \gamma) + \mu_o \frac{M_o}{r} p(r/R)^{p-1} \cos p(\phi - \gamma) \\ &= 0 \end{aligned}$$

There is no magnetic volume charge density. All the charge density is on the surface

$$\sigma_m = \mu_o M_r \big|_{r=R} = \mu_o M_o \cos p(\phi - \gamma)$$

This magnetic surface charge density produces $\mu_o \mathbf{H}$ just like σ_s produces $\epsilon_o \mathbf{E}$ (EQS). We set

$$\Psi = \begin{cases} A(R/r)^p \cos p(\phi - \gamma) & r > R \\ B(r/R)^p \cos p(\phi - \gamma) & r < R \end{cases}$$

Because there is no current present, Ψ is continuous at $r = R$ and thus

$$A = B$$

On the surface

$$-\mu_o \frac{\partial \Psi}{\partial r}\Big|_{r=R_+} + \mu_o \frac{\partial \Psi}{\partial r}\Big|_{r=R_-} = \sigma_m = \mu_o M_o \cos p(\phi - \gamma)$$

We find

$$2p\frac{A}{R} = M_o \qquad A = \frac{R}{2p} M_o$$

(b) The radial field at $r = d + R$ is

$$\mu_o H_r(r = d + R) = \mu_o \frac{M_o}{2} \cos p(\phi - \gamma)\left(\frac{R}{R+d}\right)^{p+1}$$

The flux linkage is

$$\lambda = \mu_o N^2 H_r al = \frac{\mu_o N^2 M_o}{2} al\left(\frac{R}{R+d}\right)^{p+1} \cos p\left(\frac{\pi}{2} - \Omega t\right)$$

The voltage is

$$\frac{d\lambda}{dt} = \frac{p\Omega \mu_o N^2 M_o al}{2}\left(\frac{R}{R+d}\right)^{p+1} \cos p\Omega t$$

(c) If p is high, then

$$\left(\frac{R}{R+d}\right)^{p+1} \ll 1$$

unless d is made very small.

9.4 MAGNETIZATION CONSTITUTIVE LAWS

9.4.1 (a) With the understanding that **B** and **H** are collinear, the magnitude of **B** is related to that of **H** by the constitutive law

$$B = \mu_o[H + M_o \tanh(\alpha H)] \tag{1}$$

For small argument, the tanh function is approximately its argument. Thus, like the saturation law of Fig. 9.4.4, in the neighborhood of the origin, for $\alpha H \ll 1$, the curve is a straight line with slope $\mu_o(1 + \alpha M_o)$. In the range of $\alpha H \approx 1$ the curve makes a transition to a lesser slope μ_o.

(b) It follows from (9.4.1) and (1) that

$$B = \mu_o\left[\frac{N_1 i}{2\pi R} + M_o \tanh\left(\frac{\alpha N_1 i}{2\pi R}\right)\right] \tag{2}$$

and in turn from (9.4.2) that

$$\lambda_2 = \frac{\pi w^2 N_2 \mu_o}{4}\left[\frac{N_1 i}{2\pi R} + M_o \tanh\left(\frac{\alpha N_1 i}{2\pi R}\right)\right] \tag{3}$$

Thus, the voltage is $v = d\lambda_2/dt$, the given expression.

9.4.2 The flux linkage is according to (9.4.2)

$$\lambda_2 = \frac{\pi w^2}{4} N_2 B \tag{1}$$

The field intensity is according to (9.4.1)

$$H_\phi = \frac{N_1 i}{2\pi R}$$

Therefore

$$\frac{d\lambda_2}{dt} = \frac{\pi w^2}{4} N_2 \frac{dB}{dt}$$

where we need the dispersion diagram to relate H_ϕ (i.e. i) to B (see Fig. S9.4.2).

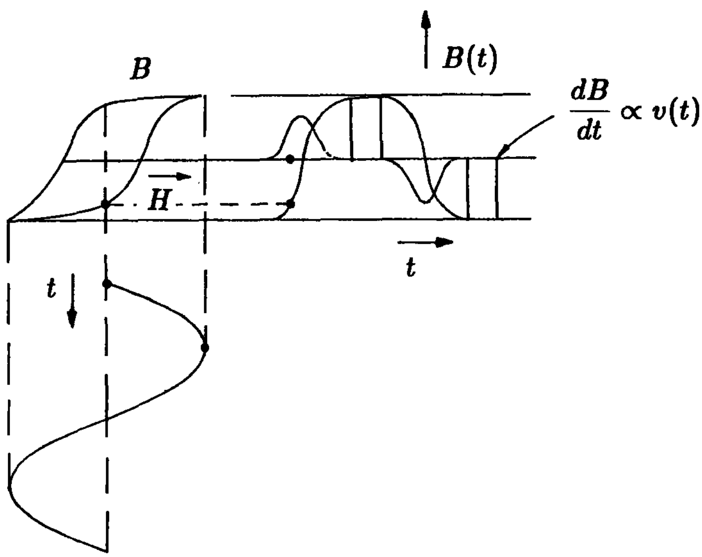

Figure S9.4.2

9.5 FIELDS IN THE PRESENCE OF MAGNETICALLY LINEAR INSULATING MATERIALS

9.5.1 The postulated uniform H field satisfies (9.5.1) and (9.5.2) everywhere inside the regions of uniform permeability. It also satisfies the continuity conditions, (9.5.3) and (9.5.4). Finally, with no H outside the conductors, (9.5.3) is satisfied. The only way in which the permeable materials can alter the uniform field that exists in their absence is by having a component collinear with the permeability gradient. As shown by (9.5.21), only then is there induced the magnetic charge necessary to altering the distribution of H. Here, such a component would be perpendicular to the interface between permeable materials, where it would produce a surface magnetic charge in accordance with (9.5.22). Because H is simply i/w throughout, the total flux linking the one turn circuit is simply

$$\lambda = \int_{A_a} \mu_a H da + \int_{A_b} \mu_b H da = (\mu_a A_a + \mu_b A_b)H = (\mu_a A_a + \mu_b A_b)i/w \qquad (1)$$

and hence, because $\lambda = Li$, the inductance is as given.

9.5.2 From Ampère's law applied to a circular contour around the inner cylinder, anywhere within the region $b < r < a$, one finds

$$H_\phi = \frac{i}{2\pi r}$$

where $\mathbf{i}_\phi$ points in the clock-wise direction, and z along the axis of the cylinder. The flux densities are

$$B_\phi = \frac{\mu_a i}{2\pi r} \quad \text{and} \quad B_\phi = \frac{\mu_b i}{2\pi r}$$

in the two media. The flux linkage is

$$\begin{aligned}\lambda &= l\left\{\int_b^R \frac{\mu_b i}{2\pi r} dr + \int_R^a \frac{\mu_a i}{2\pi r} dr\right\} \\ &= \frac{l}{2\pi}[\mu_b ln(R/b) + \mu_a ln(a/R)]i\end{aligned}$$

The inductance is

$$L = \frac{\lambda}{i} = \frac{l}{2\pi}[\mu_b ln(R/b) + \mu_a ln(a/R)]$$

9.5.3 For the reasons given in the solution to Prob. 9.5.1, the H field is simply $(i/w)\mathbf{i}_z$. Thus, the magnetic flux density is

$$B = \mu H = -\left(\frac{\mu_m x}{l}\right)\frac{i}{w} \qquad (1)$$

and the total flux linked by the one turn is

$$\lambda = \int_S B_z dy dx = d\int_{-l}^0 \left(\frac{-\mu_m x}{l}\right)\frac{i}{w} dx = \frac{\mu_m l d}{2w} i \qquad (2)$$

By definition, $\lambda = Li$, so it follows that L is as given.

9.5.4 The magnetic field does not change from that of Prob. 9.5.2. The flux linkage is

$$\lambda = l\int_b^a \mu_m(r/b)\frac{i}{2\pi r}dr = \mu_m l\left(\frac{a-b}{b}\right)i$$

The inductance is

$$L = \mu_m l\frac{a-b}{b}$$

9.5.5 (a) The postulated fields have the r dependence of the H produced by a line current i on the z axis, as can be seen using Ampère's integral law (Fig. 1.4.4). Direct substitution into (9.5.1) and (9.5.2) written in polar coordinates also shows that fields in this form satisfy Ampère's law and the continuity condition everywhere in the regions of uniform permeability.

(b) Using the postulated fields, (9.5.4) requires that

$$\frac{\mu_a A}{r} = \frac{\mu_b C}{r} \Rightarrow C = \frac{\mu_a}{\mu_b}A \tag{1}$$

(c) For a contour that encloses the interior conductor, which carries the total current i, Ampère's integral law requires that ($\beta \equiv 2\pi - \alpha$)

$$\oint_C H_\phi r dr = i = \alpha r\frac{A}{r} + \beta r\frac{C}{r} = \alpha A + \beta C \tag{2}$$

Thus, from (1),

$$i = A\left(\alpha + \beta\frac{\mu_a}{\mu_b}\right) \Rightarrow A = \frac{i}{\alpha + \beta\frac{\mu_a}{\mu_b}};$$

$$C = \frac{(\mu_a/\mu_b)i}{\alpha + \beta\frac{\mu_a}{\mu_b}} \tag{3}$$

(d) The inductance follows by integrating the flux density over the gap. Note that the same answer must be obtained from integrating over the gap region occupied by either of the permeable materials. Integration over a surface in region a gives

$$\lambda = l\int_b^a \frac{\mu_a A}{r}dr = l\mu_a A ln(a/b) = \frac{l\mu_a ln(a/b)i}{\alpha + (2\pi - \alpha)(\mu_a/\mu_b)} \tag{4}$$

Because $\lambda = Li$, it follows that the inductance of the shorted coaxial section is as given.

(e) Since the field inside the volume of the inner conductor is zero, it follows from Ampère's continuity condition, (9.5.3), that

$$K_z = H_\phi \Rightarrow K_z = \begin{cases} A/b = i/b\left[\alpha + \beta\frac{\mu_a}{\mu_b}\right]; & \text{region (a)} \\ C/b = i(\mu_a/\mu_b)/b\left(\alpha + \beta\frac{\mu_a}{\mu_b}\right); & \text{region (b)} \end{cases} \tag{5}$$

Note that these surface current densities are not equal, but are consistent with having the total current in the inner conductor equal to i.

$$i = \frac{A}{b}(\alpha b) + \frac{C}{b}\beta b = \frac{i\alpha}{\alpha + \beta(\frac{\mu_a}{\mu_b})} + \frac{i(\mu_a/\mu_b)\beta}{\alpha + \beta\frac{\mu_a}{\mu_b}} \tag{6}$$

9.5.6 The H-field changes as one proceeds from medium μ_a to the medium μ_b. For the contour shown, Ampère's law gives (see Fig. S9.5.6):

$$H_z^a a + H_z^b (w - a) = i$$

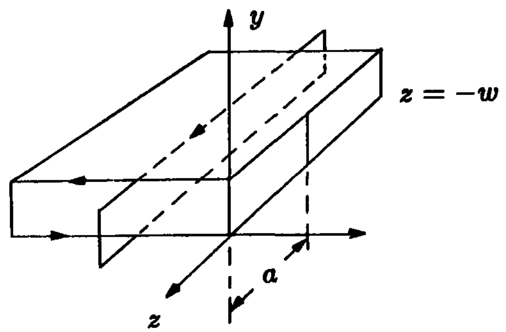

Figure S9.5.6

The flux continuity gives

$$\mu_a H_z^a = \mu_b H_z^b$$

Therefore

$$H_z^a \left[a + \frac{\mu_a}{\mu_b}(w - a)\right] = i$$

and the flux linkage is

$$\lambda = dl\mu_a H_z^a = \frac{\mu_a dli}{a + \frac{\mu_a}{\mu_b}(w - a)}$$

and the inductance is

$$L = \frac{\lambda}{i} = \frac{dl}{\frac{a}{\mu_a} + \frac{w-a}{\mu_b}}$$

9.6 FIELDS IN PIECE-WISE UNIFORM MAGNETICALLY LINEAR MATERIALS

9.6.1 (a) At the interface, Ampère's law and flux continuity require the boundary conditions

$$H_z^a - H_z^b = -\frac{\partial \Psi^a}{\partial z} + \frac{\partial \Psi^b}{\partial z} = K_o \cos \beta z \tag{1}$$

$$\mu_o H_y^a - \mu H_y^b = -\mu_o \frac{\partial \Psi^a}{\partial y} + \mu \frac{\partial \Psi^b}{\partial y} = 0 \tag{2}$$

The z dependence of the surface current density in (1) suggests that the magnetic potential be taken as the solutions to Laplace's equation

$$\Psi = \begin{cases} Ae^{-\beta y} \sin \beta z \\ Ce^{\beta y} \sin \beta z \end{cases} \tag{3}$$

Substitution of these relations into (1) and (2) gives

$$\begin{bmatrix} -\beta & \beta \\ \mu_o \beta & \mu \beta \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} K_o \\ 0 \end{bmatrix} \tag{4}$$

and hence

$$A = -\frac{\mu}{\mu_o} \frac{K_o}{\beta \left[1 + \frac{\mu}{\mu_o}\right]}; \quad C = -\frac{\mu_o}{\mu} A \tag{5}$$

Thus, the magnetic potential is as given.

(b) In the limit where the lower region is infinitely permeable, the boundary condition at $y = 0$ for the upper region becomes

$$H_z^a(y = 0) = -\frac{\partial \Psi^a}{\partial z}(y = 0) = K_o \cos \beta z \tag{6}$$

This suggests a solution in the form of (3a). Substitution gives

$$A = -K_o/\beta \tag{7}$$

which is the same as the limit $\mu/\mu_o \to \infty$ of (5a).

(c) Given the solution in the upper region, flux continuity determines the field in the lower region. In the lower region, the condition at $y = 0$ is

$$\frac{\partial \Psi^b}{\partial y}(y = 0) = \frac{\mu_o}{\mu} \frac{\partial \Psi^a}{\partial y}(y = 0) = \frac{\mu_o}{\mu} K_o \sin \beta z \tag{8}$$

and it follows that

$$\beta C \sin \beta z = \frac{\mu_o}{\mu} K_o \sin \beta z \Rightarrow C = \frac{\mu_o}{\mu} K_o/\beta \tag{9}$$

which agrees with (5) in the limit where $\mu/\mu_o \gg 1$.

9.6.2 (a) The H-field is the gradient of a Laplacian potential to the left and right of the current sheet. Because $\mathbf{n}\times\mathbf{H}=0$ at $y=\pm d$, $\Psi=$ const.

(b) At the sheet

$$\mathbf{n}\times(\mathbf{H}^a-\mathbf{H}^b)=\mathbf{K} \tag{1}$$

and thus

$$-\frac{\partial\Psi^a}{\partial y}+\frac{\partial\Psi^b}{\partial y}=K_o\sin\Big(\frac{\pi y}{2d}\Big) \tag{2}$$

From flux density continuity we obtain

$$\mu_o\frac{\partial\Psi^a}{\partial x}=\mu_o\frac{\partial\Psi^b}{\partial x} \tag{3}$$

From (2) we see that Ψ^a and $\Psi^b\propto\cos(\pi y/2d)$ and thus

$$\Psi^a=A\cos\Big(\frac{\pi y}{2d}\Big)e^{-\pi x/2d} \tag{4a}$$

$$\Psi^b=B\cos\Big(\frac{\pi y}{2d}\Big)e^{\pi x/2d} \tag{4b}$$

This satisfies $\Psi=$ const at $y=\pm d$. We have from (3)

$$-\frac{\pi}{2d}A=\frac{\pi}{2d}B$$

and from (2)

$$\frac{\pi}{2d}A-\frac{\pi}{2d}B=K_o$$

giving

$$A=-B=\frac{K_o}{(\pi/d)}$$

Therefore

$$\Psi^a_b=\pm\frac{K_o}{(\pi/d)}\cos\Big(\frac{\pi y}{2d}\Big)e^{\mp\pi x/2d}$$

9.6.3 (a) Boundary conditions at $r=R$ are

$$H^a_\phi-H^b_\phi=-\frac{1}{R}\frac{\partial\Psi^a}{\partial\phi}+\frac{1}{R}\frac{\partial\Psi^b}{\partial\phi}=\frac{Ni}{2R}\sin\phi \tag{1}$$

$$B^a_r-B^b_r=-\mu\frac{\partial\Psi^a}{\partial r}+\mu_o\frac{\partial\Psi^b}{\partial r}=0 \tag{2}$$

To satisfy these, it is appropriate to choose as solutions to Laplace's equation outside and inside the winding

$$\Psi=\begin{cases}(A/r)\cos\phi; & R<r\\ Cr\cos\phi; & r<R\end{cases} \tag{3}$$

Substitution of these relations in (1) and (2) shows that the coefficients are

$$A = \frac{NiR}{2[1+(\mu/\mu_o)]}; \quad C = -\frac{\mu}{\mu_o}\frac{A}{R^2} \tag{4}$$

and substitution of these into (3) results in the given expressions for the magnetic potential.

(b) The magnetic field intensity inside is uniform and x directed. Thus, the integration over the area of the loop amounts to a multiplication by the area. The component normal to the loop is $H_x \cos\alpha, H_x = -C$. Therefore,

$$\lambda = n\mu_o H_x \cos\alpha(2al) = -n\mu_o C\cos\alpha(2al) \tag{5}$$

With no current in the rotating loop, the flux linkage-current relation reduces to $\lambda = L_m i$, so the desired mutual inductance multiplies i in (5).

9.6.4 (a) It is best to find the H-field first, then determine the vector potential. The vector potential can then be used to find the flux according to 8.6.5. Look at stator field first ($r = a$). The scalar potential of the stator that vanishes at $r = b$ is

$$\Psi^s = A\cos\phi\left(\frac{r}{b} - \frac{b}{r}\right) \tag{1}$$

On surface of stator

$$\mathbf{n}\times\mathbf{H}^s = \mathbf{K} \tag{2}$$

where $\mathbf{n} = -\mathbf{i}_r$.

$$\mathbf{K} = \mathbf{i}_z i_1 N_s \sin\phi \tag{3}$$

where the stator wire density N_s is

$$N_s = \frac{N_1}{2a}$$

with N_1 the total number of turns. Since

$$\mathbf{n}\times\mathbf{H}^s = \frac{1}{r}\frac{\partial\Psi}{\partial\phi}\Big|_{r=a}\mathbf{i}_z = -\frac{1}{a}A\sin\phi\left(\frac{a}{b} - \frac{b}{a}\right)\mathbf{i}_z$$

We find

$$A = -\frac{N_1}{2}i_1\frac{ab}{a^2-b^2} \tag{5}$$

The $\mathbf{H}$ field due to stator windings is:

$$\mathbf{H}^s = \frac{N_1 i_1}{2}\frac{a}{a^2-b^2}\left[\left(1+\frac{b^2}{r^2}\right)\cos\phi\mathbf{i}_r - \left(1-\frac{b^2}{r^2}\right)\sin\phi\mathbf{i}_\phi\right] \tag{6}$$

The rotor potential is

$$\Psi^r = B\cos(\phi-\theta)\left(\frac{r}{a} - \frac{a}{r}\right) \tag{7}$$

We find similarly,

$$B = \frac{N_2}{2} i_2 \frac{ab}{b^2 - a^2} \tag{8}$$

The H-field is

$$\mathbf{H}^r = \frac{N_2 i_2}{2} \frac{b}{a^2 - b^2} \left[\left(1 + \frac{a^2}{r^2}\right) \cos(\phi - \theta) \mathbf{i_r} - \left(1 - \frac{a^2}{r^2}\right) \sin(\phi - \theta) \mathbf{i}_\phi \right] \tag{9}$$

Fluxes linking the windings can be obtained by evaluating $\int_S \mathbf{B} \cdot d\mathbf{a}$ or by use of the vector potential A_z. Here we use A_z. The vector potential is z-directed and is related to the $\mathbf{B}$ field by

$$\nabla \times \mathbf{A} = \mathbf{B} = \mu_o \mathbf{H} = \frac{1}{r} \frac{\partial A_z}{\partial \phi} \mathbf{i_r} - \frac{\partial A_z}{\partial r} \mathbf{i}_\phi \tag{10}$$

From the r-components of $\mathbf{H}$ we find by inspection

$$\begin{aligned} A_z = {} & \mu_o \frac{N_1 i_1}{2} \frac{ab}{a^2 - b^2} \left(\frac{r}{b} + \frac{b}{r}\right) \sin \phi \\ & + \mu_o \frac{N_2 i_2}{2} \frac{ba}{a^2 - b^2} \left(\frac{r}{a} + \frac{a}{r}\right) \sin(\phi - \theta) \end{aligned} \tag{11}$$

Of course, the ϕ component gives the same result.

(b) The inductances follow from evaluation of the flux linkages. The flux of one stator turn, extending from $\phi = -\phi_o$ to $\phi = \pi - \phi_o$ is

$$\begin{aligned} \Phi_\lambda^{ss}(\phi_o) &= l[A_z^s(\pi - \phi_o) - A_z^s(-\phi_o)]_{r=a} \\ &= \mu_o l \frac{N_1 i_1}{2} \frac{ab}{a^2 - b^2} \left(\frac{a}{b} + \frac{b}{a}\right) 2 \sin \phi_o \end{aligned} \tag{12}$$

The inductance is obtained by computing the flux linkage

$$\lambda_{11} = \int_{\phi_o = 0}^{\pi} \frac{N_1 \Phi_\lambda^{ss}(\phi_o)}{2a} a d\phi_o = \mu_o l N_1^2 i_1 \frac{a^2 + b^2}{a^2 - b^2} \tag{13}$$

The inductance is

$$L_{11} = \frac{\lambda_{11}}{i_1} = \mu_o l N_1^2 \frac{a^2 + b^2}{a^2 - b^2} \tag{14}$$

In a similar way we find

$$L_{22} = \frac{\lambda_{22}}{i_2} = \mu_o l N_2^2 \frac{a^2 + b^2}{a^2 - b^2} \tag{15}$$

The mutual inductance is evaluated from Φ_λ^{rs}, the flux due to the field produced by the stator, passing a turn of the rotor extending from $-\phi_o + \theta$ to $\pi - \phi_o + \theta$

$$\begin{aligned} \Phi_\lambda^{rs} &= l[A_z^s(\pi - \phi_o + \theta) - A_z^s(-\phi_o + \theta)]_{r=b} \\ &= \mu_o l N_1 i_1 \frac{2ab}{a^2 - b^2} \sin(\phi_o - \theta) \end{aligned} \tag{16}$$

The mutual flux linkage is

$$\lambda_{21} = \int_{\phi_o=0}^{\pi} \frac{N_2}{2b}\Phi_\lambda^{rs} b\,d\phi_s = \mu_o l N_1 N_2 i_1 \frac{2ab}{a^2-b^2}\cos\theta \tag{17}$$

$$L_{21} = \mu_o l N_1 N_2 \frac{2ab}{a^2-b^2}\cos\theta$$

A similar analysis gives L_{12} which is found equal to L_{21}. From energy arguments presented in Chap. 11, it can be proven that $L_{12} = L_{21}$ is a necessity. Note that

$$L_{21}^2 = L_{12}L_{21} \le L_{11}L_{22}$$

9.6.5 (a) The vector potential of the wire carrying a current I is

$$A_z = -\frac{\mu_o I}{2\pi} ln(\frac{r_1}{a}) \tag{1}$$

where

$$r_1 = \sqrt{(y-h)^2 + x^2}$$

and a is a reference radius. If we mount an image of magnitude i_b at the position $x = 0, y = -h$, we have

$$A_z = -\frac{\mu_o i_b}{2\pi} ln(\frac{r_2}{a}) \tag{2}$$

where

$$r_2 = \sqrt{(y+h)^2 + x^2}$$

The field in the μ-material is represented by the vector potential

$$A_z = -\frac{\mu_o i_a}{2\pi} ln(\frac{r_1}{a}) \tag{3}$$

where i_a is to be determined. We find for the $\mathbf{B} = \mu\mathbf{H}$ field

$$\begin{aligned}\mu_o\mathbf{H} = \nabla\times\mathbf{A} &= \mathbf{i_x}\frac{\partial A_z}{\partial y} - \mathbf{i_y}\frac{\partial A_z}{\partial x}\\ &= -\frac{\mu_o}{2\pi}\Big\{\mathbf{i_x}\Big(I\frac{y-h}{\sqrt{(y-h)^2+x^2}^3} + i_b\frac{y+h}{\sqrt{(y+h)^2+x^2}^3}\Big)\\ &\quad - \mathbf{i_y}\Big(I\frac{x}{\sqrt{(y-h)^2+x^2}^3} + i_b\frac{x}{\sqrt{(y+h)^2+x^2}^3}\Big); \qquad y>0\end{aligned} \tag{4a}$$

$$\mu\mathbf{H} = -\frac{\mu_o i_a}{2\pi}\frac{1}{\sqrt{(y-h)^2+x^2}^3}\{\mathbf{i_x}(y-h) - \mathbf{i_y}x\}; \qquad y<0 \tag{4b}$$

At $y = 0$ we match H_x and μH_y obtaining

$$I - i_b = \frac{\mu_o i_a}{\mu} \tag{5}$$

$$I + i_b = i_a \tag{6}$$

By adding the two equations we obtain:

$$i_a = \frac{2I}{1 + \frac{\mu_o}{\mu}} \tag{7}$$

and thus

$$i_b = \frac{1 - \frac{\mu_o}{\mu}}{1 + \frac{\mu_o}{\mu}} \tag{8}$$

(b) When $\mu \gg \mu_o$, then $\mathbf{H}_{\tan} \simeq 0$ on the interface. We need an image that cancels the tangential magnetic field, i.e.

$$i_b = I$$

(c) We have a normal flux as found in (4a) for $i_b = I$

$$\mu_o H_y = \frac{\mu_o}{2\pi} 2I \frac{x}{\sqrt{h^2 + x^2}^3}$$

This normal flux must be continuous. It can be produced by a fictitious source at $y = h$ of magnitude $i_a = 2I$. The field is (compare (4b))

$$H = -\frac{\mu_o}{\mu}\frac{I}{\pi}\frac{1}{\sqrt{(y-h)^2 + x^2}^3}\{\mathbf{i_x}(y-h) - \mathbf{i_y} x\}$$

(d) When $\mu \gg \mu_o$, we find from (2) and (8)

$$i_a \simeq 2I$$

$$i_b \simeq I$$

in concordance with the above!

9.6.6 The field in the upper region can be taken as the sum of the field due to the wire, a particular solution, and the field of an image current at the position $y = -h, x = 0$, a homogeneous solution. The polarity of this latter current is determined by which of the two physical situations is of interest.

(a) If the material is perfectly conducting, there is no flux density normal to its surface in the upper region. In this case, the image current must be in the $-z$ direction so that its y directed field is in the opposite direction to that of the actual current in the plane $y = 0$. The field at $y = h, x = 0$ due to this image current is

$$\mu_o \mathbf{H} = \frac{\mu_o i}{(2\pi)(2h)} \mathbf{i}_x \tag{1}$$

and therefore the force per unit length is as given. The wire is repelled by a perfectly conducting wall.

(b) In this case, there is no tangential magnetic field intensity at the interface, so the image current is in the same direction as the actual current. As a result, the field intensity of the image current, evaluated at the position of the actual current, is the negative of that given by (1). The resulting force is also the negative of that for the perfect conductor, as given. The wire is attracted by a permeable wall.

9.6.7 (a) In this version of an "inside-outside" problem, the "inside" region is the highly permeable one. The field intensity must be $H_o \mathbf{i}_z$ in that region and have no tangential component in the plane $z = 0$. The latter condition is satisfied by taking the configuration as being that of a spherical cavity centered at the origin with the surrounding highly permeable material extending to infinity in the $\pm z$ directions. At the surface where $r = a$, the normal flux density in the highly permeable material tends to be zero. Thus, the approximate field takes the form

$$\Psi^a = -H_o r \cos\theta + A \frac{\cos\theta}{r^2} \tag{1}$$

where the coefficient A is adjusted to make

$$\mathbf{n} \cdot \mathbf{B}\big|_{r=a} = 0 \Rightarrow \frac{\partial \Psi^a}{\partial r}(r = a) = 0 \tag{2}$$

Substitution of (1) into (2) gives $A = -a^3 H_o/2$ and hence the given magnetic potential.

(b) Because there is no surface current density at $r = A$, the magnetic potential (the tangential field intensity) is continuous there. Thus, for the field inside

$$\Psi^b(r = a) = \Psi^a(r = a) = -3H_o a/2 \tag{3}$$

To satisfy this condition, the interior magnetic scalar potential is taken to have the form

$$\Psi^b = C r \cos\theta = C z \tag{4}$$

Substitution of this expression into (3) to evaluate $C = -3H_o/2$ results in the given expression.

9.6.8 The perfectly permeable walls force the boundary condition $\Psi = 0$ on the surfaces. The bottom magnetic surface charge density is neutralized by the image charges in the wall (see Fig. S9.6.8). The top magnetic surface charge density produces a magnetic potential Ψ that is

$$\Psi = A \sinh \beta(y - a) \cos \beta x \qquad y > d/2 \tag{1a}$$

and

$$\Psi = B \sinh \beta\big(y + \frac{d}{2}\big) \cos \beta x \qquad y < d/2 \tag{1b}$$

At the interface at $y = d/2, \Psi$ is continuous

$$A \sinh \beta\big(\frac{d}{2} - a\big) = B \sinh \beta d \tag{2}$$

and thus

$$B = -A \frac{\sinh \beta\big(a - \frac{d}{2}\big)}{\sinh \beta d} \tag{3}$$

The magnetic surface charge density at $y = d/2$ is

$$\sigma_m = \mu_o M_o \cos \beta x \tag{4}$$

It forces a jump of $\partial\Psi/\partial y$ at $y = d/2$:

$$-\frac{\partial \Psi}{\partial y}\bigg|_{y=d/2_+} + \frac{\partial \Psi}{\partial y}\bigg|_{y=d/2_-} = M_o \cos \beta x \tag{5}$$

and we find

$$-A \cosh \beta\big(\frac{d}{2} - a\big) + B \cosh \beta d = \frac{M_o}{\beta} \tag{6}$$

Using (3) we obtain

$$\begin{aligned} A &= -\frac{M_o}{\beta} \frac{\sinh \beta d}{\cosh \beta\big(\frac{d}{2} - a\big) \sinh \beta d - \cosh \beta d \sinh \beta\big(\frac{d}{2} - a\big)} \\ &= -\frac{M_o}{\beta} \frac{\sinh \beta d}{\sinh \beta\big(\frac{d}{2} + a\big)} \end{aligned} \tag{7}$$

The vertical component of B, B_y, above the tape, for $y > d/2$, is

$$B_y = -\mu_o \frac{\partial \Psi}{\partial y} = \mu_o M_o \frac{\sinh \beta d}{\sinh \beta\big(\frac{d}{2} + a\big)} \cosh \beta(y - a) \cos \beta x \tag{8}$$

Note that in the limit $a \to d/2$, the flux is simply $\mu_o M_o$ as expected. If the tape moves, $\cos \beta x$ has to be expressed as $\cos \beta(x' - Ut)$. The flux is

$$\lambda = wN\mu_o M_o \frac{\sinh \beta d}{\sinh \beta\big(\frac{d}{2} + a\big)} \cosh \beta\big(h + \frac{d}{2} - a\big) \times \int_{-l/2}^{l/2} \cos \beta(x' - Ut) dx' \tag{9}$$

The integral evalues to

$$\frac{1}{\beta}\big[\sin \beta\big(\frac{l}{2} - Ut\big) + \sin \beta\big(\frac{l}{2} + Ut\big)\big] = \frac{2}{\beta} \sin \beta \frac{l}{2} \cos \beta U t \tag{10}$$

and from here on one proceeds as in the Example 9.3.2.

$$v_o = \frac{d\lambda}{dt}$$

9.6.9 In terms of the magnetic scalar potential, boundary conditions are

$$\Psi(x,b)=0;\quad \Psi(x,0)=0 \tag{1}$$

$$H_y=-\frac{\partial\Psi}{\partial y}(0,y)=-K_o\cos\frac{\pi y}{a};\quad \frac{\partial\Psi}{\partial y}(b,y)=K_o\cos\frac{\pi y}{a} \tag{2}$$

To satisfy the first pair of these while matching the y dependence of the second pair, the potential is taken as having the y dependence $\sin(\pi y/a)$. In terms of Ψ, the conditions at the surfaces $x=0$ and $x=b$ are even with respect to $x=b/2$. Thus, the combination of $\exp(\pm\pi x/a)$ chosen to complete the solution to Laplace's equation is even with respect to $x=b/2$.

$$\Psi=A\cosh\left[\frac{\pi}{a}\left(x-\frac{b}{2}\right)\right]\sin\left(\frac{\pi y}{a}\right) \tag{3}$$

Thus, both of the relations (2) are satisfied by making the coefficient A equal to

$$A=\frac{aK_o}{\pi\cosh(\pi b/2a)} \tag{4}$$

9.6.10 The solution can be divided into a particular part due to the current density in the wire and a homogeneous part associated with the field that is uniformly applied at infinity. Because of the axial symmetry in the absence of the applied field, the particular part can be found using Ampère's integral law. Thus, from an integration at a constant radius r, it follows that

$$\begin{aligned} H_{\phi p}2\pi r&=\pi r^2 J_o;\quad r<R\\ H_{\phi p}2\pi r&=\pi R^2 J_o;\quad R<r \end{aligned} \tag{1}$$

so that the particular field intensity is

$$H_{\phi p}=\begin{cases} rJ_o/2; & r<R\\ R^2J_o/2r; & R<r \end{cases} \tag{2}$$

in polar coordinates

$$\mathbf{H}=\frac{1}{\mu}\left(\frac{1}{r}\frac{\partial A_z}{\partial\phi}\mathbf{i}_r-\frac{\partial A_z}{\partial r}\mathbf{i}_\phi\right) \tag{3}$$

and it follows from (2), integrated in accordance with (3), that

$$A_{zp}=\begin{cases} -\mu_b r^2J_o/4; & r<R\\ -\frac{1}{2}\mu_a J_oR^2 ln(r/R)-\frac{1}{4}\mu_b R^2J_o; & R<r \end{cases} \tag{4}$$

In view of the applied field, the homogeneous solution is assumed to take the form

$$A_{zh}=\begin{cases} Dr\sin\phi; & r<R\\ -\mu_a H_o r\sin\phi+C\frac{\sin\phi}{r}; & R<r \end{cases} \tag{5}$$

The coefficients C and D are adjusted to satisfy the boundary conditions at $r = R$,

$$A_z^a - A_z^b = 0 \tag{6}$$

$$-\frac{1}{\mu_a}\frac{\partial A_z^a}{\partial r} + \frac{1}{\mu_b}\frac{\partial A_z^b}{\partial r} = 0 \tag{7}$$

The first of these guarantees that the flux density normal to the surface is continuous at $r = R$ while the second requires continuity of the tangential magnetic field intensity. Substitution of (5) into these relations gives a pair of equations that can be solved for the coefficients C and D.

$$\begin{bmatrix} 1/R & -R \\ 1/\mu_a R^2 & 1/\mu_b \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} \mu_a H_o R \\ -H_o \end{bmatrix} \tag{8}$$

The coefficients which follow are substituted into (5) and those expressions respectively added to (4) provide the given expressions.

9.6.11 (a) Given the magnetization, the associated H is found by first finding the distribution of magnetic charge. There is none in the volume, where $\mathbf{M}$ is uniform. The surface magnetization charge density at the surface, say at $r = R$, is

$$\sigma_{sm} = -\mu_o \mathbf{n} \cdot (\mathbf{M}^a - \mathbf{M}^b) = \mu_o M \mathbf{n} \cdot \mathbf{i_r} = \mu_o M \cos\theta \tag{1}$$

Thus, boundary conditions to be satisfied at $r = R$ by the scalar magnetic potential are

$$\Psi^a - \Psi^b = 0 \tag{2}$$

$$-\mu_o \frac{\partial \Psi^a}{\partial r} + \mu_o \frac{\partial \Psi^b}{\partial r} = \mu_o M \cos\theta \tag{3}$$

From the θ dependence in (3), it is reasonable to assume that the fields outside and inside the sphere take the form

$$\Psi = \begin{cases} -H_o r\cos\theta + A\frac{\cos\theta}{r^2} \\ -Hr\cos\theta \end{cases} \tag{4}$$

Substitution of these expressions into (2) and (3) gives

$$H = H_o - \frac{1}{3}M \Rightarrow M = 3(H_o - H) \tag{5}$$

Thus, it follows that

$$B \equiv \mu_o(H + M) = \mu_o(-2H + 3H_o) \tag{6}$$

(b) This relation between B and H is linear and therefore a straight line in the $B - H$ plane. Where $B = 0$ in (6), $H = 3H_o/2$ and where $H = 0, B = 3\mu_o H_o$. Thus, the load line is as shown in Fig. S9.6.11.

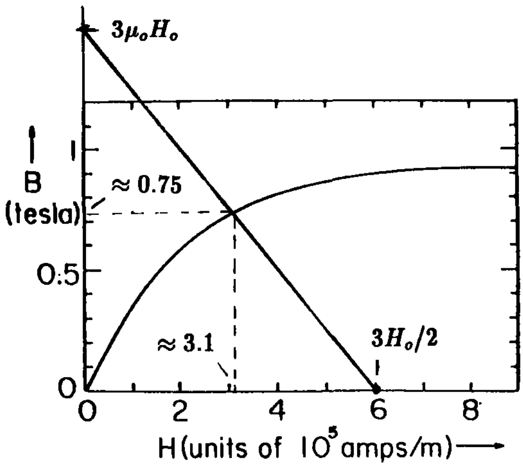

Figure S9.6.11

(c) The values of B and H within the sphere are given by the intersection of the load line with the saturation curve representing the constitutive law for the magnetization of the sphere.

(d) For the specific values given, the load line is as shown in Fig. S9.6.11. The values of B and H deduced from the intersection are also indicated in the figure.

9.6.12 We assume that the field is uniform inside the cylinder and then confirm the correctness of the assumption. The scalar potentials inside and outside the cylinder are

$$\Psi = \begin{cases} -H_o R\cos\phi(r/R) + A\cos\phi(R/r) & r > R \\ C\cos\phi(r/R) & r < R \end{cases} \tag{1}$$

Because Ψ is continuous at $r = R$

$$-H_o R + A = C \tag{2}$$

If there is an internal uniform magnetization $\mathbf{M} = M\mathbf{i}_x$, then

$$\mathbf{n}\cdot\mathbf{M} = M\cos\phi \tag{3}$$

The boundary condition for the normal component of $\mu_o\mathbf{H}$ at $r = R$ gives

$$\left(\mu_o H_o + \mu_o\frac{A}{R}\right) + \mu_o\frac{C}{R} = \mu_o M \tag{4}$$

Therefore, from (2) and (4)

$$\frac{C}{R} = -H_o + \frac{M}{2} \tag{5}$$

and the internal $(r < R)\mathbf{H}$ field is (we use no subscripts to denote the field internal to cylinder):

$$\mathbf{H} = (H_o - \frac{M}{2})\mathbf{i_x} \tag{6}$$

The magnetization causes a "demagnetization" field of magnitude $M/2$. We can construct "load line" to find internal $\mathbf{B}$ graphically. Since

$$B = \mu_o(H + M) \tag{7}$$

we find from (6) for the magnitude of the internal H field

$$H = \left(H_o - \frac{M+H}{2} + \frac{H}{2}\right) = H_o - \frac{B}{2\mu_o} + \frac{H}{2} \tag{8}$$

or

$$H = 2H_o - \frac{B}{\mu_o} \tag{9}$$

The two intersection points are (see Fig. S9.6.12)

$$H = 2H_o \quad for \quad B = 0$$

and

$$B = 2\mu_o H_o \quad \text{for} \quad H = 0$$

We read off the graph: $B = 0.67$ tesla, $H = 2.5 \times 10^5$ amps/m.

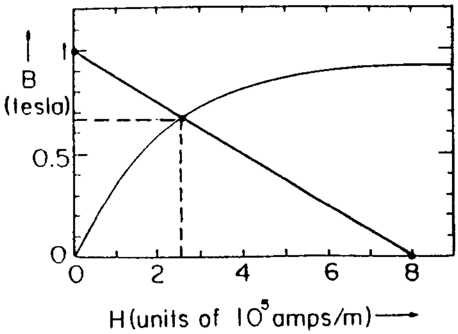

Figure S9.6.12

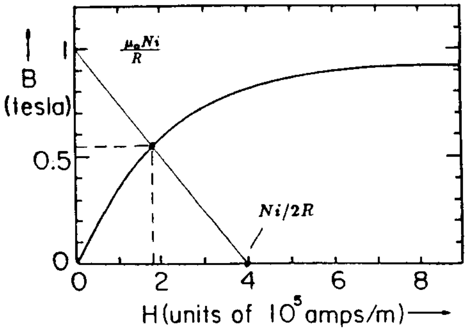

Figure S9.6.13

9.6.13 The relation between the current in the winding and H and M in the sphere are given by (9.6.15).

$$M = 3(\frac{Ni}{3R} - H) \tag{1}$$

From this, the load line follows as

$$B \equiv \mu_o(H + M) = \mu_o(\frac{Ni}{R} - 2H) \tag{2}$$

The intercepts that can be used to plot this straight line are shown in Fig. S9.6.13. The line shown is for the given specific numbers. Thus, within the sphere, $B \approx 0.54$ and $H \approx 1.8$.

9.7 MAGNETIC CIRCUITS

9.7.1 (a) Because of the high core permeability, the fields are approximated by taking an "inside-outside" approach. First, the field inside the core is approximately subject to the condition that

$$\mathbf{n}\cdot\mathbf{B} = 0 \quad \text{at} \quad r = a \quad \text{and} \quad r = b \tag{1}$$

which is satisfied because the given field distribution has no radial component. Further, Ampère's integral law requires that

$$\int_0^{2\pi} H_\phi r d\phi = Ni = \int_0^{2\pi} \frac{Ni}{2\pi r} r d\phi = Ni \tag{2}$$

In terms of the magnetic scalar potential, with the integration constant adjusted to define the potential as zero at $\phi = \pi$,

$$\begin{aligned} -\frac{1}{r}\frac{\partial \Psi}{\partial \phi} = \frac{Ni}{2\pi r} \Rightarrow \Psi &= -\frac{Ni}{2\pi}\phi + \text{const} \\ &= \frac{Ni}{2}\left(1 - \frac{\phi}{\pi}\right) \end{aligned} \tag{3}$$

This potential satisfies Laplace's equation, has no radial derivative on the inside and outside walls, suffers a discontinuity at $\phi = 0$ that is Ni and has a continuous derivative normal to the plane of the wires at $\phi = 0$ (as required by flux continuity). Thus, the proposed solution meets the required conditions and is uniquely specified.

(b) In the interior region, the potential given by (3), evaluated at $r = b$, provides a boundary condition on the field. This potential (and actually any other potential condition at $r = b$) can be represented by a Fourier series, so we represent the solution for $r < b$ by solutions to Laplace's equation taking the form

$$\Psi = \sum_{m=1}^{\infty} \psi_m \sin m\phi \left(\frac{r}{b}\right)^m \tag{4}$$

Because the region includes the origin, solutions r^{-m} are omitted. Thus, at the boundary, we require that

$$\frac{Ni}{2}\left(1 - \frac{\phi}{\pi}\right) = \sum_{m=1}^{\infty} \psi_m \sin m\phi \tag{5}$$

Multiplication by $\sin n\phi$ and integration gives

$$\begin{aligned} \int_0^{2\pi} \frac{Ni}{2}\left(1 - \frac{\phi}{\pi}\right) \sin(n\phi) d\phi &= \int_0^{2\pi} \sum_{m=1}^{\infty} \psi_m \sin m\phi \sin n\phi d\phi \\ &= \psi_n \pi \end{aligned} \tag{6}$$

Thus,

$$\psi_m = \frac{Ni}{2\pi} \int_0^{2\pi} \left(1 - \frac{\phi}{\pi}\right) \sin m\phi d\phi = \frac{Ni}{m\pi} \tag{7}$$

Substitution of this coefficient into (4) results in the given solution.

9.7.2 The approximate magnetic potential on the outer surface is

$$\Psi = \sum_{m=1}^{\infty} \frac{Ni}{m\pi} \sin m\phi \tag{1}$$

according to (b) of Prob. 9.7.1. The outside potential is a solution to Laplace's equation that must match (1) and decays to zero as $r \to \infty$. This is clearly

$$\Psi = \sum_{m=1}^{\infty} \frac{Ni}{m\pi} (a/r)^m \sin m\phi \tag{2}$$

9.7.3 Using contours C_1 and C_2 respectively, as defined in Fig. S9.7.3, Ampère's integral law gives

$$H_a a = Ni \Rightarrow H_a = Ni/a \tag{1}$$

$$H_b b = Ni \Rightarrow H_b = Ni/b \tag{2}$$

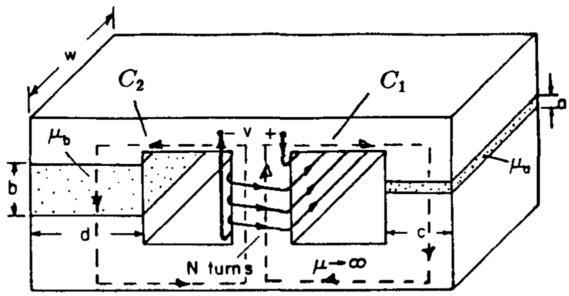

Figure S9.7.3

From the integral form of flux continuity, for a closed surface S that intersects the middle leg and passes through the gaps to right and left, we know that the flux through the middle leg is equal to the sum of those through the gaps. This flux is linked N times, so

$$\lambda = N(cw\mu_a H_a + dw\mu_b H_b) \tag{3}$$

Substitution of (1) and (2) into this expression gives

$$\lambda = N^2 w\left(\frac{c\mu_a}{a} + \frac{d\mu_b}{b}\right)i \tag{4}$$

where the coefficient of i is the given inductance.

9.7.4 The field in the gap due to the coil of N turns is approximately uniform because the hemisphere is small. From Ampère's law

$$Hh = Ni \tag{1}$$

where $\mathbf{H}$ directed downward is defined positive. This field is distorted by the sphere. The scalar magnetic potential around the sphere is

$$\Psi = R\frac{Ni}{h}\cos\theta[(r/R) - (R/r)^2] \tag{2}$$

where θ is the angle measured from the vertical axis. The field is

$$\mathbf{H} = -\frac{Ni}{h}\{\mathbf{i}_r\cos\theta[1 + 2(R/r)^2] - \mathbf{i}_\theta\sin\theta[1 - (R/r)^2]\} \tag{3}$$

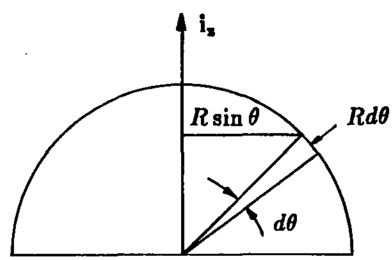

Figure S9.7.4

The flux linked by one turn at angle α is (see Fig. S9.7.4)

$$\begin{aligned}\Phi_\lambda &= \int_0^\alpha \mu_o H_r 2\pi R^2 \sin\theta d\theta\\ &= -3\mu_o\frac{Ni}{h}2\pi R^2\int_0^\alpha \sin\theta\cos\theta d\theta\\ &= -\frac{3\mu_o}{2}\frac{Ni}{h}\pi R^2(1-\cos 2\alpha)\end{aligned} \tag{4}$$

But $1 - \cos 2\alpha = 2\sin^2\alpha$ which will be used below. The flux linkage is λ_{21} where 1 stands for the coil on the $\pi/2$ leg of the "circuit", 2 for the hemispherical coil

$$\begin{aligned}\lambda_{21} &= \int_0^{\pi/2}\Phi_\lambda\frac{n}{R}\sin\alpha R d\alpha\\ &= -\frac{3}{4}\mu_o\frac{Nn}{h}i\pi R^2\int_0^{\pi/2}\sin^3\alpha d\alpha\\ &= -\mu_o\frac{Nn}{2h}\pi R^2 i\end{aligned} \tag{5}$$

The mutual inductance is

$$L_{21} = \frac{\lambda_{21}}{i} = -\mu_o\frac{Nn}{2h}\pi R^2 \tag{6}$$

9.7.5 In terms of the air-gap magnetic field intensities defined in Fig. S9.7.5, Ampère's integral law for a contour passing around the magnetic circuit through the two windings and across the two air-gaps, requires that

$$N_1 i_1 + N_2 i_2 = H_a x + H_b x \tag{1}$$

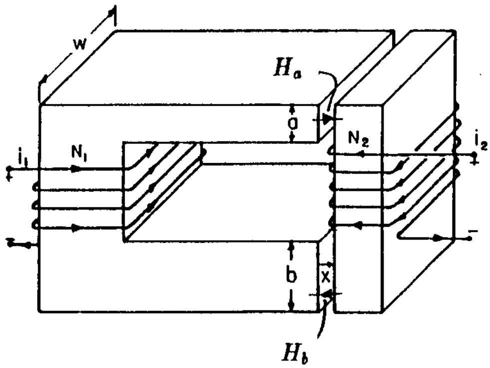

Figure S9.7.5

In terms of these same field intensities, flux continuity for a surface S that encloses the movable member requires that

$$aw\mu_o H_a = bw\mu_o H_b \Rightarrow H_b = \frac{a}{b} H_a \tag{2}$$

From these relations, it follows that

$$H_a = (N_1 i_1 + N_2 i_2)/x\left(1 + \frac{a}{b}\right) \tag{3}$$

The flux linking the first winding is that through either of the gaps, say the upper one, multiplied by N_1

$$\lambda_1 = N_1 aw\mu_o H_a = L_o(N_1^2 i_1 + N_1 N_2 i_2) \tag{4}$$

The second equation has been written using (3). Similarly, the flux linking the second coil is that crossing the upper gap multiplied by N_2.

$$\lambda_2 = N_2 aw\mu_o H_a = L_o(N_2 N_1 i_1 + N_2^2 i_2) \tag{5}$$

Identification of the coefficients of the respective currents in these two relations results in the given self and mutual inductances.

9.7.6 Denoting the H field in the gap of width x by H_x and that in the gap g by H_g, Ampère's integral law gives

$$\oint \mathbf{H}\cdot d\mathbf{s} = xH_x + gH_g = Ni \tag{1}$$

where flux continuity requires

$$\mu_o H_x \pi a^2 = \mu_o H_g 2\pi a d \tag{2}$$

Thus

$$xH_x + \frac{\pi a^2}{2\pi a d} gH_x = Ni \tag{3}$$

The flux is

$$\Phi_\lambda = \mu_o \pi a^2 H_x = \frac{\mu_o N i}{\frac{x}{\pi a^2} + \frac{g}{2\pi a d}}$$

The inductance is

$$L = \frac{N\Phi_\lambda}{i} = \frac{\mu_o N^2}{\frac{x}{\pi a^2} + \frac{g}{2\pi a d}}$$

9.7.7 We pick two contours (Fig. S9.7.7) to find the H field which is indicated in the three gaps as H_a, H_b and H_c. The fields are defined positive if they point radially outward. From contour C_1:

$$(-H_a + H_b)g = N_1 i_1 \tag{1}$$

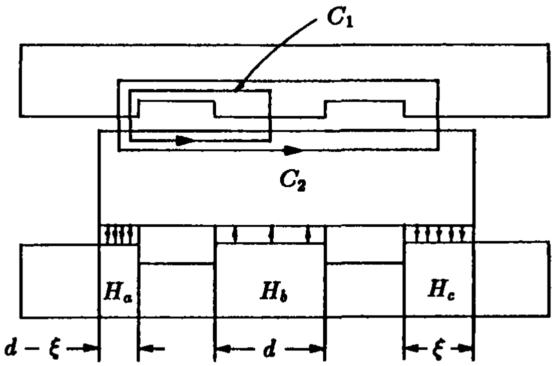

Figure S9.7.7

From contour C_2

$$(-H_a + H_c)g = N_1 i_1 + N_2 i_2 \tag{2}$$

The flux must be continuous so that

$$2\pi a[(d-\xi)\mu_o H_a + d\mu_o H_b + \xi \mu_o H_c] = 0 \tag{3}$$

We find from these three equations

$$H_a = -\frac{d+\xi}{2d}\frac{N_1 i_1}{g} - \frac{\xi}{2d}\frac{N_2 i_2}{g} \tag{4}$$

$$H_b = \frac{d-\xi}{2d}\frac{N_1 i_1}{g} - \frac{\xi}{2d}\frac{N_2 i_s}{g} \tag{5}$$

$$H_c = \frac{d-\xi}{2d}\frac{N_1 i_1}{g} + \frac{2d-\xi}{2d}\frac{N_2 i_2}{g} \tag{6}$$

The flux linkage of coil (1) is:

$$\begin{aligned}\lambda_1 &= -N_1 2\pi a(d-\xi)\mu_o H_a\\ &= \mu_o \pi a\left[\frac{d^2-\xi^2}{d}\frac{N_1^2 i_1}{g} + \frac{\xi(d-\xi)}{d}\frac{N_1 N_2 i_2}{g}\right]\end{aligned}$$

The flux linkage of coil (2) is:

$$\begin{aligned}\lambda_2 &= N_2 2\pi a\xi\mu_o H_c\\ &= \mu_o \pi a\left[\frac{\xi(2d-\xi)}{d}\frac{N_2^2 i_2}{g} + \frac{\xi(d-\xi)}{d}\frac{N_1 N_2 i_1}{g}\right]\end{aligned}$$

The inductance matrix is, by inspection

$$\begin{aligned}L_{11} &= \mu_o \pi a\frac{d^2-\xi^2}{dg}N_1^2\\ L_{22} &= \mu_o \pi a\frac{\xi(2d-\xi)}{dg}N_2^2\\ L_{12} = L_{21} &= \mu_o \pi a\frac{\xi(d-\xi)}{dg}N_1 N_2\end{aligned}$$

9.7.8 (a) Ψ must be constant over the surfaces of the central leg at $x = \mp l/2$ where we have perfectly permeable surfaces. In solving for the field internal to the central leg we assume that $\partial\Psi/\partial n = 0$ on the interfaces with μ_o.

(b) If we assume an essentially uniform field H_μ in the central leg, Ampère's integral law applied to a contour following the central leg and closing around the upper part of the magnetic circuit gives

$$H_\mu l = N_1 i_1 + N_2 i_2 \tag{1}$$

Therefore

$$\Psi(x = -l/2) = \frac{N_1 i_1 + N_2 i_2}{2} \tag{2}$$

$$\Psi(x = l/2) = -\frac{N_1 i_1 + N_2 i_2}{2} \tag{3}$$

(c) In region a, at $y = 0$, Ψ must decrease linearly from the value (2) to the value (1)

$$\Psi = -(N_1 i_1 + N_2 i_2)\frac{x}{l} \tag{4}$$

At

$$y = a, \quad \Psi = 0 \tag{5}$$

At $x = \pm l/2, 0 < y < a, \Psi$ must change linearly from (2) and (3) respectively, to zero

$$\Psi\Big(x = -\frac{l}{2}, y\Big) = \frac{N_1 i_1 + N_2 i_2}{2}\frac{(a - y)}{a} \tag{6}$$

$$\Psi\Big(x = \frac{l}{2}, y\Big) = -\frac{N_1 i_1 + N_2 i_2}{2}\frac{(a - y)}{a} \tag{7}$$

(d) Ψ must obey Laplace's equation and match boundary conditions that vary linearly with x and y. An obvious solution is

$$\Psi = Axy + Bx + Cy$$

We have, at $y = 0$

$$Bx = -(N_1 i_1 + N_2 i_2)\frac{x}{l}$$

and thus

$$B = -\frac{N_1 i_1 + N_2 i_2}{l}$$

In a similar way we find at $y = a$

$$Aax + Bx + Ca = 0$$

and thus

$$C = 0, \quad Aa = -B$$

which gives

$$\Psi = \frac{(N_1 i_1 + N_2 i_2)}{la}[xy - ax]$$

9.7.9 From Ampère's integral law we find for the H fields

$$l_1 H_1 + l_2 H_2 = Ni + Kl_1 \tag{1}$$

where K is the ("surface-") current in the thin sheet. This surface current is driven by the electric field induced by Faraday's law

$$\begin{aligned} 2\frac{K}{\sigma\Delta}(3a + w) &= \oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d}{dt}\int \mu_o \mathbf{H} \cdot d\mathbf{a} \\ &= -\mu a w \frac{dH_1}{dt} \end{aligned} \tag{2}$$

Finally, the flux is continuous so that

$$\mu H_1 3aw = \mu H_2 aw \tag{3}$$

and

$$H_2 = 3H_1 \tag{4}$$

When we introduce complex notation and use (4) in (1) we find

$$\hat{H}_1(l_1 + 3l_2) = N\hat{i}_o + \hat{K}l_1 \tag{5}$$

and

$$\hat{K} = -\frac{j\omega\mu aw}{2(3a+w)}\sigma\Delta\hat{H}_1 \tag{6}$$

Introducing (6) into (5) yields

$$\hat{H}_1 = \frac{N\hat{i}_o}{(l_1 + 3l_2)}\frac{1}{1 + j\omega\tau_m} \tag{7}$$

where

$$\tau_m = \mu\sigma\Delta\frac{awl_1}{(l_1 + 3l_2)(6a + 2w)}$$

9.7.10 The cross-sectional areas of the legs to either side are half of that through the center leg. Thus, the flux density, B, tends to be the same over the cross-sections of all parts of the magnetic circuit. For this reason, we can expect that each point within the core will tend to be at the same operating point on the given magnetization characteristic. Thus, with H_g defined as the air-gap field intensity and H defined as the field intensity at each point in the core, Ampère's integral law requires that

$$2Ni = (l_1 + l_2)H + dH_g \tag{1}$$

In the gap, the flux density is $\mu_o H_g$ and that must be equal to the flux density just inside the adjacent pole faces.

$$\mu_o H_g = B \tag{2}$$

The given load-line is obtained by combining these relations. Evaluation of the intercepts of this line gives the line shown in Fig. S9.7.10. Thus, in the core, $B \approx 0.75$ Tesla and $H \approx 0.3 \times 10^4$ A/m.

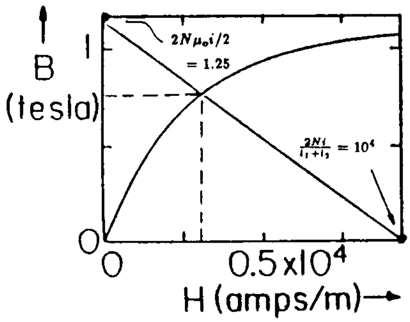

Figure S9.7.10

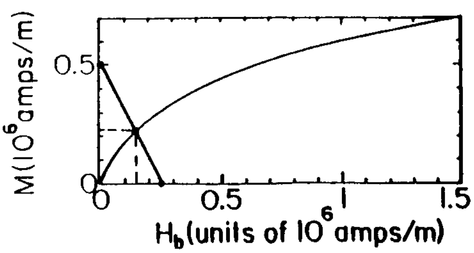

Figure S9.7.11

9.7.11 (a) From Ampère's integral law we obtain for the field H_b in the μ material and H_a in the air gap:

$$bH_b + aH_a = Ni \tag{1}$$

Further, from flux continuity

$$AB_b = A\mu_o H_a \tag{2}$$

and thus

$$H_b = \frac{Ni}{b} - \frac{a}{b}\frac{B_b}{\mu_o} \tag{3}$$

Now $B_b = \mu_o(H_b + M)$ and thus

$$H_b = \frac{Ni}{b} - \frac{a}{b}(H_b + M) \tag{4}$$

or

$$H_b = \frac{Ni}{a+b} - \frac{b}{a+b}M \tag{5}$$

This is the load line.

(b) The intercepts are at $M = 0$

$$H_b = \frac{Ni}{a+b} = \frac{Ni}{2a} = 0.25 \times 10^6$$

and at $H_b = 0$

$$M = \frac{Ni}{b} = 0.5 \times 10^6$$

We find

$$M = 0.22 \times 10^6 \text{ A/m}$$

$$H_b = 0.13 \times 10^6 \text{ A/m}$$

The B field is

$$\mu_o(H_b + M) = 4\pi \times 10^{-7}(0.13 + 0.22) \times 10^6 = 0.44\,\text{tesla}$$