ANALYSIS OF CONTINUOUS SYSTEMS; DIFFERENTIAL AND VARIATIONAL FORMULATIONS

LECTURE 2

59 MINUTES

LECTURE 2

Basic concepts in the analysis of continuous systems

Differential and variational formulations

Essential and natural boundary conditions

Definition of C^{m-1} variational problem

Principle of virtual displacements

Relation between stationarity of total potential, the principle of virtual displacements, and the differential formulation

Weighted residual methods, Galerkin, least squares methods

Ritz analysis method

Properties of the weighted residual and Ritz methods

Example analysis of a nonuniform bar, solution accuracy, introduction to the finite element method

TEXTBOOK: Sections: 3.3.1, 3.3.2, 3.3.3

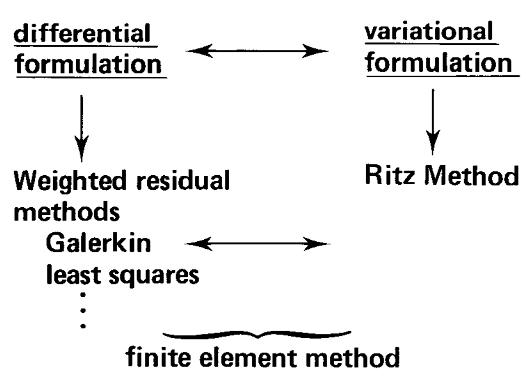
Examples: 3.15, 3.16, 3.17, 3.18, 3.19, 3.20, 3.21, 3.22, 3.23, 3.24, 3.25

BASIC CONCEPTS OF FINITE ELEMENT ANALYSIS – CONTINUOUS SYSTEMS

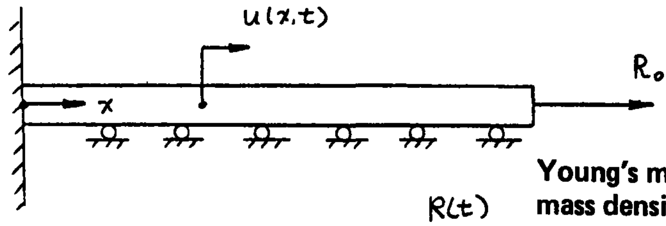
- We discussed some basic concepts of analysis of discrete systems

- Some additional basic concepts are used in analysis of continuous systems

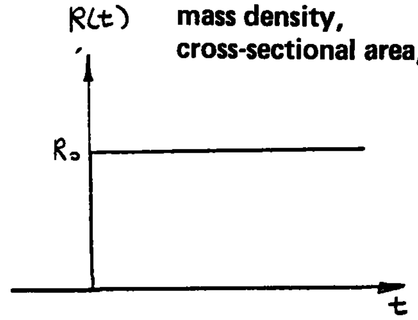
CONTINUOUS SYSTEMS



Example - Differential formulation



Young's modulus, E
mass density,
cross-sectional area, A

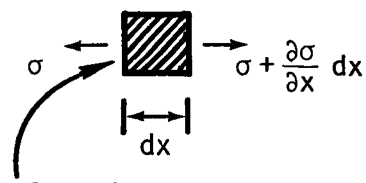


The problem governing differential equation is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}\ , \quad c = \sqrt{\frac{E}{\rho}}$$

Derivation of differential equation

The element force equilibrium requirement of a typical differential element is using d'Alembert's principle



Area A, mass density ρ .

$$\sigma A\big|_x + A\left.\frac{\partial\sigma}{\partial x}\right|_x dx - \sigma A\big|_x = \rho A \frac{\partial^2 u}{\partial t^2}$$

The constitutive relation is

$$\sigma = E\frac{\partial u}{\partial x}$$

Combining the two equations above we obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

The boundary conditions are

$$u(0,t) = 0 \quad \Rightarrow \text{essential (displ.) B.C.}$$

$$EA\,\frac{\partial u}{\partial x}(L,t) = R_0 \quad \Rightarrow \text{natural (force) B.C.}$$

with initial conditions

$$u(x,0) = 0$$

$$\frac{\partial u}{\partial t}(x,0) = 0$$

In general, we have

highest order of (spatial) derivatives in problem-governing differential equation is 2m.

highest order of (spatial) derivatives in essential b.c. is (m-1)

highest order of spatial derivatives in natural b.c. is (2m-1)

<u>Definition:</u>

We call this problem a C^{m-1} variational problem.

Example – Variational formulation

We have in general

$$\Pi = \mathcal{U} - \mathcal{W}$$

For the rod

$$\Pi = \int_0^L \frac{1}{2} EA \left(\frac{\partial u}{\partial x}\right)^2 dx - \int_0^L u\, f^B\, dx - u_L\, R$$

and

$$u_0 = 0$$

and we have $\delta \Pi = 0$

The stationary condition $\delta\Pi = 0$ gives

$$\int_0^L (EA \frac{\partial u}{\partial x})(\delta \frac{\partial u}{\partial x})\, dx - \int_0^L \delta u\, f^B\, dx - \delta u_L\, R = 0$$

This is the principle of virtual displacements governing the problem. In general, we write this principle as

$$\int_V \delta \underline{\epsilon}^T \underline{\tau}\, dV = \int_V \delta \underline{U}^T \underline{f}^B\, dV + \int_S \delta \underline{U}^{S^T} \underline{f}^S\, dS$$

or

$$\int_V \underline{\overline{\epsilon}}^T \underline{\tau}\, dV = \int_V \underline{\overline{U}}^T \underline{f}^B\, dV + \int_S \underline{\overline{U}}^{S^T} \underline{f}^S\, dS$$

(see also Lecture 3)

However, we can now derive the differential equation of equilibrium and the b.c. at $x = L$.

Writing $\frac{\partial \delta u}{\partial x}$ for $\frac{\delta \partial u}{\partial x}$, recalling that EA is constant and using integration by parts yields

$$-\int_0^L \left(EA \frac{\partial^2 u}{\partial x^2} + f^B\right) \delta u \, dx + \left[EA \left.\frac{\partial u}{\partial x}\right|_{x=L} - R\right] \delta u_L$$

$$- EA \left.\frac{\partial u}{\partial x}\right|_{x=0}$$

Since δu_0 is zero but δu is arbitrary at all other points, we must have

$$EA \frac{\partial^2 u}{\partial x^2} + f^B = 0$$

and

$$EA \left.\frac{\partial u}{\partial x}\right|_{x=L} = R$$

Also, $f^B = -A \rho \frac{\partial^2 u}{\partial t^2}$ **and**

hence we have

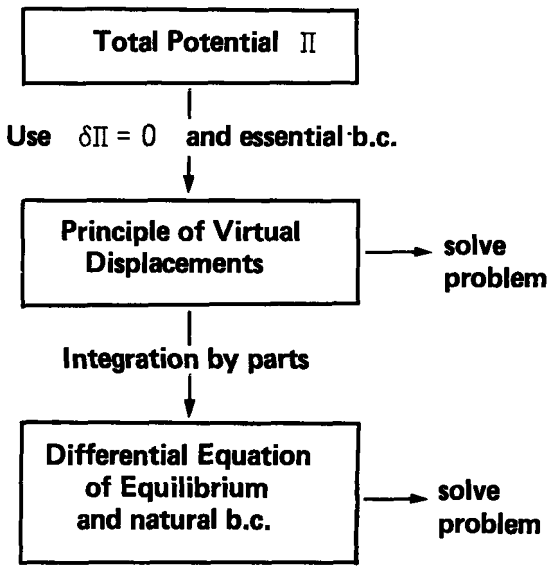
$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \; ; \; c = \sqrt{\frac{E}{\rho}}$$

The important point is that invoking $\delta\Pi = 0$ and using the essential b.c. only we generate

- the principle of virtual displacements
- the problem-governing differential equation
- the natural b.c. (these are in essence "contained in" Π, i.e., in w).

In the derivation of the problem-governing differential equation we used integration by parts

- the highest spatial derivative in Π is of order m.
- We use integration by parts m-times.



Weighted Residual Methods

Consider the steady-state problem

$$L_{2m}[\phi] = r \qquad (3.6)$$

with the B.C.

$$B_i[\phi] = q_i \Big|_{\text{at boundary}} \qquad i = 1,2,\ldots \qquad (3.7)$$

The basic step in the weighted residual (and the Ritz analysis) is to assume a solution of the form

$$\bar{\phi} = \sum_{i=1}^{n} a_i f_i \qquad (3.10)$$

where the f_i are linearly independent trial functions and the a_i are multipliers that are determined in the analysis.

Using the weighted residual methods, we choose the functions f_i in (3.10) so as to satisfy all boundary conditions in (3.7) and we then calculate the residual,

$$R = r - L_{2m}[\sum_{i=1}^{n} a_i\ f_i] \qquad (3.11)$$

The various weighted residual methods differ in the criterion that they employ to calculate the a_i such that R is small. In all techniques we determine the a_i so as to make a weighted average of R vanish.

Galerkin method

In this technique the parameters a_i are determined from the n equations

$$\int_D f_i \; R \; dD = 0 \quad i = 1,2,\ldots,n \qquad (3.12)$$

Least squares method

In this technique the integral of the square of the residual is minimized with respect to the parameters a_i ,

$$\frac{\partial}{\partial a_i} \int_D R^2 dD = 0 \qquad i = 1,2,\ldots,n$$

[The methods can be extended to operate also on the natural boundary conditions, if these are not satisfied by the trial functions.]

RITZ ANALYSIS METHOD

Let Π be the functional of the C^{m-1} variational problem that is equivalent to the differential formulation given in (3.6) and (3.7). In the Ritz method we substitute the trial functions $\bar{\Phi}$ given in (3.10) into Π and generate n simultaneous equations for the parameters a_i using the stationary condition on Π ,

$$\frac{\partial \Pi}{\partial a_i} = 0 \qquad i = 1,2,\ldots,n \qquad (3.14)$$

Properties

- The trial functions used in the Ritz analysis need only satisfy the essential b.c.

- Since the application of $\delta\Pi = 0$ generates the principle of virtual displacements, we in effect use this principle in the Ritz analysis.

- By invoking $\delta\Pi = 0$ we minimize the violation of the internal equilibrium requirements and the violation of the natural b.c.

- A symmetric coefficient matrix is generated, of form

$$\underline{K}\,\underline{U} = \underline{R}$$

Example

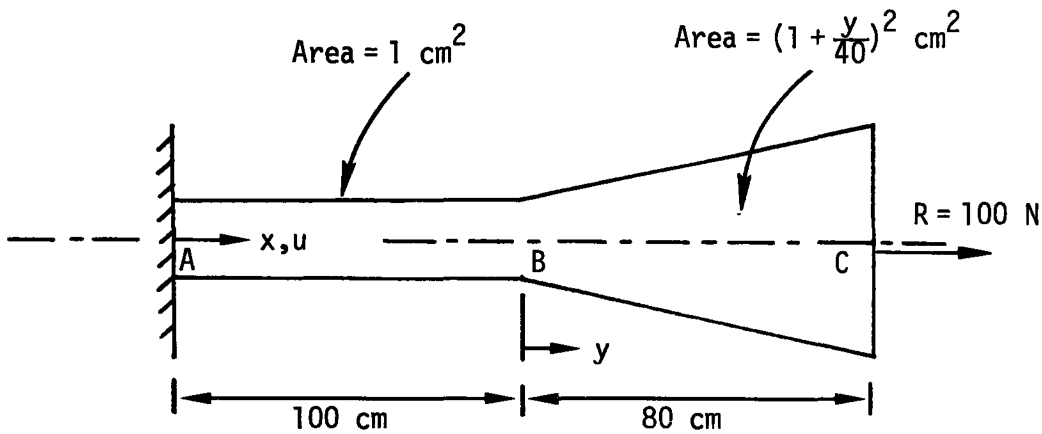


Fig. 3.19. Bar subjected to concentrated end force.

Here we have

$$\Pi = \int_0^{180} \frac{1}{2} EA\left(\frac{\partial u}{\partial x}\right)^2 dx - 100\, u\Big|_{x=180}$$

and the essential boundary condition is $u\big|_{x=0} = 0$

Let us assume the displacements

Case 1

$$u = a_1 x + a_2 x^2$$

Case 2

$$u = \frac{x\, u_B}{100} \qquad 0 \le x \le 100$$

$$u = \left(1 - \frac{x-100}{80}\right) u_B + \left(\frac{x-100}{80}\right) u_C$$

$$100 \le x \le 180$$

We note that invoking $\delta\Pi = 0$ we obtain

$$\delta\Pi = \int_0^{180} \left(EA \frac{\partial u}{\partial x}\right) \delta\left(\frac{\partial u}{\partial x}\right) dx - 100\, \delta u\Big|_{x=180} = 0$$

or the principle of virtual displacements

$$\int_0^{180} \left(\frac{\partial \delta u}{\partial x}\right)\left(EA \frac{\partial u}{\partial x}\right) dx = 100\, \delta u\Big|_{x=180}$$

$$\int_V \bar{\underline{\varepsilon}}^T \underline{\tau}\, dV = \overline{U}_i F_i$$

Exact Solution

Using integration by parts we obtain

$$\frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right) = 0$$

$$EA\left.\frac{\partial u}{\partial x}\right|_{x=180} = 100$$

The solution is

$$u = \frac{100}{E}x \; ; \; 0 \le x \le 100$$

$$u = \frac{10000}{E} + \frac{4000}{E} - \frac{4000}{E\left(1+\frac{x-100}{40}\right)} \; ;$$

$$100 \le x \le 180$$

The stresses in the bar are

$$\sigma = 100 \; ; \quad 0 \le x \le 100$$

$$\sigma = \frac{100}{\left(1+\frac{x-100}{40}\right)^2} \; ; \; 100 \le x \le 180$$

Performing now the Ritz analysis:

Case 1

$$\Pi = \frac{E}{2}\int_0^{100} (a_1+2a_2x)^2\, dx + \frac{E}{2}\int_{100}^{180} \left(1+\frac{x-100}{40}\right)^2 (a_1+2a_2x)^2\, dx - 100\, u\Big|_{x=180}$$

Invoking that $\delta\Pi = 0$ we obtain

$$E\begin{bmatrix} 0.4467 & 116 \\ 116 & 34076 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 18 \\ 3240 \end{bmatrix}$$

and

$$a_1 = \frac{128.6}{E} \; ; \quad a_2 = -\frac{0.341}{E}$$

Hence, we have the approximate solution

$$u = \frac{128.6}{E}\, x - \frac{0.341}{E}\, x^2$$

$$\sigma = 128.6 - 0.682\, x$$

Case 2

Here we have

$$\Pi = \frac{E}{2}\int_0^{100} \left(\frac{1}{100}\, u_B\right)^2 dx + \frac{E}{2}\int_{100}^{180} \left(1+\frac{x-100}{40}\right)^2$$

$$\left(-\frac{1}{80}\, u_B + \frac{1}{80}\, u_C\right)^2 dx$$

Invoking again $\delta\Pi = 0$ we obtain

$$\frac{E}{240}\begin{bmatrix} 15.4 & -13 \\ -13 & 13 \end{bmatrix}\begin{bmatrix} u_B \\ u_C \end{bmatrix} = \begin{bmatrix} 0 \\ 100 \end{bmatrix}$$

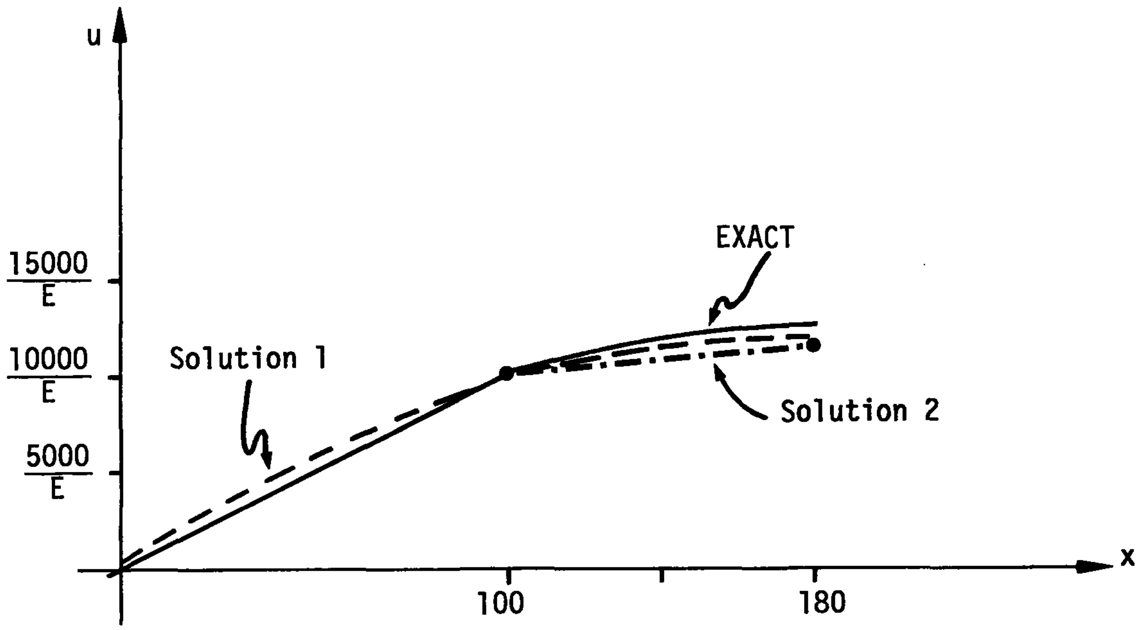
Hence, we now have

$$u_B = \frac{10000}{E} \quad ; \quad u_C = \frac{11846.2}{E}$$

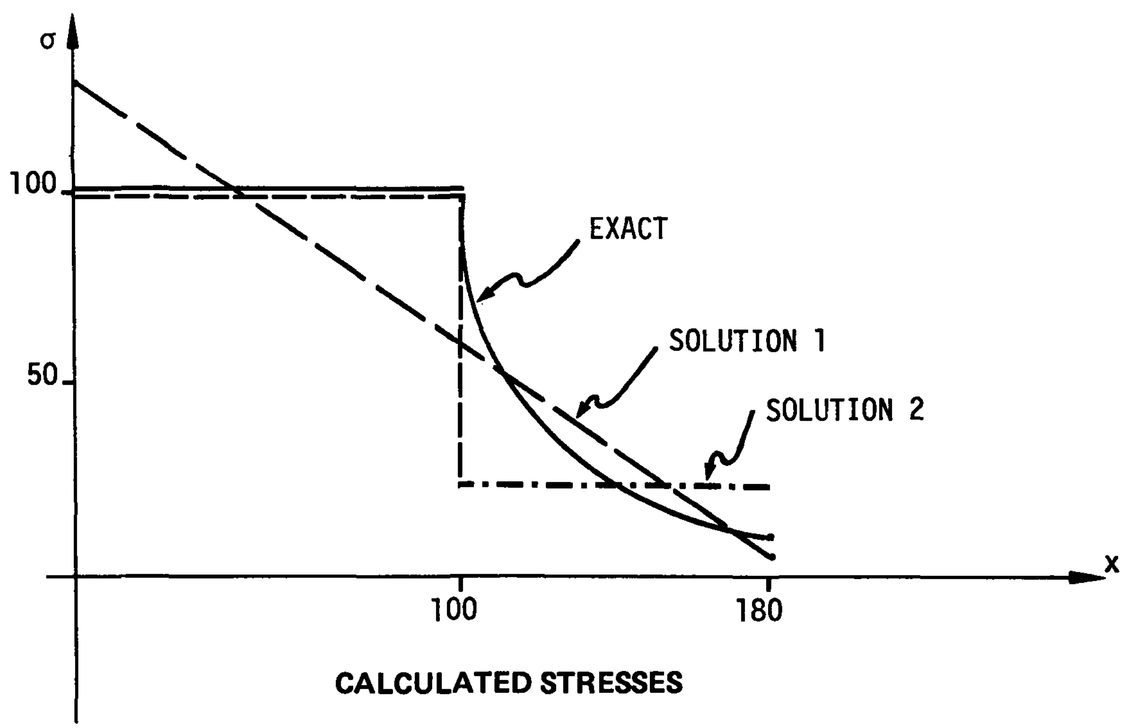
and

$$\sigma = 100 \quad ; \quad 0 \le x \le 100$$

$$\sigma = \frac{1846.2}{80} = 23.08 \qquad x \ge 100$$



CALCULATED DISPLACEMENTS



CALCULATED STRESSES

We note that in this last analysis

- we used trial functions that do not satisfy the natural b.c.
- the trial functions themselves are continuous, but the derivatives are discontinuous at point B.
  for a C^{m-1} variational problem we only need continuity in the (m-1)st derivatives of the functions; in this problem $m = 1$.
- domains A - B and B - C are finite elements and WE PERFORMED A FINITE ELEMENT ANALYSIS.