FORMULATION OF STRUCTURAL ELEMENTS

LECTURE 7

52 MINUTES

LECTURE 7 Formulation and calculation of isoparametric structural elements

Beam, plate and shell elements

Formulation using Mindlin plate theory and unified general continuum formulation

Assumptions used including shear deformations

Demonstrative examples: two-dimensional beam, plate elements

Discussion of general variable-number-nodes elements

Transition elements between structural and continuum elements

Low- versus high-order elements

TEXTBOOK: Sections: 5.4.1, 5.4.2, 5.5.2, 5.6.1

Examples: 5.20, 5.21, 5.22, 5.23, 5.24, 5.25, 5.26, 5.27

FORMULATION OF STRUCTURAL ELEMENTS

- beam, plate and shell elements
- isoparametric approach for interpolations

Strength of Materials Approach

- straight beam elements

  use beam theory including shear effects
- plate elements

  use plate theory including shear effects
  (Reissner/Mindlin)

Continuum Approach

Use the general principle of virtual displacements, but

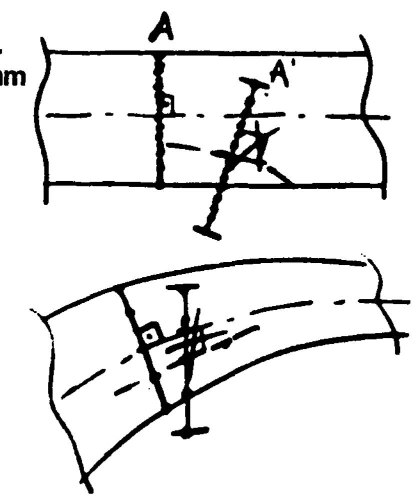
-- exclude the stress components not applicable

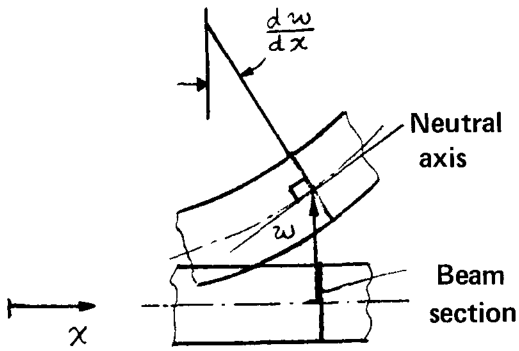
-- use kinematic constraints for particles on sections originally normal to the mid-surface

" particles remain on a straight line during deformation"

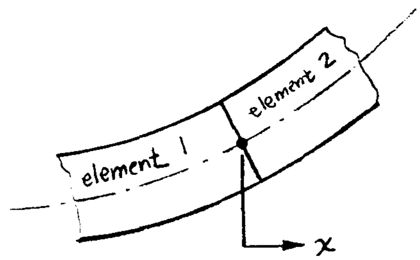
e.g. beam

e.g. shell





Deformation of cross-section

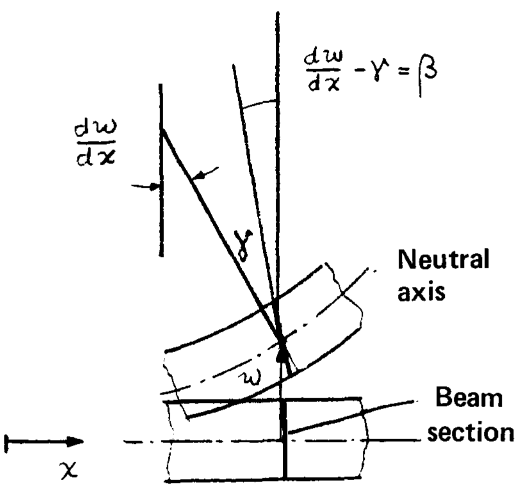


Boundary conditions between beam elements

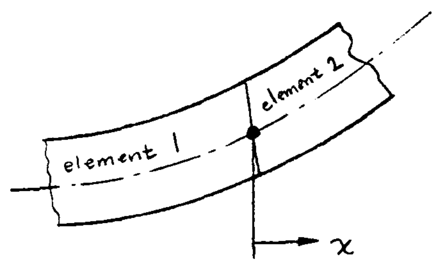
$$w\Big|_{x-0} = w\Big|_{x+0} \; ; \; \frac{dw}{dx}\Big|_{x-0} = \frac{dw}{dx}\Big|_{x+0}$$

a) Beam deformations excluding shear effect

Fig. 5.29. Beam deformation mechanisms



Deformation of cross-section



$$w\Big|_{x-0} = w\Big|_{x+0}$$

$$\beta\Big|_{x-0} = \beta\Big|_{x+0}$$

Boundary conditions between beam elements

b) Beam deformations including shear effect

Fig. 5.29. Beam deformation mechanisms

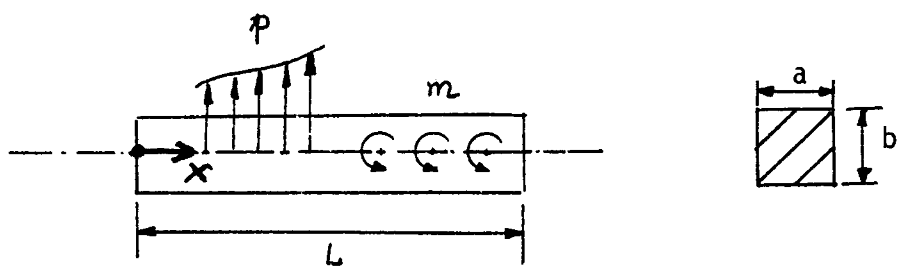
We use

$$\beta = \frac{dw}{dx} - \gamma \qquad (5.48)$$

$$\tau = \frac{V}{A_s} \; ; \; \gamma = \frac{\tau}{G} \; ; \; k = \frac{A_s}{A} \qquad (5.49)$$

$$\Pi = \frac{EI}{2} \int_0^L \left(\frac{d\beta}{dx}\right)^2 dx + \frac{GAk}{2} \int_0^L \left(\frac{dw}{dx} - \beta\right)^2 dx - \int_0^L pw \; dx - \int_0^L m \; \beta \; dx \qquad (5.50)$$

$$EI \int_0^L \left(\frac{d\beta}{dx}\right) \; \delta\left(\frac{d\beta}{dx}\right) dx + GAk \int_0^L \left(\frac{dw}{dx} - \beta\right) \; \delta\left(\frac{dw}{dx} - \beta\right) dx - \int_0^L p \; \delta w \; dx - \int_0^L m \; \delta\beta \; dx = 0 \qquad (5.51)$$

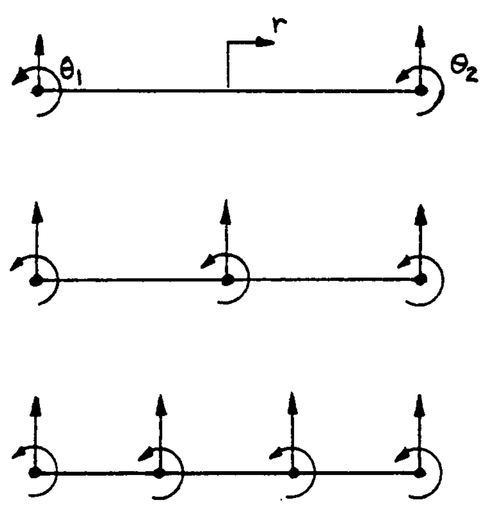


(a) Beam with applied loading

E = Young's modulus, G = shear modulus

$k = \frac{5}{6}$, $A = ab$, $I = \frac{ab^3}{12}$

Fig. 5.30. Formulation of two-dimensional beam element



(b) Two, three- and four-node models; $\theta_i = \beta_i$, $i=1,...,q$ (Interpolation functions are given in Fig. 5.4)

Fig. 5.30. Formulation of two-dimensional beam element

The interpolations are now

$$w = \sum_{i=1}^{q} h_i w_i \quad ; \quad \beta = \sum_{i=1}^{q} h_i \theta_i \qquad (5.52)$$

$$w = \underline{H}_w \, \underline{U} \; ; \qquad \beta = \underline{H}_\beta \, \underline{U}$$

$$(5.53)$$

$$\frac{\partial w}{\partial x} = \underline{B}_w \, \underline{U} \; ; \quad \frac{\partial \beta}{\partial x} = \underline{B}_\beta \, \underline{U}$$

Where

$$\underline{U}^T = [w_1 \, \ldots \, w_q \;\; \theta_1 \, \ldots \, \theta_q]$$

$$\underline{H}_w = [h_1 \, \ldots \, h_q \;\; 0 \ldots 0]$$

$$\underline{H}_\beta = [0 \; \ldots \; 0 \;\; h_1 \; \ldots \; h_q] \qquad (5.54)$$

and

$$\underline{B}_w = J^{-1} \left[\frac{\partial h_1}{\partial r} \ldots \frac{\partial h_q}{\partial r} \;\; 0 \ldots 0 \right]$$

$$\underline{B}_\beta = J^{-1} \left[0 \ldots 0 \;\; \frac{\partial h_1}{\partial r} \ldots \frac{\partial h_q}{\partial r} \right] (5.55)$$

So that

$$\underline{K} = EI \int_{-1}^{1} \underline{B}_\beta^T \underline{B}_\beta \det J \, dr + GAk \int_{-1}^{1} (\underline{B}_w - \underline{H}_\beta)^T (\underline{B}_w - \underline{H}_\beta) \det J \, dr \qquad (5.56)$$

and

$$\underline{R} = \int_{-1}^{1} \underline{H}_w^T \, p \det J \, dr + \int_{-1}^{1} \underline{H}_\beta^T \, m \det J \, dr \qquad (5.57)$$

Considering the order of interpolations required, we study

$$\Pi = \int_0^L \left(\frac{d\beta}{dx}\right)^2 dx + \alpha \int_0^L \left(\frac{dw}{dx} - \beta\right)^2 dx \; ; \qquad \alpha = \frac{GAk}{EI} \qquad (5.60)$$

Hence

- use parabolic (or higher-order) elements
- discrete Kirchhoff theory
- reduced numerical integration

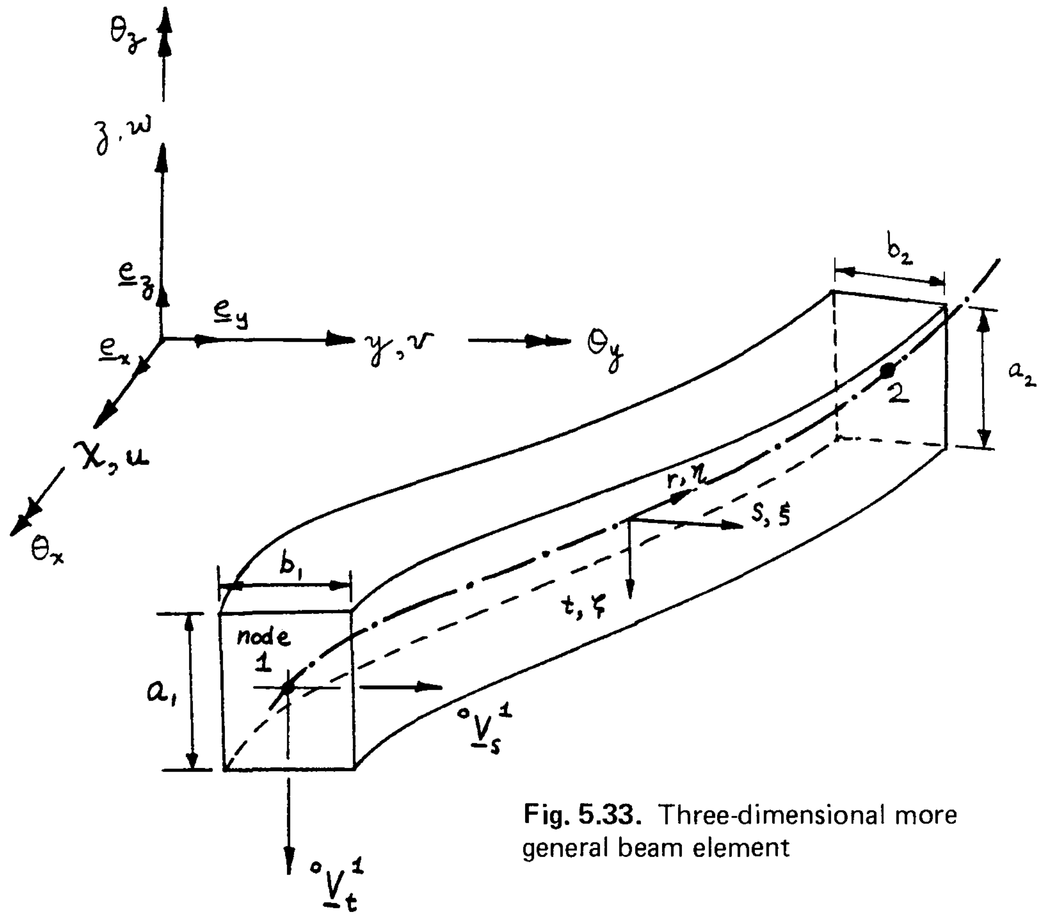


Fig. 5.33. Three-dimensional more general beam element

Here we use

$$
\begin{aligned}
{}^{\ell}x(r,s,t) &= \sum_{k=1}^{q} h_k\, {}^{\ell}x_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k\, {}^{\ell}V_{tx}^{k} \\
&\quad + \frac{s}{2} \sum_{k=1}^{q} b_k h_k\, {}^{\ell}V_{sx}^{k} \\
{}^{\ell}y(r,s,t) &= \sum_{k=1}^{q} h_k\, {}^{\ell}y_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k\, {}^{\ell}V_{ty}^{k} \\
&\quad + \frac{s}{2} \sum_{k=1}^{q} b_k h_k\, {}^{\ell}V_{sy}^{k} \qquad (5.61) \\
{}^{\ell}z(r,s,t) &= \sum_{k=1}^{q} h_k\, {}^{\ell}z_k + \frac{t}{2} \sum_{k=1}^{q} a_k\, h_k\, {}^{\ell}V_{tz}^{k} \\
&\quad + \frac{s}{2} \sum_{k=1}^{q} b_k h_k\, {}^{\ell}V_{sz}^{k}
\end{aligned}
$$

So that

$$u\ (r,s,t) = {}^1x - {}^0x$$

$$v\ (r,s,t) = {}^1y - {}^0y \qquad (5.62)$$

$$w\ (r,s,t) = {}^1z - {}^0z$$

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and

$$u(r,s,t) = \sum_{k=1}^{q} h_k u_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k\ V_{tx}^{k} + \frac{s}{2} \sum_{k=1}^{q} b_k h_k\ V_{sx}^{k}$$

$$v(r,s,t) = \sum_{k=1}^{q} h_k v_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k\ V_{ty}^{k} + \frac{s}{2} \sum_{k=1}^{q} b_k h_k\ V_{sy}^{k}$$

$$w(r,s,t) = \sum_{k=1}^{q} h_k w_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k\ V_{tz}^{k} + \frac{s}{2} \sum_{k=1}^{q} b_k h_k\ V_{sz}^{k}$$

(5.63)

Finally, we express the vectors $\underline{V}_t^k$ and $\underline{V}_s^k$ in terms of rotations about the Cartesian axes x , y , z ,

$$\underline{V}_t^k = \underline{\theta}_k \times {}^0\underline{V}_t^k$$

$$\underline{V}_s^k = \underline{\theta}_k \times {}^0\underline{V}_s^k \qquad (5.65)$$

where

$$\underline{\theta}_k = \begin{bmatrix} \theta_x^k \\ \theta_y^k \\ \theta_z^k \end{bmatrix} \qquad (5.66)$$

We can now find

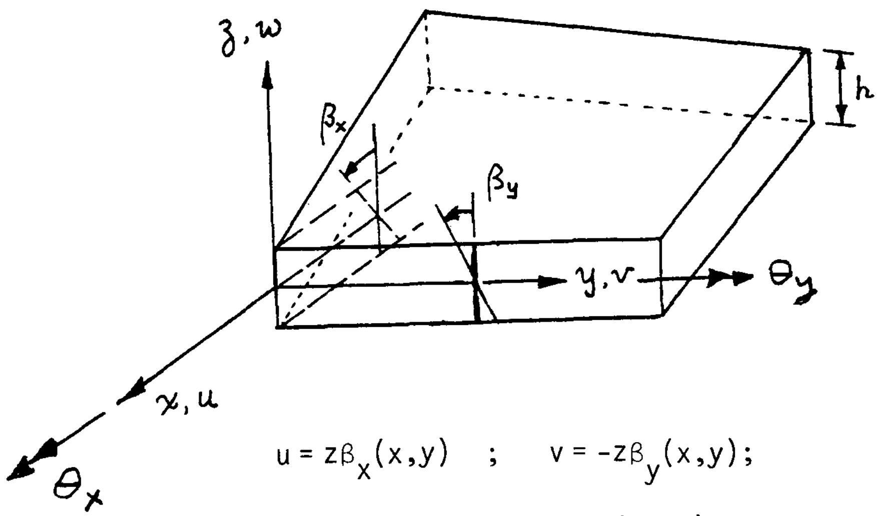
$$\begin{bmatrix} \varepsilon_{\eta\eta} \\ \gamma_{\eta\xi} \\ \gamma_{\eta\zeta} \end{bmatrix} = \sum_{k=1}^{q} \underline{B}_k \, \underline{u}_k \qquad (5.67)$$

where

$$\underline{u}_k^T = [u_k \; v_k \; w_k \;\; \theta_x^k \, \theta_y^k \, \theta_z^k] \qquad (5.68)$$

and then also have

$$\begin{bmatrix} \tau_{\eta\eta} \\ \tau_{\eta\xi} \\ \tau_{\eta\zeta} \end{bmatrix} = \begin{bmatrix} E & 0 & 0 \\ 0 & Gk & 0 \\ 0 & 0 & Gk \end{bmatrix} \begin{bmatrix} \varepsilon_{\eta\eta} \\ \gamma_{\eta\xi} \\ \gamma_{\eta\zeta} \end{bmatrix} \qquad (5.77)$$



$$u = z\beta_x(x,y) \quad ; \quad v = -z\beta_y(x,y);$$

$$\text{and} \quad w = w(x,y) \qquad (5.78)$$

Fig. 5.36. Deformation mechanisms in analysis of plate including shear deformations

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Hence

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = z \begin{bmatrix} \dfrac{\partial \beta_x}{\partial x} \\ -\dfrac{\partial \beta_y}{\partial y} \\ \dfrac{\partial \beta_x}{\partial y} - \dfrac{\partial \beta_y}{\partial x} \end{bmatrix} \qquad (5.79)$$

$$\begin{bmatrix} \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial w}{\partial y} - \beta_y \\ \dfrac{\partial w}{\partial x} + \beta_x \end{bmatrix} \qquad (5.80)$$

and

$$\begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{xy} \end{bmatrix} = z \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \frac{\partial \beta_x}{\partial x} \\ -\frac{\partial \beta_y}{\partial y} \\ \frac{\partial \beta_x}{\partial y} - \frac{\partial \beta_y}{\partial x} \end{bmatrix} \qquad (5.81)$$

$$\begin{bmatrix} \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \frac{E}{2(1+\nu)} \begin{bmatrix} \frac{\partial w}{\partial y} - \beta_y \\ \frac{\partial w}{\partial x} + \beta_x \end{bmatrix} \qquad (5.82)$$

The total potential for the element is:

$$\Pi = \frac{1}{2} \int_A \int_{-h/2}^{h/2} [\varepsilon_{xx} \; \varepsilon_{yy} \; \gamma_{xy}] \begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{xy} \end{bmatrix} dz \; dA + \frac{k}{2} \int_A \int_{-h/2}^{h/2} [\gamma_{yz} \; \gamma_{zx}] \begin{bmatrix} \tau_{yz} \\ \tau_{zx} \end{bmatrix} dx \; dA - \int_A w \; p \; dA \qquad (5.83)$$

or performing the integration through the thickness

$$\Pi = \frac{1}{2}\int_A \underline{\kappa}^T \underline{C}_b \underline{\kappa}\, dA + \frac{1}{2}\int_A \underline{\gamma}^T \underline{C}_s \underline{\gamma}\, dA - \int_A w\, p\, dA \qquad (5.84)$$

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where

$$\underline{\kappa} = \begin{bmatrix} \frac{\partial\beta_x}{\partial x} \\ -\frac{\partial\beta_y}{\partial y} \\ \frac{\partial\beta_x}{\partial y} - \frac{\partial\beta_y}{\partial x} \end{bmatrix} ; \quad \underline{\gamma} = \begin{bmatrix} \frac{\partial w}{\partial y} - \beta_y \\ \frac{\partial w}{\partial x} + \beta_x \end{bmatrix} \qquad (5.86)$$

$$\underline{C}_b = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} ;$$

$$\underline{C}_s = \frac{Ehk}{2(1+\nu)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (5.87)$$

Using the condition $\delta\Pi = 0$ we obtain the principle of virtual displacements for the plate element.

$$\int_A \delta\underline{\kappa}^T \underline{C}_b \, \underline{\kappa} \, dA + \int_A \delta \, \underline{\gamma}^T \underline{C}_s \, \underline{\gamma} \, dA - \int_A \delta w \, p \, dA = 0 \qquad (5.88)$$

We use the interpolations

$$w = \sum_{i=1}^{q} h_i w_i \quad ; \quad \beta_x = \sum_{i=1}^{q} h_i \, \theta_y^i$$

$$\beta_y = \sum_{i=1}^{q} h_i \, \theta_x^i \qquad (5.89)$$

and

$$x = \sum_{i=1}^{q} h_i x_i \quad ; \quad y = \sum_{i=1}^{q} h_i y_i$$

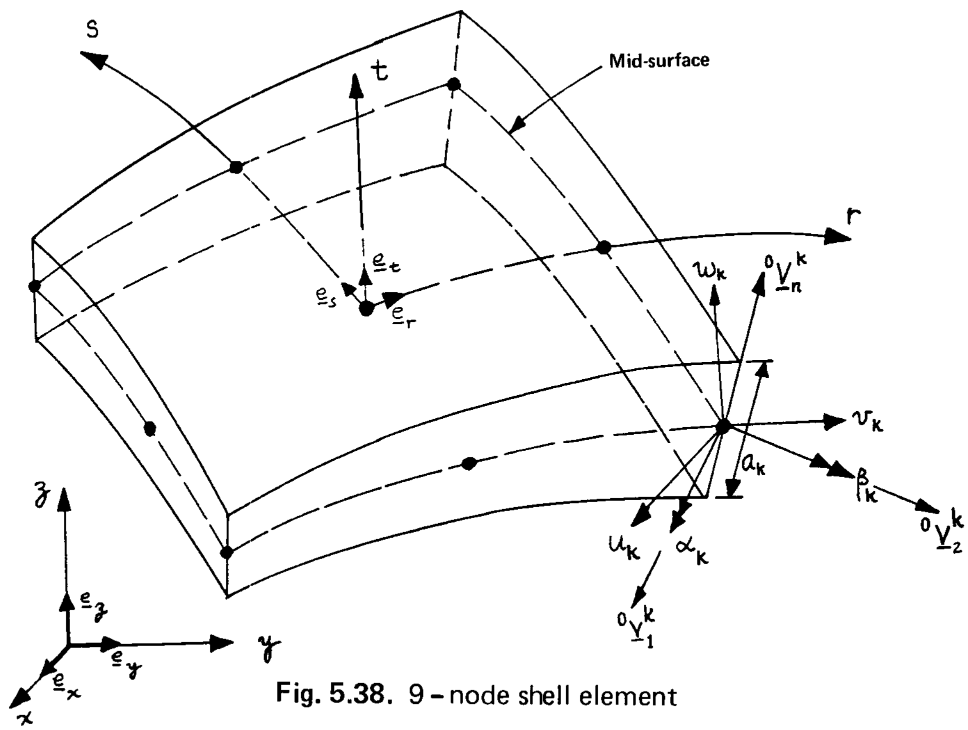


Fig. 5.38. 9 - node shell element

For shell elements we proceed as in the formulation of the general beam elements,

$${}^{\ell}x(r,s,t) = \sum_{k=1}^{q} h_k \; {}^{\ell}x_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \; {}^{\ell}V_{nx}^{k}$$

$${}^{\ell}y(r,s,t) = \sum_{k=1}^{q} h_k \; {}^{\ell}y_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \; {}^{\ell}V_{ny}^{k}$$

$${}^{\ell}z(r,s,t) = \sum_{k=1}^{q} h_k \; {}^{\ell}z_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \; {}^{\ell}V_{nz}^{k}$$

(5.90)

Therefore,

$$u(r,s,t) = \sum_{k=1}^{q} h_k u_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \, V_{nx}^k$$

$$v(r,s,t) = \sum_{k=1}^{q} h_k v_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \, V_{ny}^k$$

$$w(r,s,t) = \sum_{k=1}^{q} h_k w_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \, V_{nz}^k$$

where (5.91)

$$\underline{V}_n^k = {}^1\underline{V}_n^k - {}^0\underline{V}_n^k \qquad (5.92)$$

To express $\underline{V}_n^k$ in terms of rotations at the nodal - point k we define

$${}^0\underline{V}_1^k = \left(\underline{e}_y \times {}^0\underline{V}_n^k\right) \Big/ |\underline{e}_y \times {}^0\underline{V}_n^k| \qquad (5.93a)$$

$${}^0\underline{V}_2^k = {}^0\underline{V}_n^k \times {}^0\underline{V}_1^k \qquad (5.93b)$$

then

$$\underline{V}_n^k = -{}^0\underline{V}_2^k \, \alpha_k + {}^0\underline{V}_1^k \, \beta_k \qquad (5.94)$$

Finally, we need to recognize the use of the following stress-strain law

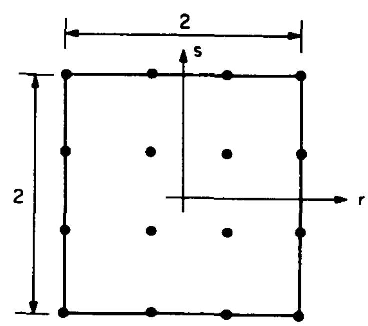
$$\underline{\tau} = \underline{C}_{sh}\,\underline{\varepsilon} \qquad (5.100)$$

$$\underline{\varepsilon}^T = [\varepsilon_{xx}\ \varepsilon_{yy}\ \varepsilon_{zz}\ \gamma_{xy}\ \gamma_{yz}\ \gamma_{zx}]$$

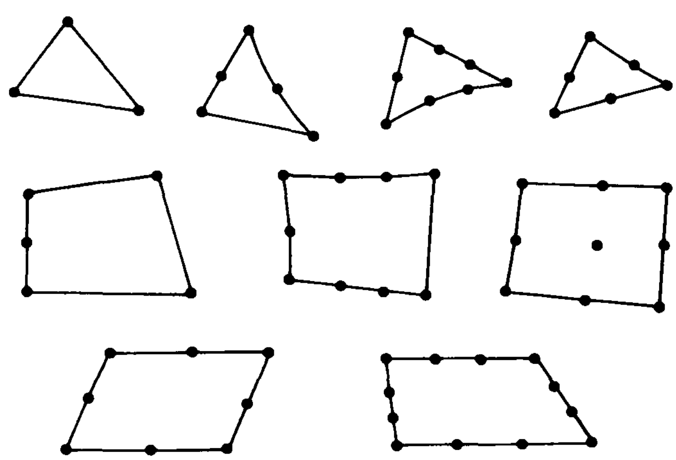
$$\underline{C}_{sh} = \underline{Q}_{sh}^T \left(\frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & \frac{1-\nu}{2} & 0 & 0 \\ & & & & \frac{1-\nu}{2} & 0 \\ \text{symmetric} & & & & & \frac{1-\nu}{2} \end{bmatrix} \right) \underline{Q}_{sh}$$

(5.101)

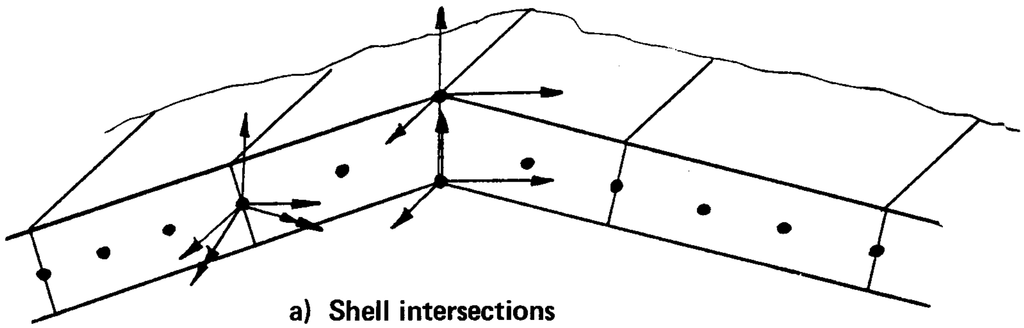
16 - node parent element with cubic interpolation



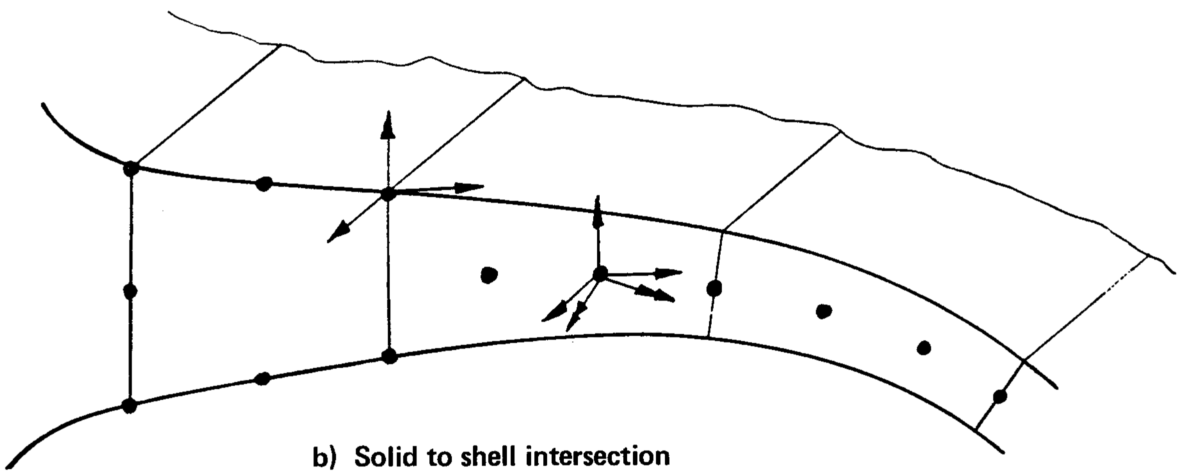
Some derived elements:



Variable - number - nodes shell element



a) Shell intersections



b) Solid to shell intersection

Fig. 5.39. Use of shell transition elements