GENERALIZED COORDINATE FINITE ELEMENT MODELS

LECTURE 4

57 MINUTES

LECTURE 4 Classification of problems; truss, plane stress, plane strain, axisymmetric, beam, plate and shell conditions; corresponding displacement, strain, and stress variables

Derivation of generalized coordinate models

One-, two-, three- dimensional elements, plate and shell elements

Example analysis of a cantilever plate, detailed derivation of element matrices

Lumped and consistent loading

Example results

Summary of the finite element solution process

Solution errors

Convergence requirements, physical explanations, the patch test

TEXTBOOK: Sections: 4.2.3, 4.2.4, 4.2.5, 4.2.6

Examples: 4.5, 4.6, 4.7, 4.8, 4.11, 4.12, 4.13, 4.14, 4.15, 4.16, 4.17, 4.18

DERIVATION OF SPECIFIC FINITE ELEMENTS

- Generalized coordinate finite element models

$$\underline{K}^{(m)} = \int_{V^{(m)}} \underline{B}^{(m)T} \underline{C}^{(m)} \underline{B}^{(m)} dV^{(m)}$$

$$\underline{R}_B^{(m)} = \int_{V^{(m)}} \underline{H}^{(m)T} \underline{f}^{B(m)} dV^{(m)}$$

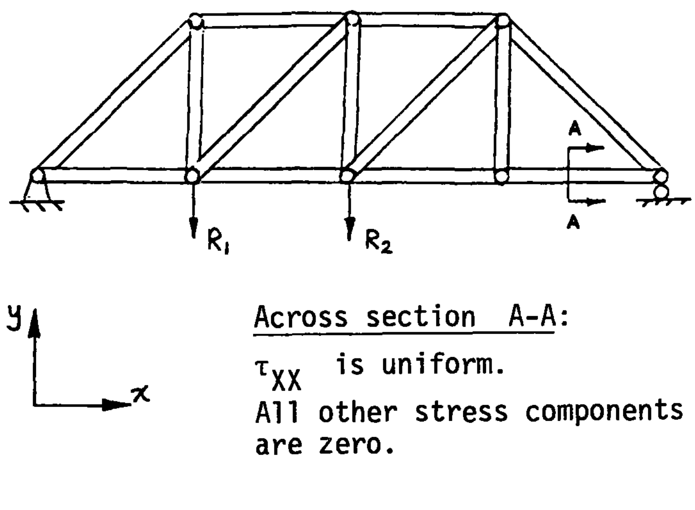
$$\underline{R}_S^{(m)} = \int_{S^{(m)}} \underline{H}^{S(m)T} \underline{f}^{S(m)} dS^{(m)}$$

etc.

In essence, we need

$\underline{H}^{(m)}$, $\underline{B}^{(m)}$, $\underline{C}^{(m)}$

- Convergence of analysis results



<u>Across section A-A:</u>

τ_{xx} is uniform.

All other stress components are zero.

Fig. 4.14. Various stress and strain conditions with illustrative examples.

(a) Uniaxial stress condition: frame under concentrated loads.

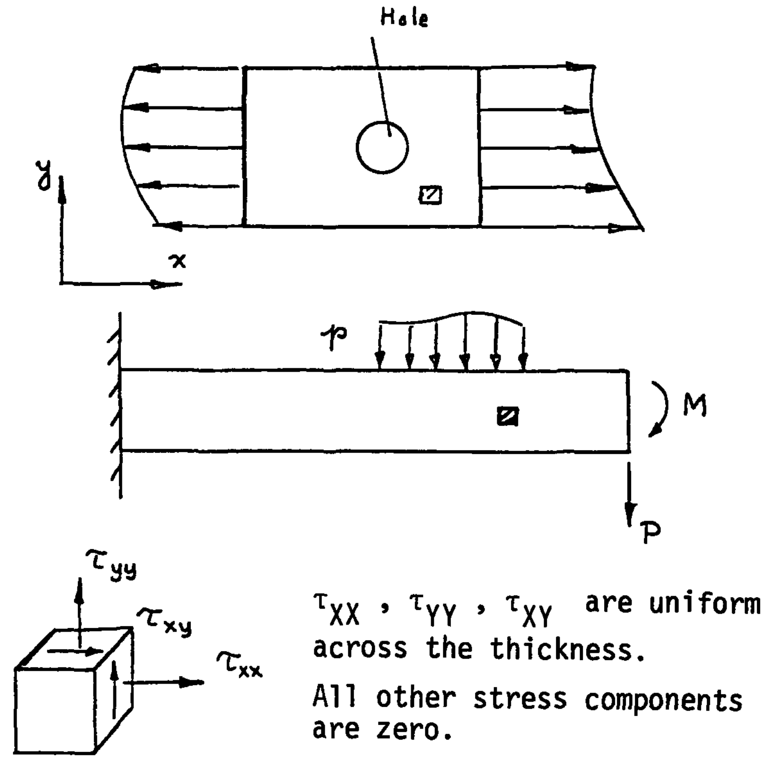
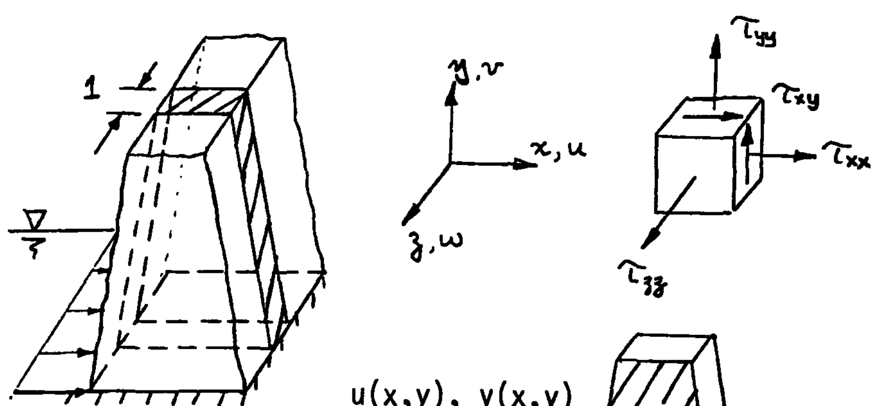


Fig. 4.14. (b) Plane stress conditions: membrane and beam under in-plane actions.



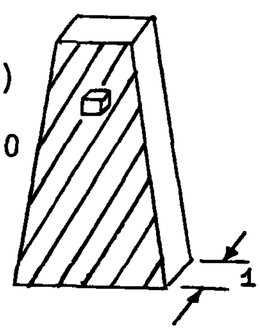


Fig. 4.14. (c) Plane strain condition: long dam subjected to water pressure.

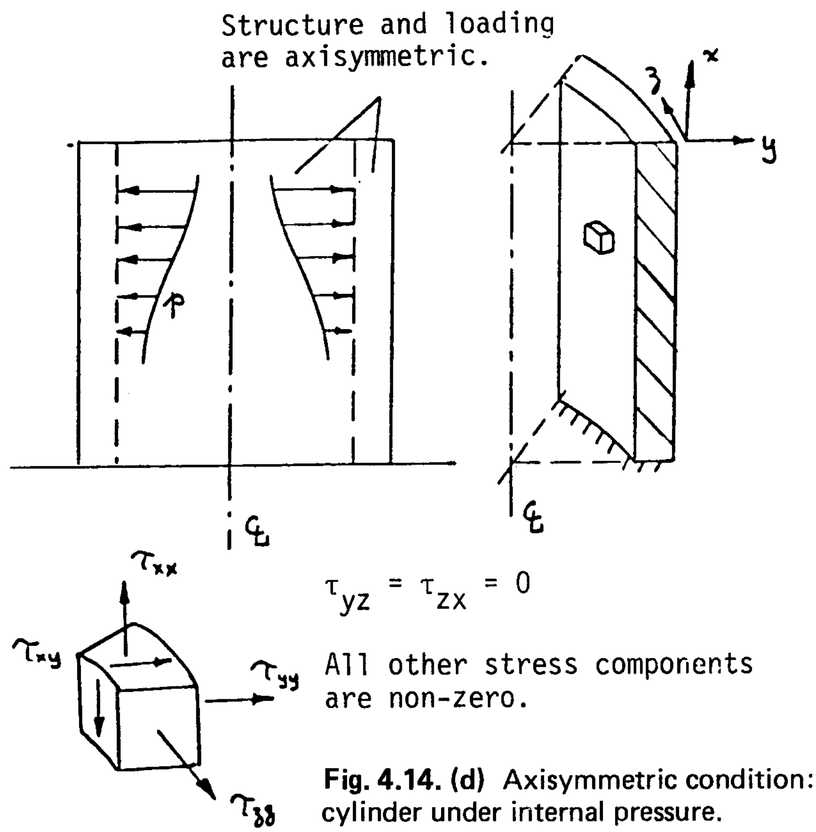


Fig. 4.14. (d) Axisymmetric condition: cylinder under internal pressure.

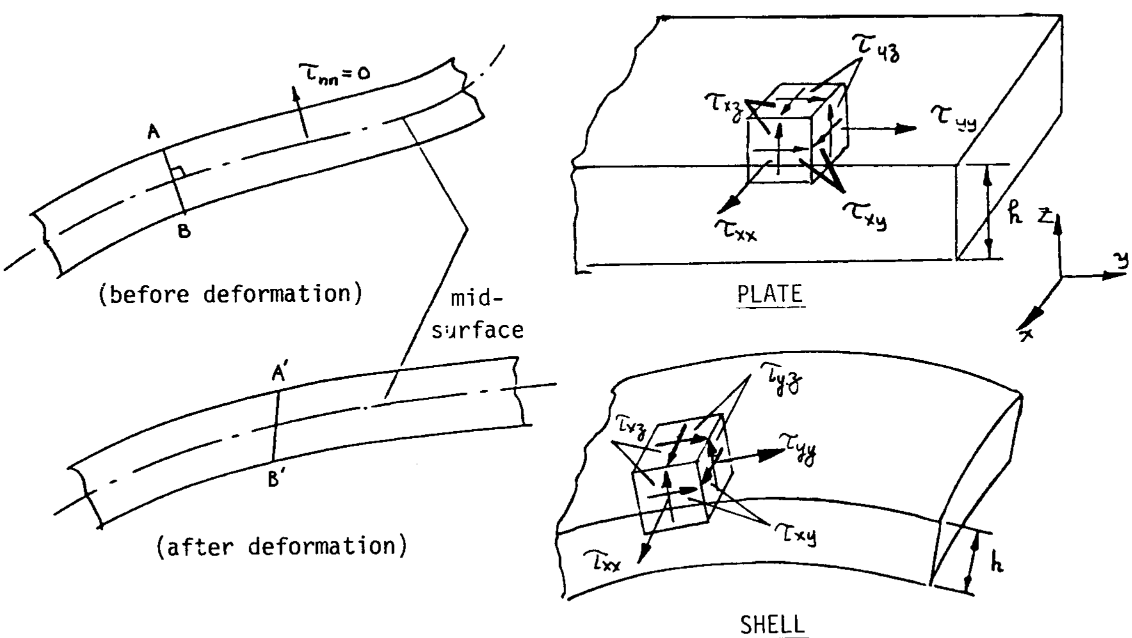


Fig. 4.14. (e) Plate and shell structures.

Problem	Displacement Components
Bar	u
Beam	w
Plane stress	u, v
Plane strain	u, v
Axisymmetric	u, v
Three-dimensional	u, v, w
Plate Bending	w

Table 4.2 (a) Corresponding Kinematic and Static Variables in Various Problems.

Problem	Strain Vector $\underline{\epsilon}^T$
Bar	$[\epsilon_{xx}]$
Beam	$[\kappa_{xx}]$
Plane stress	$[\epsilon_{xx} \quad \epsilon_{yy} \quad \gamma_{xy}]$
Plane strain	$[\epsilon_{xx} \quad \epsilon_{yy} \quad \gamma_{xy}]$
Axisymmetric	$[\epsilon_{xx} \quad \epsilon_{yy} \quad \gamma_{xy} \quad \epsilon_{zz}]$
Three-dimensional	$[\epsilon_{xx} \quad \epsilon_{yy} \quad \epsilon_{zz} \quad \gamma_{xy} \quad \gamma_{yz} \quad \gamma_{zx}]$
Plate Bending	$[\kappa_{xx} \quad \kappa_{yy} \quad \kappa_{xy}]$

Notation: $\epsilon_x = \frac{\partial u}{\partial x}, \epsilon_y = \frac{\partial v}{\partial y}, \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},$

$$\ldots, \kappa_{xx} = -\frac{\partial^2 w}{\partial x^2}, \kappa_{yy} = -\frac{\partial^2 w}{\partial y^2}, \kappa_{xy} = 2\frac{\partial^2 w}{\partial x\, \partial y}$$

Table 4.2 (b) Corresponding Kinematic and Static Variables in Various Problems.

Problem	Stress Vector $\underline{\tau}^T$
Bar	$[\tau_{xx}]$
Beam	$[M_{xx}]$
Plane stress	$[\tau_{xx} \quad \tau_{yy} \quad \tau_{xy}]$
Plane strain	$[\tau_{xx} \quad \tau_{yy} \quad \tau_{xy}]$
Axisymmetric	$[\tau_{xx} \quad \tau_{yy} \quad \tau_{xy} \quad \tau_{zz}]$
Three-dimensional	$[\tau_{xx} \quad \tau_{yy} \quad \tau_{zz} \quad \tau_{xy} \quad \tau_{yz} \quad \tau_{zx}]$
Plate Bending	$[M_{xx} \quad M_{yy} \quad M_{xy}]$

Table 4.2 (c) Corresponding Kinematic and Static Variables in Various Problems.

Problem	Material Matrix $\underline{C}$
Bar	E
Beam	EI
Plane Stress	$\dfrac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$

Table 4.3 Generalized Stress-Strain Matrices for Isotropic Materials and the Problems in Table 4.2.

ELEMENT DISPLACEMENT EXPANSIONS:

For one-dimensional bar elements

$$u(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \dots \quad (4.46)$$

For two-dimensional elements

$$u(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy + \alpha_5 x^2 + \dots$$

$$v(x,y) = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy + \beta_5 x^2 + \dots$$

(4.47)

For plate bending elements

$$w(x,y) = \gamma_1 + \gamma_2 x + \gamma_3 y + \gamma_4 xy + \gamma_5 x^2 + \dots$$

(4.48)

For three-dimensional solid elements

$$u(x,y,z) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 xy + \dots$$

$$v(x,y,z) = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 z + \beta_5 xy + \dots$$

$$w(x,y,z) = \gamma_1 + \gamma_2 x + \gamma_3 y + \gamma_4 z + \gamma_5 xy + \dots$$

(4.49)

Hence, in general

$$\underline{u} = \underline{\Phi}\,\underline{\alpha} \qquad (4.50)$$

$$\underline{\hat{u}} = \underline{A}\,\underline{\alpha}\ ;\ \underline{\alpha} = \underline{A}^{-1}\underline{\hat{u}} \qquad (4.51/52)$$

$$\underline{\epsilon} = \underline{E}\,\underline{\alpha}\ ;\ \underline{\tau} = \underline{C}\,\underline{\epsilon} \qquad (4.53/54)$$

$$\underline{H} = \underline{\Phi}\,\underline{A}^{-1}\ ;\ \underline{B} = \underline{E}\,\underline{A}^{-1} \qquad (4.55)$$

Example

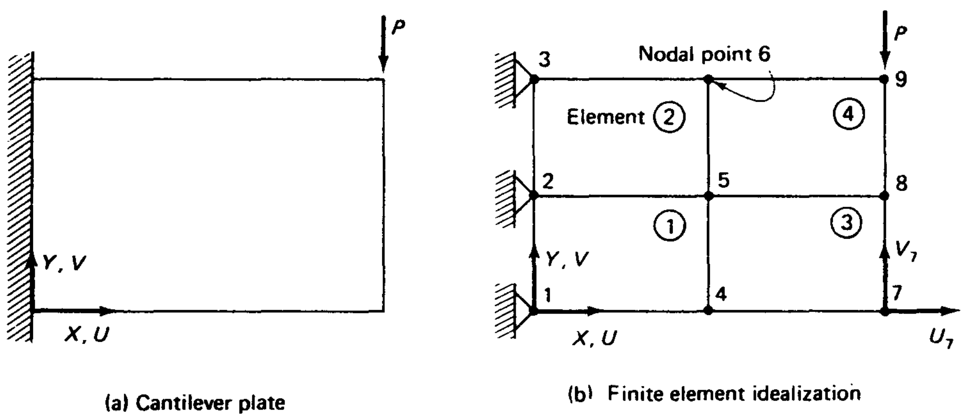


Fig. 4.5. Finite element plane stress analysis; i.e. $\tau_{ZZ} = \tau_{ZY} = \tau_{ZX} = 0$

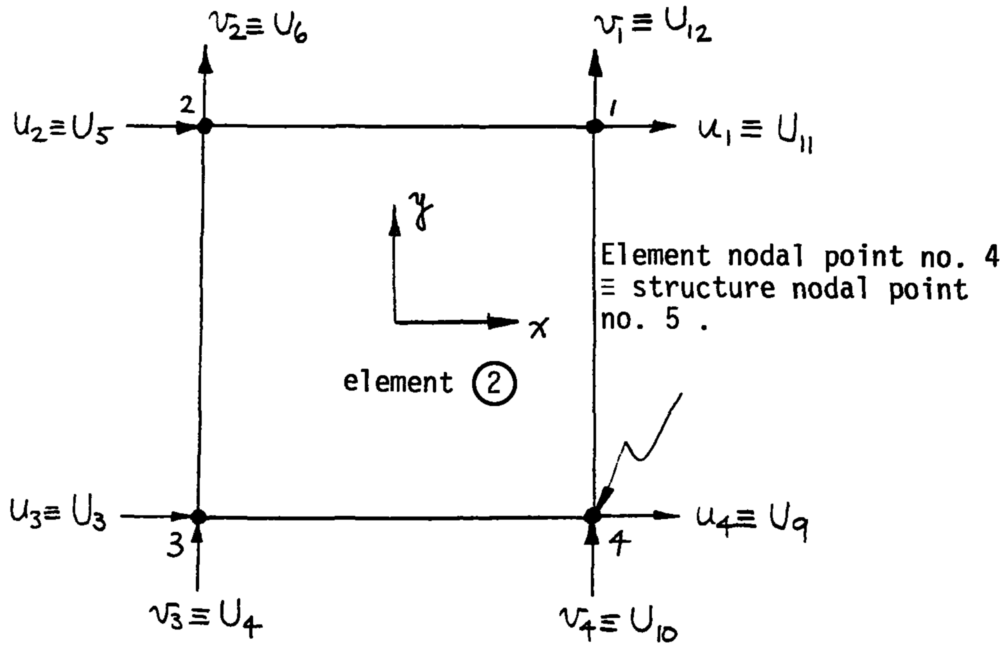


Fig. 4.6. Typical two-dimensional four-node element defined in local coordinate system.

For element 2 we have

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}^{(2)} = \underline{H}^{(2)} \underline{U}$$

where

$$\underline{U}^T = [U_1 \quad U_2 \quad U_3 \quad U_4 \; \cdots \; U_{17} \quad U_{18}]$$

To establish $\underline{H}^{(2)}$ we use:

$$u(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$

$$v(x,y) = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy$$

or

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \underline{\Phi}\ \underline{\alpha}$$

where

$$\underline{\Phi} = \begin{bmatrix} \underline{\phi} & \underline{0} \\ \underline{0} & \underline{\phi} \end{bmatrix}; \ \underline{\phi} = [1 \ \ x \ \ y \ \ xy]$$

and

$$\underline{\alpha}^T = [\alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4 \ \beta_1 \ \beta_2 \ \beta_3 \ \beta_4]$$

Defining

$$\hat{\underline{u}}^T = [u_1 \ u_2 \ u_3 \ u_4 \ \ v_1 \ v_2 \ v_3 \ v_4]$$

we have

$$\hat{\underline{u}} = \underline{A}\,\underline{\alpha}$$

Hence

$$\underline{H} = \underline{\Phi}\,\underline{A}^{-1}$$

Hence

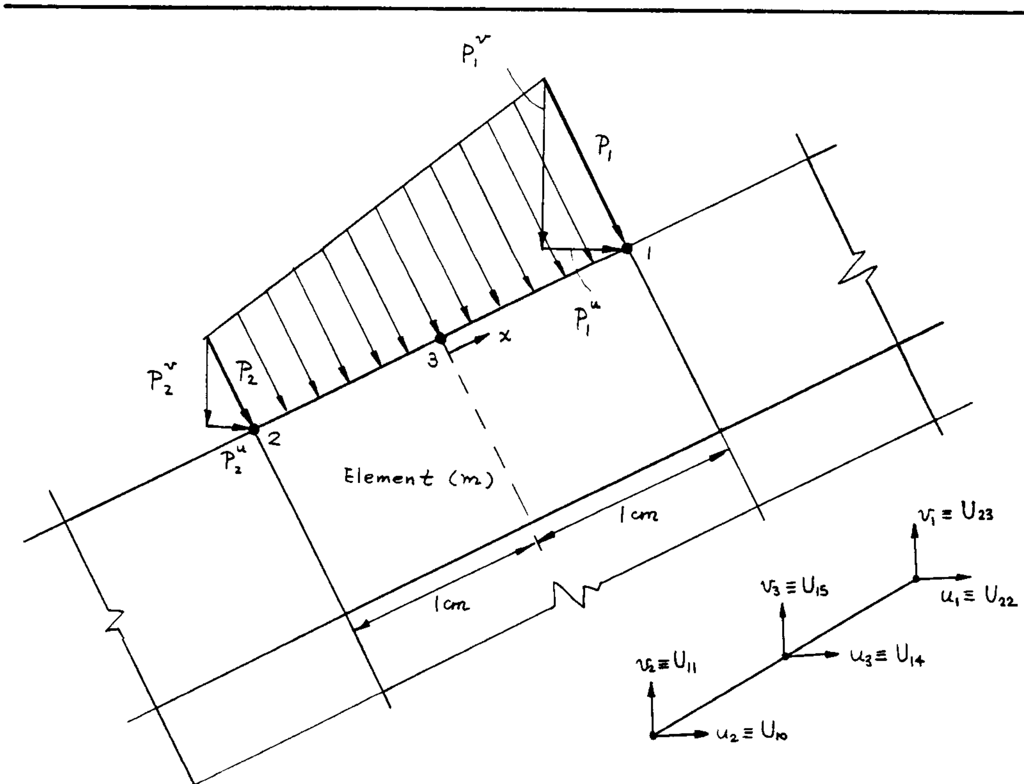
$$\underline{H} = \frac{1}{4}\left[\begin{array}{c|c|c} (1+x)(1+y) & & 0 \\ & \cdots & \\ 0 & & (1+x)(1+y) \end{array}\right]_{2\times 8}$$

and

$$\begin{array}{cccccccccc} & & u_3 & v_3 & u_2 & v_2 & & & u_4 & v_4 \\ U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} \end{array}$$

$$\underline{\mathbf{H}}^{(2)} = \left[\begin{array}{cc|cc|cc|cc|cc|cc|cc|c} 0 & 0 & H_{13} & H_{17} & H_{12} & H_{16} & 0 & 0 & H_{14} & H_{18} & H_{11} & H_{15} & 0 & 0 & \ldots\text{zeros}\ldots & 0 \\ 0 & 0 & H_{23} & H_{27} & H_{22} & H_{26} & 0 & 0 & H_{24} & H_{28} & H_{21} & H_{25} & 0 & 0 & \ldots\text{zeros}\ldots & 0 \end{array}\right]_{2\times 18}$$

$$\begin{array}{ccccc} u_1 & v_1 & \leftarrow\text{element degrees of freedom} & & \\ U_{11} & U_{12} & U_{13} & U_{14} & U_{18} \leftarrow\text{assemblage degrees of freedom} \end{array}$$



(a) Element layout (b) Local-global degrees of freedom

Fig. 4.7. Pressure loading on element (m)

In plane-stress conditions the element strains are

$$\underline{\epsilon}^T = [\epsilon_{XX} \quad \epsilon_{YY} \quad \gamma_{XY}]$$

where

$$\epsilon_{xx} = \frac{\partial u}{\partial x} \; ; \; \epsilon_{yy} = \frac{\partial v}{\partial y} \; ; \; \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Hence

$$\underline{B} = \underline{E}\,\underline{A}^{-1}$$

where

$$\underline{E} = \left[\begin{array}{cccc|cccc} 0 & 1 & 0 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x \\ 0 & 0 & 1 & x & 0 & 1 & 0 & y \end{array}\right]$$

ACTUAL PHYSICAL PROBLEM

GEOMETRIC DOMAIN
MATERIAL
LOADING
BOUNDARY CONDITIONS

↓

MECHANICAL IDEALIZATION

KINEMATICS, e.g. truss
plane stress
three-dimensional
Kirchhoff plate
etc.

MATERIAL, e.g. isotropic linear elastic
Mooney-Rivlin rubber
etc.

LOADING, e.g. concentrated
centrifugal
etc.

BOUNDARY CONDITIONS, e.g. prescribed displacements
etc.

YIELDS:
GOVERNING DIFFERENTIAL
EQUATIONS OF MOTION
e.g.

$$\frac{\partial}{\partial x}\left(EA \frac{\partial u}{\partial x}\right) = -p(x)$$

↓

FINITE ELEMENT SOLUTION

CHOICE OF ELEMENTS AND
SOLUTION PROCEDURES

YIELDS:
APPROXIMATE RESPONSE
SOLUTION OF MECHANICAL
IDEALIZATION

Fig. 4.23. Finite Element Solution Process

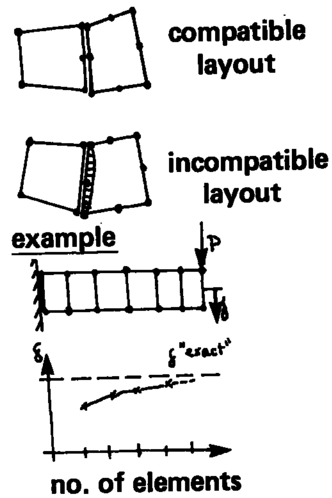
ERROR	ERROR OCCURRENCE IN	SECTION discussing error
DISCRETIZATION	use of finite element interpolations	4.2.5
NUMERICAL INTEGRATION IN SPACE	evaluation of finite element matrices using numerical integration	5.8.1 6.5.3
EVALUATION OF CONSTITUTIVE RELATIONS	use of nonlinear material models	6.4.2
SOLUTION OF DYNAMIC EQUILI-BRIUM EQUATIONS	direct time integration, mode superposition	9.2 9.4
SOLUTION OF FINITE ELEMENT EQUATIONS BY ITERATION	Gauss-Seidel, Newton-Raphson, Quasi-Newton methods, eigensolutions	8.4 8.6 9.5 10.4
ROUND-OFF	setting-up equations and their solution	8.5

Table 4.4 Finite Element Solution Errors

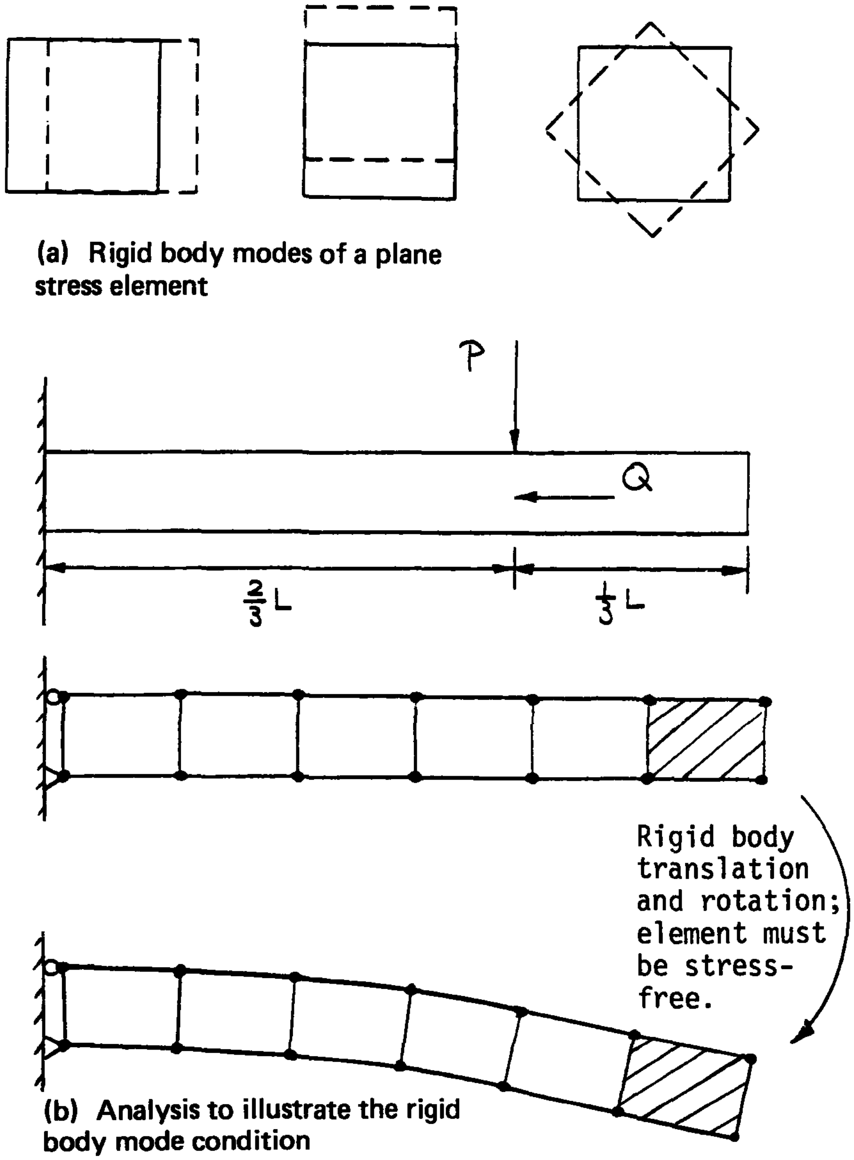
CONVERGENCE

Assume a compatible element layout is used, then we have monotonic convergence to the solution of the problem-governing differential equation, provided the elements contain:

1) all required rigid body modes
2) all required constant strain states



If an incompatible element layout is used, then in addition every patch of elements must be able to represent the constant strain states. Then we have convergence but non-monotonic convergence.



(a) Rigid body modes of a plane stress element

(b) Analysis to illustrate the rigid body mode condition

Fig. 4.24. Use of plane stress element in analysis of cantilever

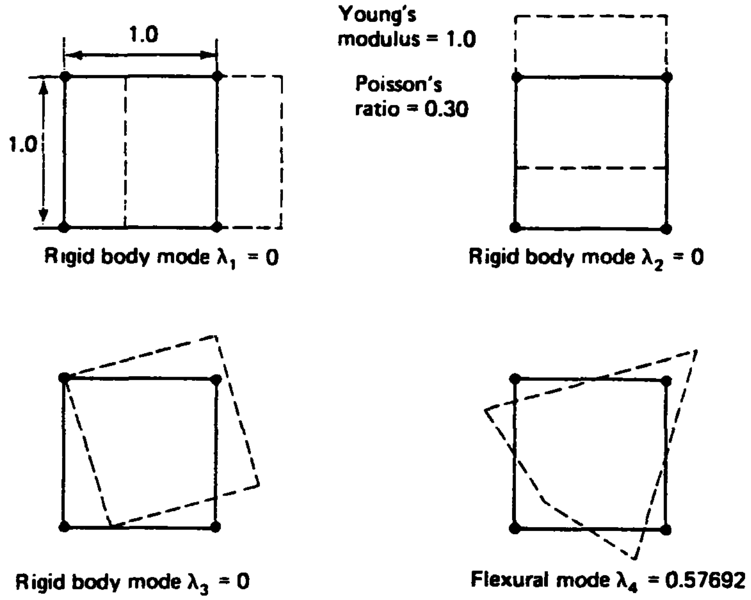


Fig. 4.25 (a) Eigenvectors and eigenvalues of four-node plane stress element

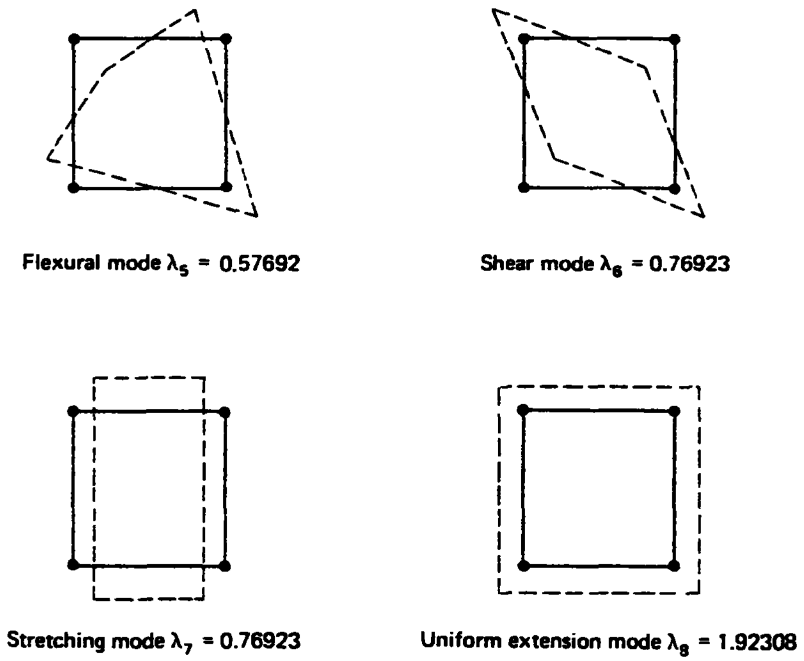
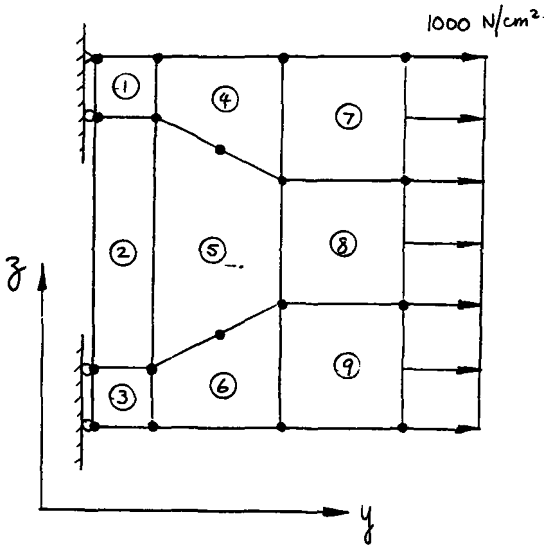
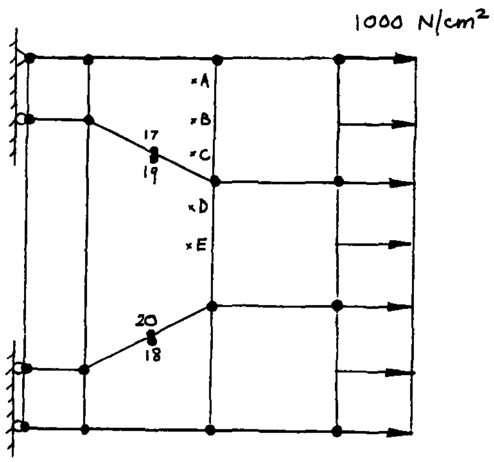


Fig. 4.25 (b) Eigenvectors and eigenvalues of four-node plane stress element



a) compatible element mesh; constant stress $\sigma_{yy} = 1000\ N/cm^2$ in each element.



b) incompatible element mesh; node 17 belongs to element 4, nodes 19 and 20 belong to element 5, and node 18 belongs to element 6.

Fig. 4.30 (a) Effect of displacement incompatibility in stress prediction

σ_{yy} stress predicted by the incompatible element mesh:

Point	$\sigma_{yy}(N/m^2)$
A	1066
B	716
C	359
D	1303
E	1303

Fig. 4.30 (b) Effect of displacement incompatibility in stress prediction