Unit 8: Complex Integration, Part I

1.8.1(L)

a. At first glance the solution to this exercise seems trivial. Most likely we would proceed just as in the real case and invoke the result that

$$\int_a^b f(x)dx = F(b) - F(a),$$

where $F' = f$. We would then simply replace x by z to obtain

$$\int_a^b f(z)dz = F(b) - F(a) \tag{1}$$

where $F'(z) = f(z)$; and a and b are now complex rather than real.

Using (1) in the present exercise we obtain, since

$$\frac{d(\frac{1}{2}z^2)}{dz} = z,$$

$$\int_0^{2i} zdz = \frac{1}{2}z^2 \Big|_{z=0}^{2} = \frac{1}{2}(2i)^2 - \frac{1}{2}(0)^2$$

$$= \frac{1}{2}(4i^2)$$

$$= -2. \tag{2}$$

Now, unless a very specific warning to the contrary were given, we might be tempted to argue that the result given in (2) was valid for the same reason that we were allowed to do this in our study of calculus of a real variable. In other words, at just about every turn in our treatment of complex variables we seemed to invoke the result that the complex numbers have the same structure as the real numbers.

The trouble is that our definition of the definite integral in complex variables is modeled after the concept of the two-dimensional line integral. That is, our limits of integration,

1.8.1(L) continued

a and b, are complex numbers, and hence points in the z-plane. Accordingly there are many paths that join a to b. Accordingly, the left side of equation (1) will, in general, depend on the path that joins a to b.

Now it happens, as we showed in the lecture, that if f(z) is analytic the value of the integral does not depend on the path that joins a to b; and in this case, the value of the integral is given by equation (1). The crucial point, however, is that if f(z) is not analytic the integral on the left side of (1) is ambiguous and its value depends on the path that joins a to b. This, in fact, is what the purpose of part (b) of this exercise is for. Namely, we shall show that $\int_0^{2i} \bar{z}\, dz$ depends on the path which joins 0 to 2i.

b. Recall that by definition

$$\int_c f(z)dz = \int_{t_0}^{t_1} f(z(t))z'(t)dt \tag{3}$$

where c is given in complex form by $z = z(t)$, $t_0 \leq t \leq t_1$*

Now in the present exercise $f(z) = \bar{z}$. Moreover, the curve c_1 is given by

$$z = it,\ 0 \leq t \leq 2. \tag{4}$$

There are other parametric forms to describe c_1, but as we mentioned in our earlier treatment of line integrals, the value of the integral does not depend on the choice of parameter.

Thus, with equation (4), equation (3) becomes

*Recall that if c is given in the xy-plane by the equation

$$\left.\begin{array}{l} x = x(t) \\ y = y(t) \end{array}\right\}$$

the complex interpretation in the z-plane is obtained by identifying x with the real part of z and y with the imaginary part. Thus, for example, the circle

$$\left.\begin{array}{l} x = \cos t \\ y = \sin t \end{array}\right\} 0 \leq t \leq 2\pi \text{ becomes } z = \cos t + i \sin t,\ 0 \leq t \leq 2\pi,$$

i.e., the set of complex numbers $\{z: z = \cos t + i \sin t,\ 0 \leq t \leq 2\pi\}$.

1.8.1(L) continued

$$\int_{c_1} \bar{z}\, dz = \int_0^2 \overline{it}\, \frac{d(it)}{dt}\, dt = \int_0^2 -it(i)\, dt$$

or

$$\int_{c_1} \bar{z}\, dz = \int_0^2 t\, dt = \frac{1}{2} t^2 \Big|_0^2 = 2. \tag{5}$$

In other words, equation (5) tells us that

$$\int_0^{2i} \bar{z}\, dz = 2$$

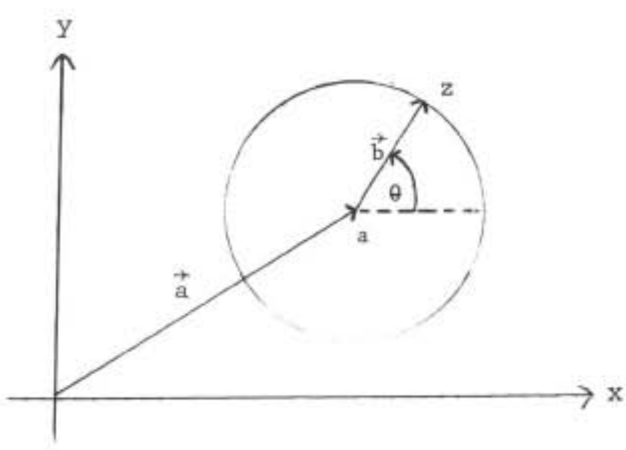
provided we are referring the straight line path which joins 0 to 2i.

On the other hand if c_2 is the set $\{z: z = i + e^{i\theta}, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\}$ * i.e., the semi-circle centered at i of radius 1 and joining 0 to 2i in the counter-clockwise direction, then f(z) is still $\bar{z}$ but since $z = i + e^{i\theta}$, $\bar{z}$ is now $\overline{i + e^{i\theta}}$ or $\bar{i} + \overline{e^{i\theta}}$ or

$$-i + e^{-i\theta}. \tag{6}$$

Using (6) with (3) we now see that

*$z = a + be^{i\theta}$ in the Argand diagram is the circle centered at a with radius b. To see this, draw the circle of radius b centered at a, and use vector rotation.



$$\begin{aligned} z &= \vec{a} + \vec{b} \\ &= a + |\vec{b}|\, e^{i\theta} \\ &= a + be^{i\theta} \end{aligned}$$

1.8.1(L) continued

$$\int_{c_2} f(z)dz = \int_{c_2} \bar{z}\, dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (-i + e^{-i\theta}) \frac{d(i + e^{i\theta})}{d\theta} d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (-i + e^{-i\theta})(ie^{i\theta}) d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (e^{i\theta} + i) d\theta$$

$$= \frac{1}{i} e^{i\theta} + i\theta \Bigg|_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= [\tfrac{1}{i} e^{i\frac{\pi}{2}} + i\ \tfrac{\pi}{2}] - [\tfrac{1}{i} e^{-i\frac{\pi}{2}} - i\ \tfrac{\pi}{2}]$$

$$= \tfrac{1}{i}\ (i) + i\ \tfrac{\pi}{2} - \tfrac{1}{i}(-i) + i\ \tfrac{\pi}{2}$$

$$= 2 + i\pi. \tag{7}$$

From (7) we conclude that

$$\int_0^{2i} \bar{z}\, dz = 2 + i\pi$$

provided we are talking about the semi-circular path c_2 which joins 0 to 2i.

Comparing (5) and (7) we see that

$$\int_0^{2i} \bar{z}\, dz$$

depends on the path that joins 0 and 2i.

It is crucial to observe that

$$\int_0^{2i} \bar{z}\, dz$$

1.8.1(L) continued

exists for each path but the value depends on the path. Moreover $\bar{z}$ is a continuous function. Hence, for

$$\int_a^b f(z)dz$$

to be well-defined, independent of path, f(z) must be analytic. Continuity is no longer enough (as it was in the real case).

By way of review, the proof that

$$\int_a^b f(z)dz$$

was independent of path if f(z) is analytic was a direct consequence of the Cuachy-Riemann conditions. Namely, in Cartesian form

$$\int_a f(z)dz = \int_c (u + iv)(dx + idy)$$

$$= \int_c [(udx - vdy) + i[vdx + udy].$$

Hence,

$$\int_c f(z)dz = \int_c \underbrace{udx - vdy}_{\substack{\text{exact} \leftrightarrow \\ u_y = -v}} + i \int_c \underbrace{vdx + udy}_{\substack{\text{exact} \leftrightarrow \\ u_x = v_y}}$$

These are the
Cauchy-Riemann conditions.

As a check on part (a), let us compute

$$\int_0^{2i} zdz$$

along the two paths c_1 and c_2 given in part (6).

We have f(z) = z and for c_1, z = ti $0 \leq t \leq 2$. Hence,

1.8.1(L) continued

$$\int_{c_1} zdz = \int_0^2 (ti)\frac{d(ti)}{dt}\,dt$$

$$= \int_0^2 (ti)i\ dt$$

$$= -\int_0^2 tdt$$

$$= -\frac{1}{2}t^2 \Big|_{t=0}^{2}$$

$$= -2 \qquad (3)$$

which checks with equation (2).

Similarly, with c_2 given by $z = i + e^{i\theta}, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ we obtain,

$$\int_{c_2} zdz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (i + e^{i\theta})\frac{d(i + e^{i\theta})}{d\theta}d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (i + e^{i\theta})\ ie^{i\theta}d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (-e^{i\theta} + ie^{2i\theta})d\theta$$

$$= -\frac{1}{i}e^{i\theta} + \frac{1}{2}e^{2i\theta} \Big|_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= ie^{i\theta} + \frac{1}{2}e^{2i\theta} \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

1.8.1(L) continued

$$= [ie^{\frac{i\pi}{2}} + \frac{1}{2}e^{\pi i}] - [ie^{-\frac{i\pi}{2}} + \frac{1}{2}e^{-\pi i}]$$

$$= [i(i) + \frac{1}{2}(-1)] - [i(-i) + \frac{1}{2}(-1)]$$

$$= 2i^2$$

$$= -2 \tag{9}$$

which again checks with equation (2).

As a final note if we want to see why the answer to part (a) is -2 without making reference to a specific path and without referring to real and imaginary parts, we have:

Let c be given by $z = z(t)$, $t_o \le t \le t_1$, where $z(t_o) = 0$ and $z(t_1) = 2i$. Then

$$\int_c z\,dz = \int_{t_o}^{t_1} z\,\frac{dz}{dt}\,dt$$

$$= \int_{t_o}^{t_1} \frac{d[\frac{1}{2}z^2]dt}{dt}$$

$$= \frac{1}{2}z^2 \Big|_{t=t_o}^{t_1}$$

$$= \frac{1}{2}[z^2(t_1) - z^2(t_o)]$$

and since $z(t_1) = 2i$ and $z(t_o) = 0$, we obtain

$$\int_c z\,dz = \frac{1}{2}[(2i)^2 - 0^2] = -2.$$

1.8.2

Since $2e^{2z}$ is analytic (i.e., $\frac{d}{dz}[2e^{2z}] = 4e^{2z}$) and since $\frac{d}{dz}(e^{2z}) = 2e^{2z}$, we have that

$$\int_1^i 2e^{2z}dz = e^{2z}\Big|_{z=1}^{i} \tag{1}$$

independently of the path which joins $z = 1$ to $z = i$. That is from (1),

$$\int_1^i 2e^{2z}dz = e^{i2} - e^2$$

$$= (\cos 2 + i \sin 2)^{*} - e^2$$

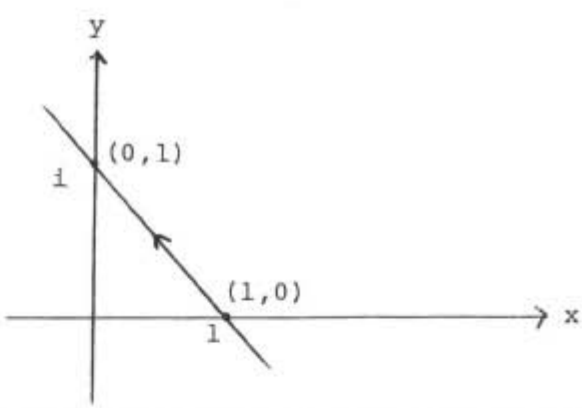
$$= (\cos 2 - e^2) + i \sin 2 \tag{2}$$

where (2) holds for each path (in particular, the given path) that joins 1 to i.

1.8.3

Since $\bar{z}^{\,2}$ is not analytic we must be prepared to have $\int_1^i \bar{z}^2 dz$ depend on the path which joins $z = 1$ to $z = i$.

a. Suppose the path c is the straight line which joins 1 to i. We then have



$$\left.\begin{array}{l}\dfrac{y-1}{x} = -1 \\ y = -x + 1\end{array}\right\} \text{x varies from 1 to 0.}$$

*As usual keep in mind that sin 2 and cos 2 use 2 as a number. Thus, if an angular interpretation is required sin 2 and cos 2 refer to sin (2 radians) and cos (2 radians).

1.8.3 continued

Therefore,

$$z = x + (-x + 1)i$$

$$dz = dx - i\,dx = (1 - i)dx$$

$$\bar{z} = x + (x - 1)i$$

$$\bar{z}^2 = x^2 + 2x(x - 1)i + (x - 1)^2 i^2$$

$$= x^2 + 2x(x - 1)i - x^2 + 2x - 1$$

$$= (2x - 1) + 2x(x - 1)i.$$

Hence,

$$\int_C \bar{z}^2 dz = \int_1^0 [(2x - 1) + 2x(x - 1)i](1 - i)dx$$

$$= -\int_0^1 \{[2x - 1 + 2x(x - 1)] + i[2x(x - 1) - (2x - 1)]\}dx$$

$$\int_C \bar{z}^2 dz = -\int_0^1 (2x^2 - 1) - i\int_0^1 (2x^2 - 4x + 1)dx$$

$$= -\frac{2}{3}x^3 - x\Big|_{x=0}^{1} - \frac{2}{3}x^3 + 2x^2 - x\Big|_{x=0}^{1} i$$

$$= \frac{1}{3} + \frac{1}{3}i$$

$$= \frac{1}{3}(1 + i).$$

b. If c is now given by $z = e^{i\theta}$, $0 \le \theta \le \frac{\pi}{2}$ we have

$$dz = ie^{i\theta}d\theta$$

$$\bar{z} = e^{-i\theta}$$

$$\bar{z}^2 = e^{-i2\theta}.$$

1.8.3 continued

Hence,

$$\int_C^{\frac{\pi}{2}} z^2 dz = \int_0^{\frac{\pi}{2}} e^{-i2\theta} ie^{i\theta} d\theta$$

$$= i\int_0^{\frac{\pi}{2}} e^{-i\theta} d\theta$$

$$= i\left[\frac{1}{-i} e^{-i\theta}\right] \Bigg|_{\theta=0}^{\frac{\pi}{2}}$$

$$= -e^{-i\theta} \Bigg|_{\theta=0}^{\frac{\pi}{2}}$$

$$= -e^{-i\frac{\pi}{2}} - (-e^{0})$$

$$= 1 - e^{-i\frac{\pi}{2}} = 1 - (-i) = 1 + i.$$

1.8.4(L)

Our main aim in this exercise, other than to supply more drill on the technique of integrating complex-valued functions, is to show how a knowledge of complex integration sheds light on certain "mysteries" of the calculus of a single real variable.

For example, in Part I of our course we saw that the improper integral

$$\int_{-1}^{2} \frac{dx}{x^2}$$

(improper because $x = 0$ is included in the interval of integration diverged). Yet, if we failed to notice this and applied the Fundamental Theorem (which didn't apply in this case since the integrand was discontinuous at $x = 0$) we obtained

$$\int_{-1}^{2} \frac{dx}{x^2} = -\frac{1}{x} \Bigg|_{x=-1}^{2} = -\frac{1}{2} - 1 = -\frac{3}{2}. \qquad (1)$$

1.8.4(L) continued

We knew that (1) was preposterous in the sense that since $\frac{1}{x^2}$ could not be negative,

$$\int_{-1}^{2} \frac{dx}{x^2}$$

could not be negative; and thus, the right side of (1) could not be correct. What we did not know at the time was what was significant about $-\frac{3}{2}$; that is, where did it come from and what did not mean? We shall now use complex integration to solve the "mystery"!

a. We have already seen that $\int_a^b f(z)dz$ is unambiguous provided that (i) a and b are points in a region R in which f is analytic and (ii) the smooth path that goes from a to b lies entirely within R. Moreover, in this case, just as in the real case, $\int_a^b f(z)dz = F(b) - F(a)$ where $F'(z) = f(z)$ for all $z \in R$.

Applying this to the present situation we see that

$$\int_{-1}^{2} \frac{dz}{z^2} = -\frac{1}{z}\Bigg|_{z=-1}^{2} = -\frac{1}{2} - 1 = -\frac{3}{2} \qquad (2)$$

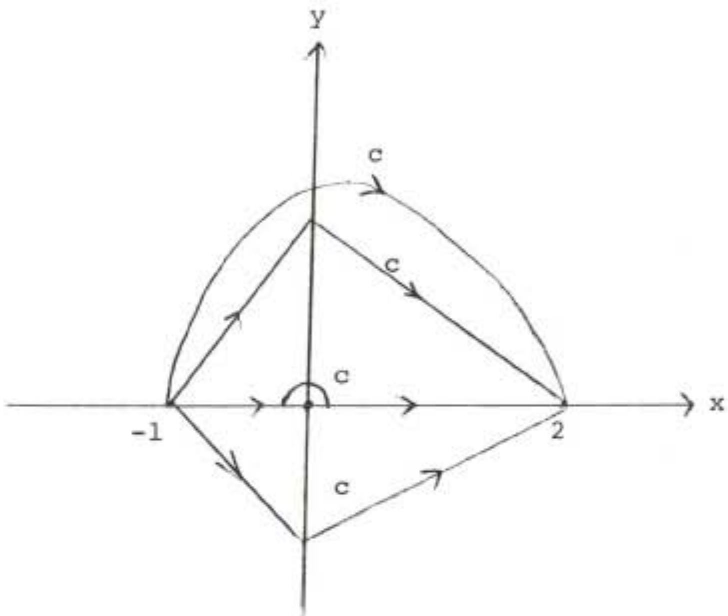
provided that the path from -1 to 2 lies in a region which does not include $z = 0$ (i.e., $f(z) = \frac{1}{z^2}$ is analytic everywhere except at $z = 0$).

Notice that (1) and (2) appear to be identical except for the symbols x and z. The big difference is that in (1) our path (which must be along the real axis since f is a [real] function of a real variable) passes through a singularity* of f while in (2) it does not since in (2) our path avoids $z = 0$.

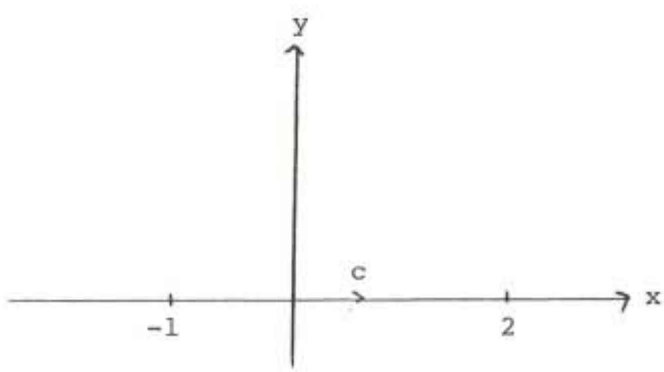
*In talking about derivatives of a function f (whether real or complex) we refer to a point at which f does not exist as a singular point (or singularity) of f.

1.8.4(L) continued

For example, equation (2) applies for each of the paths, c, shown below



Equation (2), however, does not apply to c where



since then c passes through the singular point z = 0.

More analytically, in terms of the definition

$$\int_c f(z)\,dz = \int_{t_o}^{t_1} f(z(t)z'(t)dt.$$

Suppose c is any curve which joins -1 to 2 without passing through 0. Say, c is given by

1.8.4(L) continued

$z = z(t),\ t_o \le t \le t_1$

where (3)

$z(t_o) = -1$ and $z(t_1) = 2$

Then,

$$\int_C \frac{dz}{z^2} = \int_{t_o}^{t_1} \frac{1}{z^2}\frac{dz}{dt}\,dt$$

$$= \int_{t_o}^{t_1} \frac{d[-\frac{1}{z}]}{dt}\,dt$$

or

$$\int_C \frac{dz}{z^2} = -\frac{1}{z(t)}\Bigg|_{t=t_o}^{t_1}$$

$$= -\frac{1}{z(t_1)} - [-\frac{1}{z(t_1)}]$$

$$= \frac{1}{z(t_o)} - \frac{1}{z(t_1)}$$

So that by (3),

$$\int_C \frac{dz}{z^2} = \frac{1}{-1} - [\tfrac{1}{2}] = -\tfrac{3}{2}. \quad (4)$$

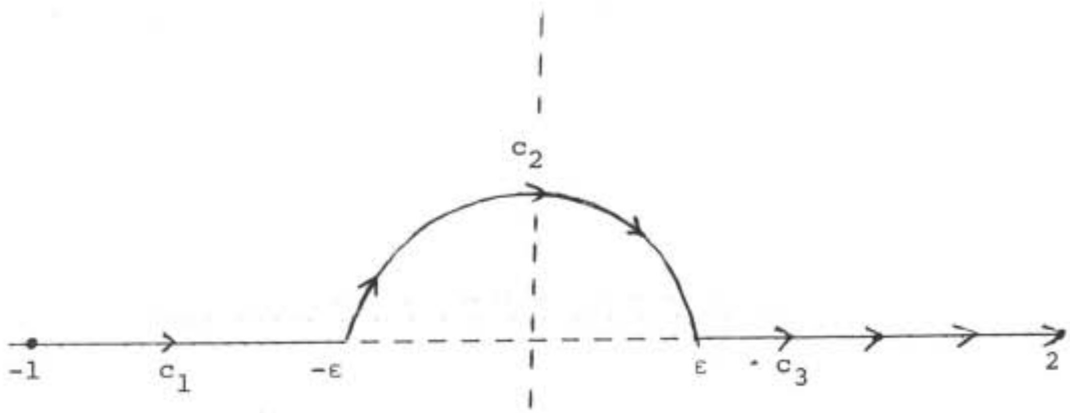
In other words, we also see from (4) that

$$\int_{-1}^{2} \frac{dz}{z^2} = -\frac{3}{2}$$

along any path joining -1 to 2 provided only that the path does not pass through the origin.

1.8.4(L) continued

b. We have



Therefore, letting $c = c_1 \cup c_2 \cup c_3$ we have

$$\int_c \frac{dz}{z^2} = \int_{c_1} \frac{dz}{z^2} + \int_{c_2} \frac{dz}{z^2} + \int_{c_3} \frac{dz}{z^2} \,. \tag{5}$$

Now, c_1 is given by $z = t \quad -1 < t \leq -\varepsilon$

Hence,

$$\int_{c_1} \frac{dz}{z^2} = \int_{-1}^{-\varepsilon} \frac{1}{t^2} \frac{d(t)}{dt} dt$$

$$= \int_{-1}^{-\varepsilon} \frac{dt}{t^2}$$

$$= -\frac{1}{t} \Bigg|_{t=-1}^{t=-\varepsilon}$$

$$= -\frac{1}{(-\varepsilon)} - [-\frac{1}{(-1)}]$$

$$= \frac{1}{\varepsilon} - 1. \tag{6}$$

1.8.4(L) continued

Next we observe that c_2 is the circle $z = e^{i\theta}$ where θ varies from π to 0 [or, in other words, $c_2 = -c_4$ where $c_4 = \{z:\varepsilon z = \varepsilon e^{i\theta}, 0 \leq \theta \leq \pi\}$. Hence

$$\int_{c_2} \frac{dz}{z^2} = \int_{\pi}^{0} \frac{1}{\varepsilon^2 e^{2i\theta}} \frac{d(\varepsilon e^{i\theta})}{d\theta}\, d\theta$$

$$= - \int_0^{\pi} \frac{\varepsilon i e^{i\theta}}{\varepsilon^2 e^{2i\theta}}\, d\theta.$$

Thus,

$$\int_{c_2} \frac{dz}{z^2} = - \int_0^{\pi} \frac{i}{\varepsilon} e^{-i\theta} d\theta$$

$$= - \frac{1}{\varepsilon}[-e^{-i\theta} \Big|_{\theta=0}^{\pi}$$

$$= \frac{1}{\varepsilon} e^{-i\theta} \Big|_0^{\pi}$$

$$= \frac{1}{\varepsilon}(e^{-i\pi} - e^{-i0})$$

$$= \frac{(-1 - 1)}{\varepsilon}$$

$$= - \frac{2}{\varepsilon}\,.$$

Finally, c_3 is given by $z = t, \varepsilon \leq t \leq 1$ so that

$$\int_{c_3} \frac{dz}{z^2} = \int_{\varepsilon}^{2} \frac{1}{z^2} \frac{d(t)}{dt}\, dt$$

$$= \int_{\varepsilon}^{2} \frac{dt}{t^2}$$

1.8.4(L) continued

$$= -\frac{1}{t}\Bigg|_{t=\varepsilon}^{2}$$

$$= \frac{1}{\varepsilon} - \frac{1}{2}. \tag{8}$$

Adding the results of (6), (7), and (8) we obtain from (5) that

$$\int_c \frac{dz}{z^2} = \frac{1}{\varepsilon} - 1 - \frac{2}{\varepsilon} + \frac{1}{\varepsilon} - \frac{1}{2}$$

$$= -\frac{3}{2} \tag{9}$$

which agrees with (2).

The crucial point in part (b) is that our result holds for <u>every</u> $\varepsilon > 0$. In particular, if we recognize that our contour c depends on ε by writing c_ε instead of c, we see from (9) that

$$\lim_{\varepsilon \to 0} \int_{c_\varepsilon} \frac{dz}{z^2} = -\frac{3}{2}\ .$$

We view $\lim_{\varepsilon \to 0} c_\varepsilon$ as being an <u>indented</u> contour. That is, we delete a tiny interval (a <u>dot</u>, so to speak) $[-\varepsilon,\varepsilon]$ and replace it by the upper half of the semi-circle having $-\varepsilon$ and ε as endpoints of a diameter.

If we then agree to identify

$$\int_{-1}^{2} \frac{dz}{x^2}$$

with

$$\int_c \frac{dz}{z^2}$$

for sufficiently small $\varepsilon > 0$ then we may say that

$$\int_{-1}^{2} \frac{dx}{x^2} = -\frac{3}{2}$$

1.8.4(L) continued

without any reference to

$$\frac{1}{x^2} = \infty$$

when $x = 0$. In other words, using an indented contour we say that

$$\int_{-1}^{2} \frac{dx}{x^2} \approx \int_{C_\varepsilon} \frac{dz}{z^2} = -\frac{3}{2}.$$

Pictorially,

1. Along the given curve, c

$$\int_{C_\varepsilon} \frac{dz}{z^2} \text{ is exactly } -\frac{3}{2}.$$

2. We view

$$\int_{-1}^{2} \frac{dx}{x^2} \text{ as being approximately } \int_{C_\varepsilon} \frac{dz}{z^2},$$

but remembering that in the real case the semi-circle is "imaginary" since the only numbers that exist are along the x-axis.

To indicate that we obtained the answer by the indented contour technique we usually use the notation

$$P\int_{-1}^{2} \frac{dx}{x^2} = -\frac{3}{2}$$

and this is called the <u>Cauchy Principal value of the improper integral</u>.

1.8.4(L) continued

The main point is that when we write

$$P\int_{-1}^{2} \frac{dx}{x^2} = -\frac{3}{2}$$

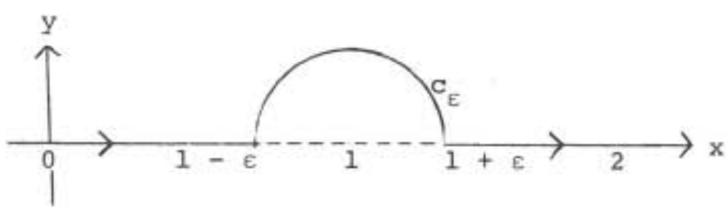
we are using a very natural definition in the z-plane which doesn't exist if the domain of f must be the real axis. It is in this sense that we have exhibited still another example in which the complex plane allows us to deduce real results which were not as obvious when all we had to work with were the real numbers.

1.8.5

a. We view

$$\int_0^2 \frac{dx}{(x-1)^2}$$

as the limiting position of c as $\varepsilon \to 0$, where



Then, since for any curve c which joins z = 0 to z = 2 without passing through z = 0,

$$\int_0^2 \frac{dz}{(z-1)^2} = -\frac{1}{z-1}\Bigg|_{z=0}^{2} = -\frac{1}{2} - 1 = -\frac{3}{2}$$

we have, in particular, that

$$\int_{c_\varepsilon} \frac{dz}{(z-1)^2} = -\frac{3}{2}\ .$$

Hence,

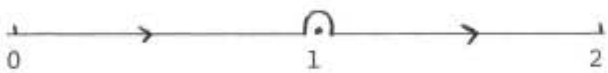
$$\lim_{\varepsilon\to 0} \int_{c_\varepsilon} \frac{dz}{(z-1)^2}\, dz = -\frac{3}{2} = P\int_0^2 \frac{dx}{(x-1)^2}\ .$$

1.8.5 continued

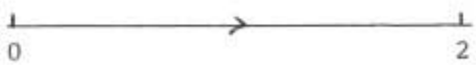
In summary,

$$\int_0^2 \frac{dx}{(x-1)^2}$$

would be $-\frac{3}{2}$ if we could view the path of integration as being



rather than



b. $$P \int_0^2 \frac{dx}{(x-1)^4} = \lim_{\varepsilon \to 0} \int_c \frac{dz}{(z-1)^4}$$

$$= \int_\varepsilon \frac{dz}{(z-1)^4},$$

where c is any path which joins z = 0 to z = 2 without passing through z = 0,

$$= -\frac{1}{3}(z-1)^{-3} \Big|_{z=0}^{2}$$

$$= [-\tfrac{1}{3}(2-1)^{-3}] - [-\tfrac{1}{3}(0-1)^{-3}]$$

$$= -\frac{1}{3} - \frac{1}{3}$$

$$= -\frac{2}{3}.$$

Pictorially, again,

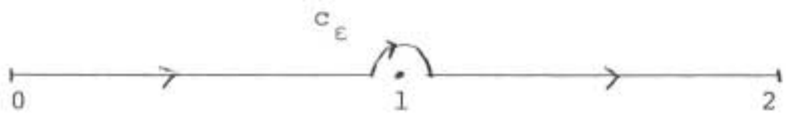
$$P\int_0^2 \frac{dx}{(x-1)^4}$$

is

$$\int_\varepsilon \frac{dz}{(z-1)^4}$$

1.8.5 continued

where



1.8.6(L)

a. Here we are deriving a result which is extremely crucial in helping us estimate the size of a complex integrals. The proof is exactly the same as the proof in the real case from an algebraic point of view, but geometrically the complex case is not as easy to picture.

Recall that in the real case our approach was along the idea that

$$\int_a^b f(x)\,dx = \lim_{\substack{\max \\ \Delta x_k \to 0}} \sum_{k=1}^{n} f(c_k)\Delta x_k,$$

hence,

$$\left|\int_a^b f(x)\,dx\right| = \left|\lim_{\substack{\max \\ \Delta x_k \to 0}} \sum_{k=1}^{n} f(c_k)\Delta x_k\right|$$

$$= \lim_{\substack{\max \\ \Delta x_k \to 0}} \left|\sum_{k=1}^{n} f(c_k)\Delta x_k\right|$$

but since the absolute value of a sum is no greater than the sum of the absolute values,

$$\left|\sum_{k=1}^{n} f(c_k)\Delta x_k\right| \le \sum_{k=1}^{n} |f(c_k)\Delta x_k|.$$

Then, since the absolute value of a product is the product of the absolute values, we saw that $|f(c_k)\Delta x_k| = |f(c_k)||\Delta x_k|$, so that

$$\left|\int_a^b f(x)\,dx\right| \le \lim_{\substack{\max \\ \Delta x_k \to 0}} \sum_{k=1}^{n} |f(c_k)||\Delta x_k| \ . \tag{1}$$

1.8.6 continued

If in (1) we now take into account that $|f(x)| \le M$ for x [a,b] we see that

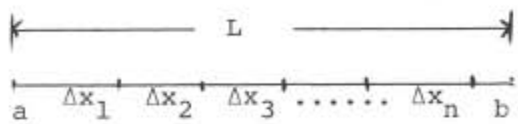
$$\left|\int_a^b f(x)\,dx\right| \le \lim_{\substack{\max \\ \Delta x_k \to 0}} \sum_{k=1}^{n} M|\Delta x_k|$$

and since M is a constant it may be taken out of the summation sign. Thus,

$$\left|\int_a^b f(x)\,dx\right| \le M \lim_{\substack{\max \\ \Delta x_k \to 0}} \sum_{k=1}^{n} |\Delta x_k|. \tag{2}$$

Finally since

$$\sum_{k=1}^{n} |\Delta x_k| = b - a = L$$

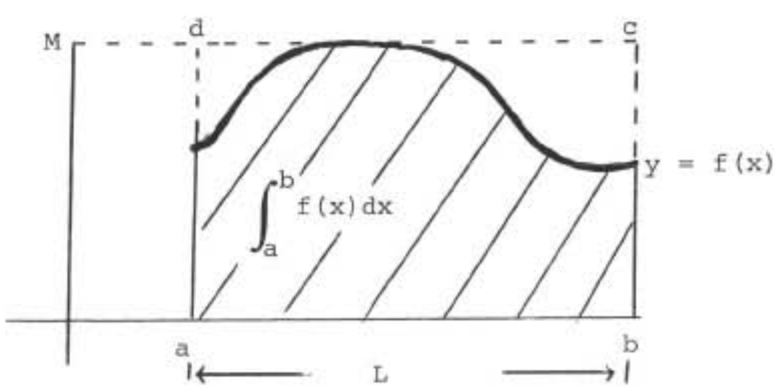
[i.e., 

; $\Delta x_1 + \ldots + \Delta x_n = L$]

we concluded from (2) that

$$\left|\int_a^b f(x)\,dx\right| \le ML. \tag{3}$$

Pictorially, for $f \ge 0$, all (3) means is that the area under the curve cannot exceed that of the greatest rectangle. That is,



ML is area of rectangle abcd.

1.8.6 continued

Now even though the geometry in the complex case is more "complex" since our diagrams involve higher dimensions, the point is that algebraically things remain the same (or at least, almost the same) as in the real case. Namely, among other things the absolute value properties are the same.

Thus we obtain

$$\left|\int_a^b f(z)dz\right|^* = \left|\lim_{\substack{\max\\ \Delta z_k \to 0}} \sum_{k=1}^{n} f(c_k)\Delta z_k\right| . \tag{4}$$

Hence,

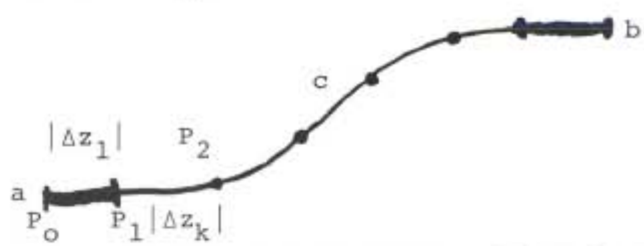
$$\left|\int_a^b f(z)dz\right| \leq \lim_{\substack{\max\\ \Delta z_k \to 0}} \sum_{k=1}^{n} |f(c_k)||\Delta z_k|$$

$$\leq M \lim_{\substack{\max\\ \Delta z_k \to 0}} \sum_{k=1}^{n} |\Delta z_k| . \tag{5}$$

The only difference (5) and (2) is that there are many curves c which join a and b when a and b are viewed as arbitrary points in the z-plane. The key property, however, is that for any such curve c,

$$\sum_{k=1}^{n} |\Delta z_k|$$

cannot exceed the length, L, of c. This follows simply from the axiom that a straight line is the shortest distance between two points. Pictorially,



*Since f(z) need not be analytic $\int_a^b f(z)dz$ is ambiguous. Thus, we should write $|\int_c f(z)dz|$ but we prefer the notation of (4) to emphasize the similarity of structure to that of the real case.

1.8.6 continued

1. Δz_k is the vector $\overline{P_{k-1}P_k}$. Hence, $|\Delta z_k|$ is the length of the line segment $P_{k-1}P_k$.

2. In any event, $|\Delta z_1| + \ldots + |\Delta z_n| \leq$ length of c since $|\Delta z_k|$ is the approximation of a segment of this curve by a straight line chord.

b. $$\left| \int_{c_r} \frac{e^{iz}}{z^2}\, dz \right| \leq \max_{z \varepsilon c_R} \left\{ \left| \frac{e^{iz}}{z^2} \right| \right\} \pi R \tag{6}$$

(i.e., $\pi R = L =$ length of c_R) .

Now $e^{iz} = e^{i(x + iy)}$

$= e^{ix - y}$

$= e^{-y}e^{ix}$.

Hence,

$|e^{iz}| = |e^{-y}||e^{ix}|$ *

$= |e^{-y}|$

$= e^{-y}$.

On c_R, y varies from 0 to R, hence

$\frac{1}{e^R} \leq \frac{1}{e^y} \leq \frac{1}{e^o} = 1$

so that

$$|e^{iz}| \leq 1 \tag{7}$$

*For any real x, $e^{ix} = \cos x + i \sin x$ implies that

$|e^{ix}| = \sqrt{\cos^2 x + \sin^2 x} = 1$

1.8.6 continued

on c_R.

We also know that for all z on c_R, $|z| = R$. Hence on c_R

$$|z^2| = |z|^2 = R^2. \tag{8}$$

Putting (7) and (8) into (6) we have that

$$\left|\int_{c_R} \frac{e^{iz}}{z^2}\,dz\right| \le \frac{1}{R^2}(\pi R)$$

or

$$\left|\int_{c_R} \frac{e^{iz}}{z^2}\,dz\right| \le \frac{\pi}{R}. \tag{9}$$

c. From (9) we have that

$$\lim_{R\to\infty} \left|\int_{c_R} \frac{e^{iz}}{z^2}\,dz\right| \le \lim_{R\to\infty} \frac{\pi}{R} = 0 \tag{10}$$

and since

$$\left|\int_{c_R} \frac{e^{iz}}{z^2}dz\right| \ge 0,$$

we conclude from (10) that

$$\lim_{R\to\infty} \left|\int_{c_R} \frac{e^{iz}}{z^2}\,dz\right| = 0 \tag{11}$$

but since $|w| = 0 \leftrightarrow w = 0$, we conclude from (11) that

$$\int_{c_R} \frac{e^{iz}dz}{z^2} \to 0 \text{ as } R \to \infty.$$

1.8.7

If c is the straight line joining z = 0 to z = 3 + 4i then the length, L, of c is

$\sqrt{3^2 + 4^2} = 5.$

Now if $f(z) = [\mathrm{Re}(z)]^2 + i[\mathrm{Im}(z)]^2$; i.e., $f(z) = x^2 + iy^2$ then on c, since $y = \frac{4}{3}x$, $f(z) = x^2 + i\,\frac{16x^2}{9}$. Hence on c,

$$\begin{aligned} f(z) &= \sqrt{x^4 + \frac{256}{81}x^4} \\ &= \frac{x^2}{9}\sqrt{337} \qquad 0 \le x \le 3. \end{aligned} \tag{1}$$

Therefore, the maximum value of $|f(z)|$ occurs when x = 3 in which case (1) tells us that

$|f(z)| \le \sqrt{337}$.

Hence,

$$\left|\int_0^{3+4i} (x^2 + iy^2)dz\right| \le 5\sqrt{337}$$

along the straight line joining z = 0 to z = 3 + 4i.

1.8.8(optional)

We have

$$F(z) = \int_{z_o}^{z} f(\zeta)d\zeta \tag{1}$$

where the value of F(z) does not depend on the path between z_o and z since f is analytic and we use ζ rather than z to emphasize that z is a chosen point whereas ζ generally names all points on the curve joining z_o to z. [This is analogous to our writing, say,

$$F(x) = \int_{x_o}^{x} f(t)dt$$

in the real case rather than

1.8.8 continued

$$F(x) = \int_{x_o}^{x} f(x)dx.]$$

We then have

$$F(z + \Delta z) = \int_{z_o}^{z + \quad z} f(\zeta)d\zeta$$

so that

$$F(z + \Delta z) - F(z) = \int_{z_o}^{z_o + \Delta z} f(\zeta)d\zeta - \int_{z_o}^{z} f(\zeta)d\zeta$$

$$= \int_{z}^{z + \Delta z} f(\zeta)d\zeta.$$

Hence,

$$\frac{F(z + \Delta z) - F(z)}{\Delta z} = \frac{1}{\Delta z} \int_{z}^{z + \Delta z} f(\zeta)\, d\zeta. \tag{2}$$

To utilize our conjecture that $F'(z) = f(z)$ we rewrite the integral in (z) by adding and subtracting $f(z)$. Thus,

$$\frac{F(z + \Delta z) - F(z)}{\Delta z} = \frac{1}{\Delta z} \int_{z}^{z + \Delta z} \{f(z) + [f(\zeta) - f(z)]\}\, d\zeta$$

$$= \frac{1}{\Delta z} \int_{z}^{z + \Delta z} f(z)d\zeta + \frac{1}{\Delta z} \int_{z}^{z + \Delta z} [f(\zeta) - f(z)]d\zeta. \tag{3}$$

Since z is fixed, $f(z)$ is a constant (ζ is the variable of integration) and accordingly,

$$\frac{1}{\Delta z} \int_{z}^{z + \Delta z} f(z)d\delta = \frac{f(z)}{\Delta z} \int_{z}^{z + \Delta z} d\zeta$$

$$= \frac{f(z)}{\Delta z} [\zeta \Big|_{\zeta = z}^{z + \Delta z}] = \frac{f(z)}{\Delta z} [(z + \Delta z) - z] = f(z).$$

1.8.8 continued

Thus, (3) becomes

$$\frac{F(z + \Delta z) - F(z)}{\Delta z} = f(z) + \frac{1}{\Delta z}\int_{z}^{z + \Delta z}[f(\zeta) - f(z)]d\zeta \tag{4}$$

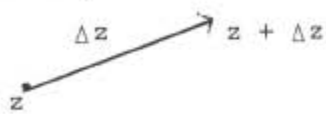
and from (4) we see that the conjectured result would hold provided

$$\lim_{\Delta z \to 0} \left[\frac{1}{\Delta z}\int_{z}^{z + \Delta z}[f(\zeta) - f(z)]d\zeta \right] = 0.$$

But from Exercise 6.8.6 we know that

$$\left| \int_{z}^{z + \Delta z}[f(\zeta) - f(z)]d\zeta \right| \le \max\{| f(\zeta) - f(z)\} \ |\Delta z| \tag{5}$$

(where we pick the straight line path which joins z to $z + \Delta z$, i.e.,



)

Hence,

$$\left| \frac{1}{\Delta z}\int_{z}^{z + \Delta z}[f(\zeta) - f(z)]d\zeta \right| \le \left| \frac{1}{\Delta z}\right| \left| \int_{z}^{z + \Delta z}[f(\zeta) - f(z)]d\zeta \right.$$

$$\le \max\{| f(\zeta) - f(z)|\} \text{ [by (5)]} \tag{6}$$

Since f is continuous* $\lim\limits_{\Delta z \to 0} f(\zeta) = f(z)$, so that

$$\lim_{\Delta z \to 0} \{ \max[|f(\zeta) - f(z)]\} = 0,$$

---

* We have not actually proved that an analytic function is continuous, but the proof is the same as in the real case, namely we write

$$f(z + \Delta z) - f(z) = \left[\frac{f(z + \Delta z) - f(z)}{\Delta z}\right]\Delta z$$

to conclude that

$$\lim_{\Delta z \to 0} [f(z + \Delta z) - f(z)] = \lim_{\Delta z \to 0} f'(z)\Delta z = 0.$$

1.8.8 continued

so from (6)

$$\lim_{\Delta z \to 0} \frac{1}{\Delta z} \int_{z}^{z+\Delta z} [f(\zeta) - f(z)]d = 0, \text{ and (4) yields}$$

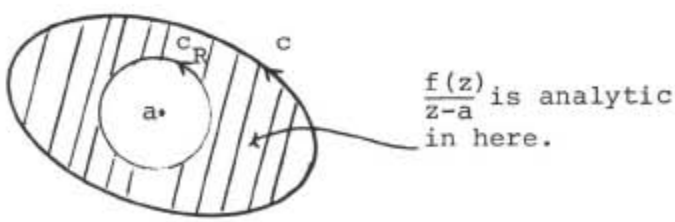
$$F'(z) = \lim_{\Delta z \to 0} \left[\frac{F(z + \Delta z) - F(z)}{\Delta z}\right] = f(z)$$

1.8.9(L)

a. Since f(z) is analytic so also is $\frac{f(z)}{z-a}$ provided we are in a region which excludes z = a. In particular if c_R denotes a circle of radius R centered at z = a such that c_R has entirely within c, we have that

$$\oint_c \frac{f(z)dz}{z - a} = \oint_{c_R} \frac{f(z)dz}{z - a} . \tag{1}$$

Pictorially



Now c_R being a circle of radius R centered at z = a is described by

$$z = a + Re^{i\theta} , \ 0 \le \theta \le 2\pi.$$

Hence, $dz = iRe^{i\theta}\, d\theta$ and $z - a = Re^{i\theta}$.

1.8.9(L) continued

Hence,

$$\oint_c \frac{f(z)dz}{z-a} = \oint_{c_R} \frac{f(z)dz}{z-a}$$

$$= \int_0^{2\pi} \frac{f(a+Re^{i\theta})\ i\ Re^{i\theta}d\theta}{Re^{i\theta}}$$

$$= i\int_0^{2\pi} f(a+Re^{i\theta})d\theta. \qquad (2)$$

b. Since (2) holds for all $R > 0$ (provided only that c_R has within c) we may look at (2) as $R \to 0$ to obtain

$$\lim_{R\to 0} \oint_c \frac{f(z)dz}{z-a} = \lim_{R\to 0}\left[i\int_0^{2\pi} f(a+Re^{i\theta})d\theta\right]. \qquad (3)$$

Since

$$\oint_c \frac{f(z)dz}{z-a}$$

does not depend on R we have that

$$\lim_{R\to 0} \oint_c \frac{f(z)dz}{z-a} = \oint_c \frac{f(z)dz}{z-a};$$

and since f is continuous (because it's analytic)

$$\lim_{R\to 0} \oint_0^{2\pi} f(a+Re^{i\theta})d\theta = \int_0^{2\pi} \lim_{R\to 0} f(a+Re^{i\theta})d\theta = \int_0^{2\pi} f(a)d\theta,$$

equation (3) becomes

$$\oint_c \frac{f(z)dz}{z-a} = i\int_0^{2\pi} f(a)d\theta$$

$$= i\ f(a)\left[\theta\Big|_{\theta=0}^{2\pi}\right]$$

$$= 2\pi i\ f(a), \qquad (4)$$

1.8.9(L) continued

or,

$$f(a) = \frac{1}{2\pi i} \oint_c \frac{f(z)dz}{z - a}. \tag{4}$$

From (4) we see that f(a) is computed by a line integral which involves only that we know how f behaves on c. In other words, the behaviour of an analytic function at some point z = a in a region R is determined once we know how f behaves on the boundary of R. That is, an analytic function has its behaviour in a region determined solely by how it behaves on the boundary of the region. This is one of the reasons that analytic functions play such a large part in boundary value problems.

c. $\oint_c \frac{e^z dz}{z - i}$

is equation (4) with $f(z) = e^z$ and $a = i$. Hence, in this case,

$$\oint_c \frac{f(z)dz}{z - d} = 2\pi i\ f(a)$$

implies

$$\oint_c \frac{e^z dz}{z - i} = 2\pi i\ e^i$$

and since $e^i = e^i = \cos 1 + i \sin 1$, we have that

$$\oint_c \frac{e^z dz}{z - i} = 2\pi i[\cos 1 + i \sin 1] = -2\pi \sin 1 + i\ 2\pi \cos 1^*.$$

d. Letting $f(z) = \sin z$ and $a = 0$, equation (4) yields

$$\oint_c \frac{\sin zdz}{z} = 2\pi i \sin (0) = 0.$$

This shows that $\oint_c f(z)dz$ may be zero even if f is not analytic everywhere within c. For example, here, $\frac{\sin z}{z}$ is not analytic at $z = 0$.

*Where 1 is a number. If we wish to use sin 1 and cos 1 with 1 being viewed as an angle we must remember that sin 1 means sin (1 radian) not sin (1°) etc.

Unit 9: Complex Integration, Part 2

1.9.1(L)

a. As seen in the previous unit, we know that

$$\oint_c \frac{dz}{(z-a)^n} = \oint_{c_R} \frac{dz}{(z-a)^n} \tag{1}$$

where c_R is the circle centered at $z = a$, with radius R, and lying within c.

For any such choice of R, we have that c_R is defined by

$$z = a + Re^{i\theta} \qquad 0 \leqslant \theta \leqslant 2\pi$$

so that

$$dz = iRe^{i\theta}d\theta$$

and

$$z - a = Re^{i\theta}, \text{ or } (z-a)^n = R^n e^{in\theta}.$$

Thus, we obtain from (1) that

$$\oint_c \frac{dz}{(z-a)^n} = \int_0^{2\pi} \frac{iRe^{i\theta}d\theta}{R^n e^{in\theta}}$$

$$= i\int_0^{2\pi} \frac{e^{-i(n-1)\theta}d\theta}{R^{n-1}}. \tag{2}$$

The integral on the right side of (2) suggests two cases. Namely, if we integrate the right side of (2), we obtain

1.9.1(L) continued

$$\oint_C \frac{dz}{(z-a)^n} = \frac{i}{R^{n-1}}\left[\frac{e^{-i(n-1)\theta}}{-i(n-1)}\right]_{\theta=0}^{2\pi}$$

$$= \left.\frac{e^{-i(n-1)\theta}}{-(n-1)R^{n-1}}\right]_{\theta=0}^{2\pi} \tag{3}$$

which is fine <u>except</u> when n = 1, in which case the right side of (3) has a zero denominator.

We observe from (3), however, that if $n \neq 1$, then

$$\oint_C \frac{dz}{(z-a)^n} \quad \frac{e^{-i(n-1)2\pi} - e^{i0}}{-(n-1)R^{n-1}}$$

$$= \frac{\left[e^{-i2\pi}\right]^{n-1} - 1}{-(n-1)R^{n-1}}$$

$$= \frac{1^{n-1} - 1}{-(n-1)R^{n-1}} \tag{4}$$

$$= 0. \tag{5}$$

[Again, by way of review, if n = 1, (4) takes on the form $\frac{0}{0}$.]

If n = 1, (2) becomes

$$\oint_C \frac{dz}{z-a} = i\int_0^{2\pi} \frac{e^{0}}{R^{0}}\, d\theta$$

$$= i\int_0^{2\pi} d\theta$$

$$= 2\pi i.^* \tag{6}$$

---

*This may be checked as a special case of our result in Exercise 1.8.9. Namely, $\oint_C \frac{f(z)}{z-a}\, dz = 2\pi i f(a)$; and letting $f(z) \equiv 1$, this yields $\oint_C \frac{dz}{z-a} = 2\pi i(1) = 2\pi i$.

1.9.1(L) continued

Combining (5) and (6), we have

$$\oint_c \frac{dz}{(z-a)^n} = \begin{cases} 0, & n \neq 1 \\ 2\pi i, & n = 1 \end{cases}$$

where c is any closed curve which includes $z = a$ in its interior.

b. The result of part (a) is often used to compute integrals of the form

$$\oint_c \frac{f(z)dz}{(z-a)^k}$$

where c is a closed curve in a region R in which f is analytic.

The general idea is that we expand f(z) in a power series about $z = a$, divide term-by-term by $(z - a)^k$ and integrate term-by-term. In this particular example, we have that

sin z

converges uniformly and absolutely to $z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots$. Hence,

$$\sin z - z^2 = \left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots\right] - z^2$$

or, by absolute convergence (so we can rearrange terms)

$$\sin z - z^2 \equiv z - z^2 - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots . \tag{1}$$

Then, since we may divide uniformly convergent series term-by term we conclude from (1) that

$$\frac{\sin z - z^2}{z^6} = \frac{1}{z^6} - \frac{1}{z^4} - \frac{1}{3!z^3} + \frac{1}{5!z} - \frac{z}{7!} + \ldots . \tag{2}$$

1.9.1(L) continued

Since the order of summation and integration may be reversed when we have uniform convergence, we may conclude from (2) that

$$\oint_c \frac{\sin z - z^2}{z^6}\,dz = \oint_c \frac{dz}{z^5} - \oint_c \frac{dz}{z^4} - \frac{1}{3!}\oint_c \frac{dz}{z^3} + \frac{1}{5!}\oint_c \frac{dz}{z} -$$

$$- \frac{1}{7!}\oint_c z\,dz + \dots\,. \tag{3}$$

Since for any positive integer n, z^n is analytic, $\oint_c z^n dz = 0$. Moreover, from part (a) (with a = 0), $\oint_c \frac{dz}{z^n} = 0$ for any integer n except n = 1 in which case $\oint \frac{dz}{z^n} = \oint \frac{dz}{z} = 2\pi i$.

Putting this information into the right side of (3), we see that every term is zero, except $\frac{1}{5!}\oint_c \frac{dz}{z}$, which is $\frac{1}{5!}(2\pi i) = \frac{\pi i}{60}$. That is, we see from (3), that

$$\oint_c \frac{\sin z - z^2}{z^6}\,dz = \frac{\pi i}{60}$$

where c is any closed curve which contains z = 0 as an interior point.

1.9.2

$$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \dots$$

implies that

$$e^{z^2} = 1 + z^2 + \frac{z^4}{2!} + \frac{z^6}{3!} + \frac{z^8}{4!} + \dots\,.$$

Hence,

1.9.2 continued

$$\frac{e^{z^2}}{z^7} = \frac{1}{z^7} + \frac{1}{z^5} + \frac{1}{2!z^3} + \frac{1}{3!z} + \frac{1}{4!}z + \ldots .$$

Therefore,

$$\oint_c \frac{e^{z^2}dz}{z^7} = \oint_c \frac{dz}{z^7} + \oint_c \frac{dz}{z^5} + \frac{1}{2!}\oint_c \frac{dz}{z^3} + \frac{1}{3!}\oint_c \frac{dz}{z} + \frac{1}{4!}\oint_c zdz + \ldots \quad (1)$$

and since $\oint_c \frac{dz}{z} = 2\pi i$* and $\oint_c z^n dz = 0$ if $n \neq -1$, equation (1) yields

$$\oint_c \frac{e^{z^2}dz}{z^7} = \frac{1}{3!}(2\pi i) = \frac{1}{3}\pi i.$$

1.9.3 (Optional)

We already know from the last unit that if f(z) is analytic in R and c is a closed curve lying in R that for any point $a \varepsilon R$

$$f(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)dz}{z-a}. \quad (1)$$

To indicate that z = a is <u>any</u> point in R, we prefer to write (1) in the form

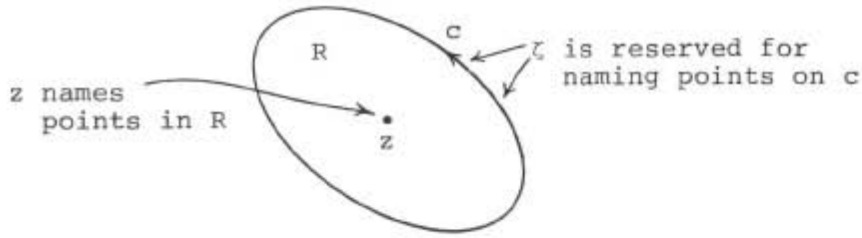
$$f(z) = \frac{1}{2\pi i}\oint_c \frac{f(\zeta)d\zeta}{\zeta - z} \quad (1')$$

where in (1') ζ denotes a point <u>on</u> the contour c while z denotes the general point <u>inside</u> c.

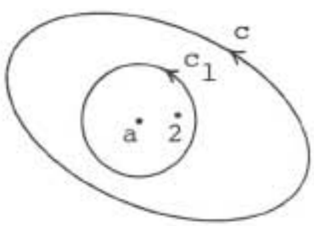
*Just a reminder, if c doesn't enclose the origin $\oint_c \frac{dz}{z} = 0$ since then $\frac{1}{z}$ is analytic inside c. It is in concluding that $\oint_c \frac{dz}{z} = 2\pi i$ where we use the hypothesis that c contains z = 0 as an interior point.

1.9.3 continued

Pictorially,



The official "trick" now is to pick a point $a \in R$ and draw a circle, c_1, centered at a which lies inside c and includes the given point z. Thus, for example,



Referring to (1'), we may now view

$$\oint_c \frac{f(\zeta)d\zeta}{\zeta - z}$$

as

$$\oint_c f(\zeta)\left[\frac{1}{\zeta - z}\right]d\zeta.$$

We then set up $\frac{1}{\zeta - z}$ as a convergent geometric series as follows.

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - a) - (z - a)} = \frac{1}{(\zeta - a)}\left[\frac{1}{1 - \frac{(z - a)}{(\zeta - a)}}\right]. \tag{2}$$

Now since z lies inside c_1 and no point ζ on c lies inside c_1, we have that as long as z is in c_1, a is closer to z than it is to ζ. That is, $|z - a| < |\zeta - a|$; i.e., $\frac{|z - a|}{|\zeta - a|} < 1$, hence,

1.9.3 continued

$$\left|\frac{z - a}{\zeta - a}\right| < 1.$$

But

$$\frac{1}{1 - u} = 1 + u + u^2 + u^3 + \ldots + u^n + \ldots$$

provided $|u| < 1$.
Hence, for $\left|\frac{z - a}{\zeta - a}\right| < 1$,

$$\frac{1}{1 - \left(\frac{z - a}{\zeta - a}\right)}$$

converges uniformly (and absolutely) to

$$1 + \frac{z - a}{\zeta - a} + \frac{(z - a)^2}{(\zeta - a)^2} + \frac{(z - a)^3}{(\zeta - a)^3} + \ldots .$$

Therefore, from (2).

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - a}\left[1 + \frac{z - a}{\zeta - a} + \frac{(z - a)^2}{(\zeta - a)^2} + \ldots\right]$$

$$= \frac{1}{\zeta - a} + \frac{z - a}{(\zeta - a)^2} + \frac{(z - a)^2}{(\zeta - a)^3} + \ldots .$$

Accordingly,

$$\oint_c \frac{f(\zeta)d\zeta}{\zeta - z} = \oint_c f(\zeta) \sum_{n=0}^{\infty} \frac{(z - a)^n}{(\zeta - a)^{n+1}}\, d\zeta,$$

and by uniform convergence,

$$\oint_c \frac{f(\zeta)d\zeta}{\zeta - z} = \sum_{n=0}^{\infty} \left[\oint_c \frac{f(\zeta)d\zeta}{(\zeta - a)^{n+1}}\right] (z - a)^n.$$

1.9.3 continued

Hence,

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)d\zeta}{\zeta - z} = \sum_{n=0}^{\infty} A_n(z - a)^n \tag{3}$$

where

$$A_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)d\zeta}{(\zeta - a)^{n+1}} \tag{4}$$

Equation (3) supplies us with what is perhaps the most outstanding feature of an analytic function. Notice that in deriving (3), we assumed only that f(z) possessed a first derivative; but from equation (3), we see that the assumption that f'(a) existed led to the conclusion that f(z) could be expanded in a series involving powers of (z - a). The convergence of this series, as our proof indicated, followed from the convergent properties of the geometric series - not from the circular-reasoning that f(z) could be expressed in a power series.

In still other words, since the series $\sum_{n=0}^{\infty} A_n(z - a)^n$ converges uniformly to f(z) in some neighborhood of z = a, we may differentiate term-by-term to obtain that

$$f'(z) = \sum_{n=1}^{\infty} n\, A_n(z - a)^{n-1}$$

$$f''(z) = \sum_{n=2}^{\infty} n(n-1)A_n(z - a)^{n-2}, \text{ etc.}$$

where each series again converges uniformly to the appropriate derivative of f(z).

The next key observation is that once we know that $f^{(n)}(a)$ exists for all whole numbers n, then f(z), if it has a power series expansion about z = a, must be given by

1.9.3 continued

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z - a)^n. \tag{5}$$

Since the power series representation is unique, we may compare (3) and (5) to conclude that

$$\frac{f^{(n)}(a)}{n!} = A_n \tag{6}$$

for all whole number, n; and now using the value of A_n given by equation (5), we see from (6) that

$$\frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i} \oint_c \frac{f(\zeta)d\zeta}{(\zeta - a)^{n+1}}$$

or

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_c \frac{f(\zeta)d\zeta}{(\zeta - a)^{n+1}} \tag{7}$$

Finally, if you feel more comfortable using z than ζ, equation (7) may, of course, be written as

$$\left.\begin{array}{l} f^{(n)}(a) = \dfrac{n!}{2\pi i} \displaystyle\oint_c \dfrac{f(z)dz}{(z - a)^{n+1}} \\ \\ \text{where } z \text{ refers to points on } c. \end{array}\right\} \tag{8}$$

In summary, then, if f(z) is differentiable at z = a then the n^{th} derivative of f(z) exists at z = a, and in fact

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_c \frac{f(z)dz}{(z - a)^n}$$

Moreover, for each z in a sufficiently small neighborhood of z = a,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z - a)^n.$$

1.9.3 continued

In particular, this proves the very important property of a differentiable complex function of a complex variable. Namely, if f is differentiable at $z = a$ then f possesses derivatives of every order at $z = a$. For this reason, we are justified in defining "analytic" to mean either that f is differentiable or that it possesses derivatives of all orders (i.e., can be expressed in a power series). This is quite different from the real case in which a function f can possess one or more derivatives at $x = a$ but not be expandable in a power series about $x = a$. For example, if $f(x) = x^{5/2}$ $f'(x)$ and $f''(x)$ exist at $x = 0$, but f cannot be expanded in a power series about $x = 0$ since $f^{(n)}(0)$ does not exist once $n \geqslant 3$.

As a partial check of equation (8), notice that when $n = 0$, we get the Cauchy Integral Formula (discussed in the previous unit), $f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{z - a}$.

b. $\frac{1}{2\pi i} \oint_C \frac{e^z + z^4}{(z - 2\pi i)^3} dz$

has the form

$$\frac{1}{2\pi i} \oint_C \frac{f(z)dz}{(z - a)^{n+1}} \tag{9}$$

with

$$f(z) = e^z + z^4, \ a = 2\pi i, \text{ and } n = 2. \tag{10}$$

To utilize equation (8), we rewrite (9) in the form

$$\frac{1}{n!}\left[\frac{n!}{2\pi i} \oint_C \frac{f(z)dz}{(z - a)^{n+1}}\right]$$

and replace $\frac{n!}{2\pi i} \oint_C \frac{f(z)dz}{(z - a)^{n+1}}$ by $f^{(n)}(a)$ to conclude that

1.9.3 continued

$$\frac{1}{2\pi i} \oint_c \frac{f(z)dz}{(z-a)^{n+1}} = \frac{f^{(n)}(a)}{n!},$$

so that from (10)

$$\frac{1}{2\pi i} \oint_c \frac{(e^z + z^4)dz}{(z-2\pi i)^3} = \frac{\left.\frac{d^2(e^z + z^4)}{dz^2}\right|_{z=2\pi i}}{2!}$$

Hence,

$$\frac{1}{2\pi i} \oint_c \frac{(e^z + z^4)dz}{(z-2\pi i)^3} = \frac{\left. e^z + 12z^2 \right|_{z=2\pi i}}{2!}$$

$$= \frac{e^{2\pi i} + 12(2\pi i)^2}{2}$$

$$= \frac{1 + 12(-4\pi^2)}{2}$$

$$= \frac{1 - 48\pi^2}{2}. \tag{11}$$

As a check of (11), we may use the procedure of Exercise 1.9.1 and expand $e^z + z^4$ in powers of $(z - 2\pi i)$. This yields

(i) $$e^z = \sum_{n=0}^{\infty} \frac{\left.\frac{d(e^z)}{dz}\right|_{z=2\pi i}}{n!} (z-2\pi i)^n$$

$$= \sum_{n=0}^{\infty} \frac{e^{2\pi i}}{n!} (z-2\pi i)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} (z-2\pi i)^n$$

1.9.3 continued

(ii) $g(z) = z^4 \rightarrow g'(z) = 4z^3$, $g''(z) = 12z^2$, $g'''(z) = 24z$,

$$g^{(4)}(z) = 24 \text{ and } g^{(n)}(z) \equiv 0 \text{ for } n > 4.$$

Hence,

$g(2\pi i) = (2\pi i)^4$, $g'(2\pi i) = 4(2\pi i)^3$, $g''(2\pi i) = 12(2\pi i)^2$

$g'''(2\pi i) = 24(2\pi i)$, $g^{(4)}(2\pi i) = 24$, $g^{(n)}(2\pi i) = 0$, $n > 4$.

Thus,

$$z^4 = \sum_{n=0}^{\infty} \frac{g^{(n)}(2\pi i)}{n!}(z - 2\pi i)^n$$

$$= (2\pi i)^4 + 4(2\pi i)^3(z - 2\pi i) + \frac{12(2\pi i)^2}{2!}(z - 2\pi i)^2 + \frac{24(2\pi i)}{3!}$$

$$(z - 2\pi i)^3 + \frac{24}{4!}(z - 2\pi i)^4$$

$$= 16\pi^4 - 32\pi^3 i(z - 2\pi i) - 24\pi^2(z - 2\pi i)^2 + 8\pi i(z - 2\pi i)^3 +$$

$$+ (z - 2\pi i)^4.$$

Hence,

$$e^z + z^4 = 1 + (z - 2\pi i) + \frac{(z - 2\pi i)^2}{2!} + \frac{(z - 2\pi i)^3}{3!} + \frac{(z - 2\pi i)^4}{4!} +$$

$$+ \frac{(z - 2\pi i)^5}{5!} + \ldots + 16\pi^4 - 32\pi^3 i(z - 2\pi i) - 24\pi^2$$

$$(z - 2\pi i)^2 + 8\pi i(z - 2\pi i)^3 + (z - 2\pi i)^4$$

$$= (1 + 16\pi^4) + (1 - 32\pi^3 i)(z - 2\pi i) + \left(\frac{1}{2!} - 24\pi^2\right)(z - 2\pi i)^2 +$$

$$+ \left(\frac{1}{3!} + 8\pi i\right)(z - 2\pi i)^3 + \left(\frac{1}{4!} + 1\right)(z - 2\pi i)^4 + \sum_{n=5}^{\infty} \frac{(z - 2\pi i)^n}{n!}$$

1.9.3 continued

Therefore,

$$\frac{e^z + z^4}{(z - 2\pi i)^3} = \frac{\frac{1}{2!} - 24\pi^2}{z - 2\pi i} + \begin{Bmatrix}\text{a "bunch" of other terms} \\ \text{whose integral around c is 0*}\end{Bmatrix}$$

Hence,

$$\oint_c \frac{e^z + z^4 dz}{(z - 2\pi i)^3} = (\tfrac{1}{2!} - 24\pi^2) \oint_c \frac{dz}{z - 2\pi i} = (\tfrac{1}{2} - 24\pi^2)(2\pi i)$$

or

$$\frac{1}{2\pi i} \oint_c \frac{e^z + z^4}{(z - 2\pi i)^3} dz = \frac{1}{2} - 24\pi^2 = \frac{1 - 48\pi^2}{2}.$$

The major advantage of (7) in many applications is that we no longer have to look at the term-by-term power series expansion of $f(z)$ to evaluate $\oint_c \frac{f(z)}{(z - a)^{n+1}} dz$.

1.9.4

$e^z + \cos z + z^3$ is analytic everywhere in the z-plane. Hence, we may compute

$$\oint_c \frac{e^z + \cos z - z^3}{(z - i)^4} dz$$

by means of the recipe

$$\oint_c \frac{f(z)dz}{(z - a)^{n+1}} = \frac{2\pi i\ f^n(a)}{n!}. \qquad (1)$$

*Recall from the first exercise that $\oint_c (z - 2\pi i)^n dz = \begin{cases} 2\pi i, & \text{if } n = -1 \\ 0, & \text{otherwise}\end{cases}$

1.9.4 continued

In our example; $n = 3$, $a = i$, and $f(z) = e^z + \cos z - z^3$.
Therefore,

$$f^{(n)}(z) = \frac{d^3(e^z + \cos z - z^3)}{dz^3} = e^z + \sin z - 6$$

so,

$$f^{(n)}(a) = f^{(3)}(i) = e^i + \sin i - 6.$$

Putting these results into (1) yields

$$\oint_c \frac{e^z + \cos z - z^3}{(z-i)^4}\,dz = \frac{2\pi i(e^i + \sin i - 6)}{3!}. \tag{2}$$

We may make use of the identity $e^{iz} = \cos z + i \sin z$ (and, consequently, $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$) to rewrite (2) as

$$\oint_c \frac{(e^z + \cos z - z^3)}{(z-i)^4}\,dz = \frac{2\pi i}{6}\left[e^i + \frac{e^{i(i)} - e^{-i(i)}}{2i} - 6\right]$$

$$= (2\pi i\, e^i + \pi(e^{-1} - e^1) - 12\pi i)/6$$

$$= (2\pi i(\cos 1 + i \sin 1) - 2\pi \sin h\ 1 - 12\pi i)/6$$

$$= (-2\pi \sin 1 - 2\pi \sin h\ 1 + i(2\pi \cos 1 - 12\pi))/6$$

and in this way the integral is expressed in the form $a + bi$ with a and b real.

1.9.5(L)

$\oint_c \frac{z}{z^2+1}\,dz$ does not have the form $\oint_c \frac{f(z)}{(z-a)^n}\,dz$ which we have handled before. That is, our denominator $z^2 + 1$ is not a power of $z - a$.

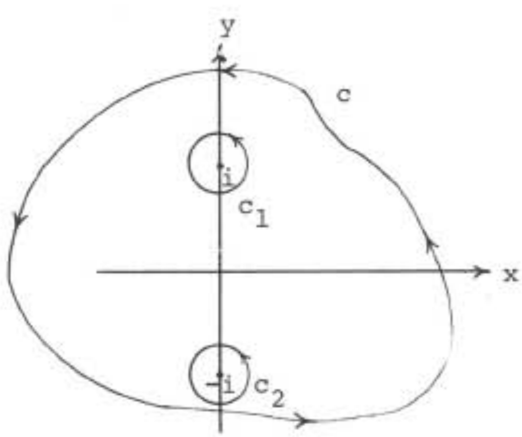
1.9.5(L) continued

What is true, however, is that

$$\oint_c \frac{zdz}{z^2+1} = \oint_c \frac{zdz}{(z+i)(z-i)}. \tag{1}$$

The right side of (1) shows us <u>two</u> "trouble spots"; namely, $z = i$ and $z = -i$, where our integrand is not well behaved. The point is that we may choose contours c_1 and c_2 which do not intersect; one of which, say c_1, encloses i and the other, $-i$.

Pictorially,



[c_1 and c_2 do not have to be circles, although we may choose them to be if we so wish.
The key point is that if c_3 is a circle centered at $z = i$ and c_3 lies within c_1, then

$$\oint_{c_1} \frac{zdz}{z^2+1} = \oint_{c_3} \frac{zdz}{z^2+1}$$

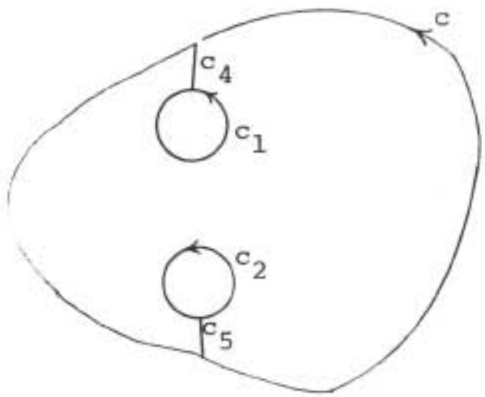
since $\frac{z}{z^2+1}$ is analytic (on and) between c_1 and c_3.]

The key step now, as in our discussion of simply-connected regions in Block 5 of Part 2, is to observe that

$$\oint_c \frac{zdz}{z^2+1} = \oint_{c_1} \frac{zdz}{z^2+1} + \oint_{c_2} \frac{zdz}{z^2+1}. \tag{2}$$

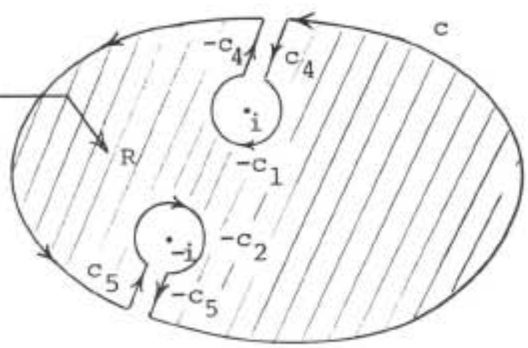
To see this, we "slit" our region to make it simply-connected. Thus,

1.9.5(L) continued



I.e.,

This region R is simple-connected and $\frac{z}{z^2+1}$ is analytic in R.



c_4 and $-c_4$ (as do c_5 and $-c_5$) really coincide, but we spread them apart to emphasize the simply-connected region in question.

Letting c denote the boundary of R, we have that

$$c = c \cup c_4 \cup -c_1 \cup -c_4 \cup c_5 \cup -c_2 \cup -c_5$$

and since (i) the integral along c_4 cancels the one along $-c_4$ (i.e., the paths are the same but with opposite sense), (ii) the integral along c_5 cancels the one along $-c_5$, and (iii) $\oint_c \frac{zdz}{z^2+1} = 0$ because $\frac{z}{z^2+1}$ is analytic in and on c, we conclude that

$$0 = \oint_c \frac{zdz}{z^2+1} = \oint_c \frac{zdz}{z^2+1} + \oint_{-c_1} \frac{zdz}{z^2+1} + \oint_{-c_2} \frac{zdz}{z^2+1}$$

$$= \oint_c \frac{zdz}{z^2+1} - \oint_{c_1} \frac{zdz}{z^2+1} - \oint_{c_2} \frac{zdz}{z^2+1}$$

1.9.5(L) continued

or,

$$\oint_c \frac{zdz}{z^2+1} = \oint_{c_1} \frac{zdz}{z^2+1} + \oint_{c_2} \frac{zdz}{z^2+1}$$

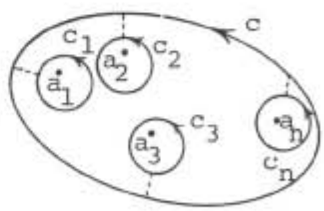
This method generalizes as follows.

Suppose $f(z)$ is defined in a region R whose boundary is the simple closed curve c; and suppose that f is analytic everywhere on the boundary curve c, while in R it is analytic everywhere except at the n isolated* points $a_1, \ldots,$ and a_n in R.

Then if we surround $a_1, \ldots,$ and a_n by non-intersecting contours $c_1, \ldots,$ and c_n (the fact that $a_1, \ldots,$ and a_n are isolated means that we can surround them by disjoint contours) all lying within c, it follows that

$$\oint_c f(z)dz = \oint_{c_1} f(z)dz + \ldots + \oint_{c_n} f(z)dz.$$

Pictorially,



Returning to the main stream of our discussion, the advantage of equation (2) is that the integrals around c_1 and c_2 are of the type discussed in the previous exercises. For example, inside c_1, $\frac{z}{z^2+1}$ is analytic except at $z = i$. Thus, if we write $\frac{z}{z^2+1}$ in the form

*See note at the end of this exercise.

1.9.5(L) continued

$$\frac{z}{(z+i)(z-i)} = \frac{\left[\frac{z}{z+i}\right]}{z-i},$$

we observe that $\frac{z}{z+i}$ is analytic in and on c_1 so that $\oint_{c_1} \frac{zdz}{z^2+1}$ has the form $\oint_{c_1} \frac{f(z)dz}{z-i}$ where $f(z) = \frac{z}{z+i}$ is analytic in and on c_1.

We may, therefore, use the formula $\oint_{c_1} \frac{f(z)}{z-i} = 2\pi i\ f(i)$ to conclude that

$$\begin{aligned}\oint_{c_1} \frac{f(z)dz}{z-i} &= 2\pi i\ f(i)\\ &= 2\pi i\ \left(\frac{i}{i+i}\right)\\ &= \pi i.\end{aligned}$$

That is,

$$\oint_{c_1} \frac{zdz}{z^2+1} = \pi i. \tag{3}$$

Similarly,

$$\begin{aligned}\oint_{c_2} \frac{zdz}{z^2+1} &= \oint_{c_2} \frac{zdz}{(z-i)(z+i)}\\ &= \oint_{c_2} \frac{\left[\frac{z}{z-i}\right]dz}{z+i}\\ &= \oint_{c_2} \frac{g(z)dz}{z-(-i)}\end{aligned}$$

where $g(z) = \frac{z}{z-i}$.

1.9.5(L) continued

Since g(z) is analytic in and on c_2, we have that

$$\oint_{c_2} \frac{g(z)dz}{z - (-i)} = 2\pi i\ g(-i)$$

$$= 2\pi i \left[\frac{-i}{-i - i}\right]$$

$$= \pi i$$

so that

$$\oint_{c_2} \frac{zdz}{z^2 + 1} = \pi i. \qquad (4)$$

Combining (3) and (4) with (2), we conclude that

$$\oint_{c} \frac{zdz}{z^2 + 1} = 2\pi i.$$

A NOTE ON SINGULAR POINTS

It should be clear by now that in the study of complex variables functions which are not differentiable (or if they are differentiable only at isolated points in the plane) are of very minor importance compared with those which are differentiable (analytic).

Yet there are "shades of gray" between the extremes of being analytic and being non-analytic. One type of non-analytic function is one that is analytic everywhere in a region, except at most at a set of points of measure 0.* If such a function exists, we refer to the points at which the derivative fails to exist as singular points.

*This tells us that we don't even have to limit our discussion to a finite number of "bad" points, but that rather we can include infinite numbers provided only that the order of infinity is not too great. We do not wish to make too big an issue over this point, but we do want to mention it in connection with our discussion of essential singularities which follows immediately.

1.9.5(L) continued

Singular points are of two types. There is the isolated singularity in which we can draw a contour around the singularity such that the function is analytic everywhere else within (and on) the contour. This was the type of singularity discussed in the present exercise. Then there is the essential singularity in which no matter how small a contour we draw about the singularity, other singular points exist within the contour.

For example, if we define f by

$$f(z) = \frac{1}{\sin(\frac{1}{z})},$$

we see that our denominator vanishes whenever $\frac{1}{z} = k\pi$ for any integer k. That is, f(z) is infinite (hence, non-analytic) whenever $z = \frac{1}{k\pi}$. In particular, f has a singularity at $z = 0$ [since $\frac{1}{\sin \frac{1}{0}}$ isn't even defined]. Moreover, if we enclose $z = 0$ by any circle, say, with arbitrarily small non-zero radius, R, this circle contains points of the form $\frac{1}{k\pi}$ in it. That is, if the radius of the circle is R, then for positive values of k,

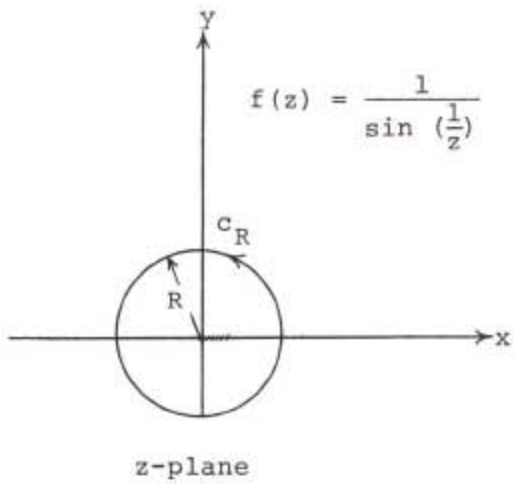
$$\frac{1}{k\pi} < R \leftrightarrow \frac{1}{k} < \pi R$$

$$\leftrightarrow k > \frac{1}{\pi R}.$$

Thus, if $R = 10^{-6}$, $k > \frac{1}{\pi(10^{-6})}$ then $\frac{1}{k\pi} \in R$. In this illustration, $\frac{1}{\pi(10^{-6})} = \frac{10^6}{\pi} < 10^5$; hence, $k > 10^5$ (for example) implies that $\frac{1}{k\pi} < 10^{-6}$.

Pictorially,

1.9.5(L) continued



① No matter how small $R > 0$ is chosen, $k > \frac{1}{\pi R}$ implies that $z = \frac{1}{k\pi} \varepsilon c_R$ and for all such z, f(z) is non-analytic, having the form $\frac{1}{0}$.

② Thus, we cannot isolate $z = 0$ from other singularities of f(z).

In our brief treatment of singularities, we shall concentrate only on isolated singularities. Even more specifically, we shall concern ourselves with those isolated singularities which are called <u>poles</u>.

More mathematically, the isolated singularity at $z = a$ of the function f(z) is called a pole if there exists a positive integer k such that $(z - a)^k f(z)$ is analytic at $z = a$. If k is the smallest integer for which $(z - a)^k f(z)$ is analytic at $z = a$, we say that $z = a$ is <u>a pole of order k</u>.

For example, if $f(z) = \frac{e^z}{(z - i)^3}$, then f has an isolated singularity singularity at $z = i$, and this singularity is a pole of order 3 since

$$(z - i)^3 \frac{e^z}{(z - i)^3}$$

is analytic at $z = i$, and $k = 3$ is the smallest exponent for which $(z - i)^k \frac{e^z}{(z - i)^3}$ is analytic at $z = i$.

Notice that not all isolated singularities are poles. For example, if $f(z) = \cos \frac{1}{z - 1}$, then f is analytic everywhere except when $z = 1$. Thus, $\cos \frac{1}{z - 1}$ has an isolated singularity at $z = 1$.

1.9.5(L) continued

Now, since

$$\cos u = \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n}}{(2n)!},$$

we see that

$$\cos\left(\frac{1}{z-1}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{z-1}\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!(z-1)^{2n}}.$$

That is,

$$\cos\left(\frac{1}{z-1}\right) = 1 - \frac{1}{2!(z-1)^2} + \frac{1}{4!(z-1)^4} - \frac{1}{6!(z-1)^6} + \cdots \qquad (1)$$

and there is no finite power k of $(z - 1)$ which makes $(z-1)^k \cos\left(\frac{1}{z-1}\right)$ analytic since our denominators in (1) are endlessly increasing powers of $z - 1$. Therefore, $z = 1$ is not a pole of $\cos\left(\frac{1}{z-1}\right)$ even though it is an isolated singularity.

The point is that in our discussion of

$$\oint_c \frac{f(z)dz}{(z-a)^n}$$

where $f(z)$ is analytic in and on c where c is a closed contour containing $z = a$; we are dealing with the function $g(z) = \frac{f(z)}{(z-a)^n}$ which has a pole of order n at $z = a$ [since then $(z-a)^n g(z) = f(z)$, which is analytic at $z = a$].

At any rate, relative to the discussion in the present exercise, if c is a simple-closed curve which is the boundary of a region R and if $f(z)$ is analytic on c and in R (except for at most a finite number of poles in R), then we may evaluate $\oint_c f(z)dz$ by the method used in the solution of this exercise.

1.9.6(L)

$$f(z) = \frac{z^3}{(z - 1)\left(z^2 + 1\right)^2} = \frac{z^3}{(z - 1)[(z + i)(z - i)]^2}$$

$$= \frac{z^3}{(z - 1)(z + i)^2(z - i)^2}. \qquad (1)$$

From (1), we see that f has three (isolated) singularities in the z-plane and these are

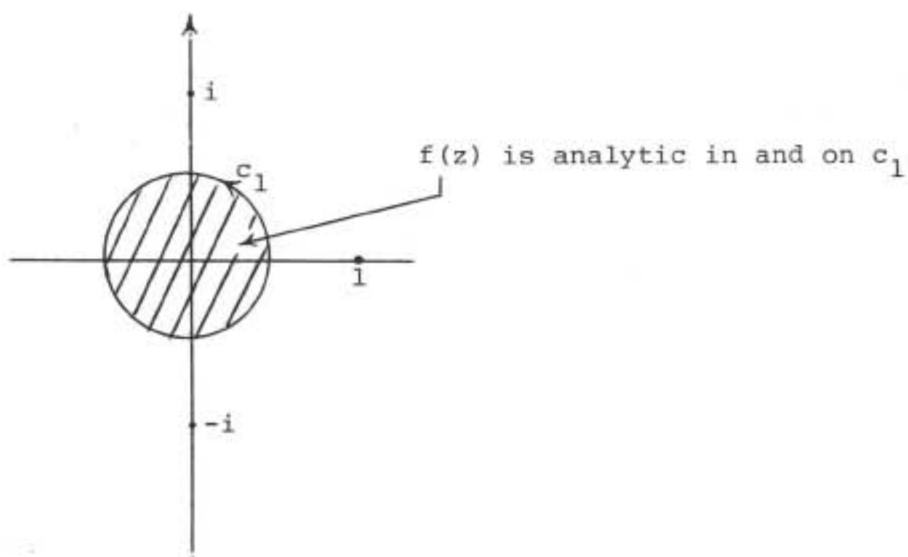
(i) a simple pole (i.e., a pole of order 1) at $z = 1$

(ii) a pole of order 2 at $z = -i$

(iii) a pole of order 2 at $z = i$.

a. Since c_1 is the circle of radius $\frac{1}{2}$ centered at $z = 0$, we see that c_1 encloses no singular points of f.

Pictorially,

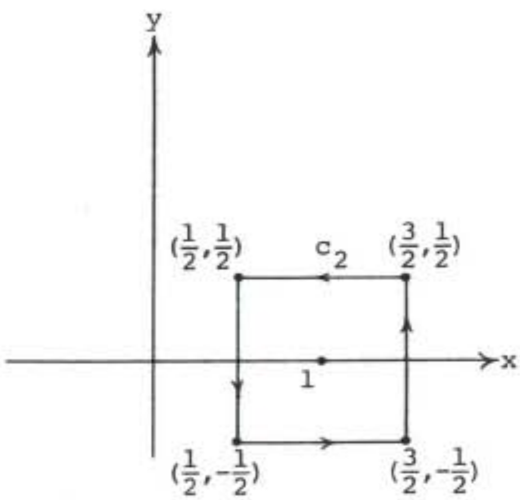


Since $\dfrac{z^3}{(z - 1)\left(z^2 + 1\right)^2}$ is analytic in and on c_1,

1.9.6(L)

$$\oint_{c_1} \frac{z^3 dz}{(z-1)\left(z^2+1\right)^2} = 0.$$

b. Since c_2 is the square with vertices at $(\frac{1}{2},-\frac{1}{2})$, $(\frac{3}{2},-\frac{1}{2})$, $(\frac{1}{2},\frac{1}{2})$, and $(\frac{3}{2},\frac{1}{2})$, we have



Thus, c_2 encloses only the singularity at $z = 1$. In this case,

$$\oint_{c_2} \frac{z^3}{(z-1)\left(z^2+1\right)^2} dz$$

may be viewed as

$$\oint_{c_2} \frac{g(z)}{(z-1)} dz$$

where

$$g(z) = \frac{z^3}{\left(z^2+1\right)^2}$$

is analytic in and on c_2.

1.9.6(L) continued

Therefore,

$$\oint_{c_2} \frac{g(z)dz}{z-1} = 2\pi i\ g(1)$$

$$= 2\pi i\left[\frac{1^3}{\left(1^2+1\right)^2}\right]$$

$$= 2\pi i\left[\frac{1}{4}\right]$$

$$= \frac{\pi i}{2}.$$

That is,

$$\oint_{c_2} \frac{z^3dz}{(z-1)\left(z^2+1\right)^2} = \frac{\pi i}{2}. \tag{2}$$

c. Since c_3 is the circle centered at $z = i$ with radius $\frac{3}{4}$, we see that the only singularity of $\dfrac{z^3}{(z-1)\left(z^2+1\right)^2}$ in or on c_3 is $z = i$. With this in mind, we write

$$\oint_{c_3} \frac{z^3dz}{(z-1)\left(z^2+1\right)^2} = \oint \frac{z^3dz}{(z-1)(z+i)^2(z-i)^2},$$

and letting $h(z) = \dfrac{z^3}{(z-1)(z+i)^2}$, we see that

$$\oint_{c_3} \frac{z^3dz}{(z-1)\left(z^2+1\right)^2} = \oint_{c_3} \frac{h(z)dz}{(z-i)^2}$$

where $h(z)$ is analytic in and on c_3.

1.9.6(L) continued

Hence,

$$\oint_{c_3} \frac{h(z)dz}{(z - i)^2} = 2\pi i\ h'(i). \tag{3}$$

Since

$$h(z) = \frac{z^3}{(z - 1)(z + i)^2}$$

$$h'(z) = \frac{[(z - 1)(z + i)^2]3z^2 - z^3[(z - 1)2(z + i) + (z + i)^2]}{(z - 1)^2(z + i)^4}$$

$$= \frac{(z + i)z^2[3(z - 1)(z + i) - z\{2(z - 1) + (z + i)\}]}{(z - 1)^2(z + i)^4}$$

$$= \frac{(z + i)z^2[3z^2 - 3i - 3z + 3zi - 3z^2 + 2z - iz]}{(z - 1)^2(z + i)^4}$$

$$= \frac{(z + i)z^2[-z(1 - 2i) - 3i]}{(z - 1)^2(z + i)^4}.$$

In particular,

$$h'(i) = \frac{(2i)i^2[-i(1 - 2i) - 3i]}{(i - 1)^2(2i)^4}$$

$$= \frac{-2i[-4i - 2]}{(i - 1)^2\ 16}$$

$$= \frac{4i(1 + 2i)}{-2i\ 16}$$

$$= -\frac{1 + 2i}{8}.$$

1.9.6(L) continued

Hence, from (3), we conclude that

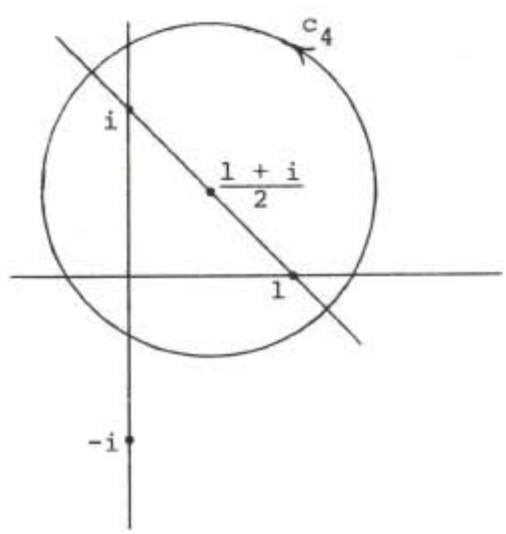
$$\oint_{c_3} \frac{z^3 dz}{(z-1)\left(z^2+1\right)^2} = \oint \frac{h(z)dz}{(z-i)^2} = 2\pi i\left(-\frac{1+2i}{8}\right)$$

$$= -\frac{\pi i(1+2i)}{4}$$

$$= \frac{\pi}{2} - \frac{\pi i}{4}. \tag{4}$$

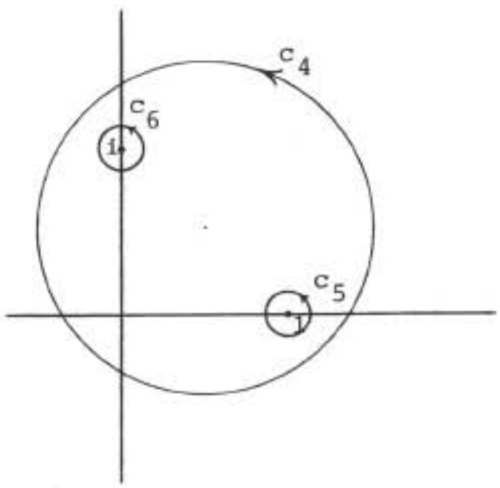
d. Since $\frac{1+i}{2}$ is the average of 1 and i, $z = \frac{1+i}{2}$ is the midpoint of the segment joining i to 1. Thus, the circle centered at $\frac{1+i}{2}$ with radius 1 encloses the poles at $z = i$ and $z = 1$, but not the pole at $z = -i$.

Pictorially,



We now enclose 1 by the small contour c_5 and i by c_6. Say,

1.9.6(L) continued



Then

$$\oint_{c_4} \frac{z^3 dz}{(z-1)\left(z^2+1\right)^2} = \oint_{c_5} \frac{z^3 dz}{(z-1)\left(z^2+1\right)^2} + \oint_{c_6} \frac{z^3 dz}{(z-1)\left(z^2+1\right)^2}. \tag{5}$$

We may now "cheat" by utilizing some previous results as follows. With c_2 as in (b), c_5 lies within c_2 and since $\dfrac{z^3}{(z-1)\left(z^2+1\right)^2}$ is analytic on the region enclosed between c_2 and c_5, we conclude that

$$\oint_{c_5} \frac{z^3 dz}{(z-1)\left(z^2+1\right)^2} = \oint_{c_2} \frac{z^3 dz}{(z-1)\left(z^2+1\right)^2}$$

$$= \frac{\pi i}{2} \quad \text{[by (2)]}. \tag{6}$$

Similarly, with c_3 as in part (c), since c_6 is contained within c_3 and $\dfrac{z^3}{(z-1)\left(z^2+1\right)^2}$ is analytic on the region bounded by c_3 and c_6, we conclude that

1.9.6(L) continued

$$\oint_{c_6} \frac{z^3 dz}{(z-1)\left(z^2+1\right)^2} = \oint_{c_3} \frac{z^3 dz}{(z-1)\left(z^2+1\right)^2}$$

$$= \frac{\pi}{2} - \frac{\pi i}{4} \quad \text{[by (4)].} \tag{7}$$

Substituting (6) and (7) into (5) yields

$$\oint_{c_4} \frac{z^3 dz}{(z-1)\left(z^2+1\right)^2} = \frac{\pi i}{2} + \frac{\pi}{2} - \frac{\pi i}{4} = \frac{\pi i}{4} + \frac{\pi}{2}.$$

1.9.7

Since c is the circle $|z| = 1$, $f(z) = \frac{1+z}{z(2-z)}$ is analytic on and within c except for a simple pole at $z = 0$. Thus, letting $g(z) = \frac{1+z}{2-z}$, we have that

$$\oint_c \frac{(1+z)dz}{z(2-z)} = \oint_c \frac{g(z)dz}{z} \tag{1}$$

and since $g(z)$ is analytic in the region enclosed by c, we may conclude that

$$\oint_c \frac{g(z)dz}{z} \left[= \oint_c \frac{g(z)dz}{z-0}\right] = 2\pi i\ g(0)$$

$$= 2\pi i\left[\frac{1+0}{2-0}\right]$$

$$= \pi i. \tag{2}$$

Combining (2) and (1) yields

$$\oint_c \frac{(1+z)dz}{z(2-z)} = \pi i.$$

1.9.8(L)

Obviously (we hope!) one would not need the calculus of complex functions to solve this problem. Indeed

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \arctan x \Big]_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

What we want to do in this exercise is start with a problem which is computationally easy and whose answer is already known to us by other techniques (to serve as a check), but which will illustrate how the use of complex variables helps us evaluate certain types of real definite integrals.

The techniques which we shall employ here works in general to evaluate any integral of the form

$$\int_{-\infty}^{\infty} \frac{P(x)dx}{Q(x)}$$

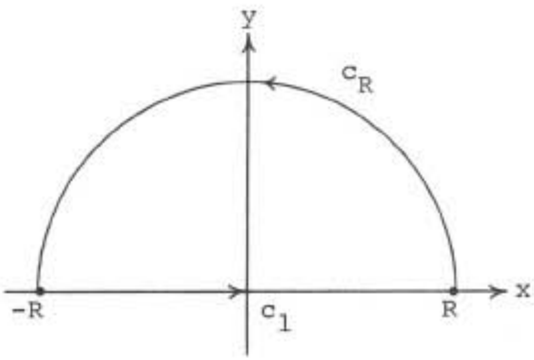
where (i) $P(x)$ and $Q(x)$ are polynomials in x, (ii) the degree of $Q(x)$ is at least 2 greater than that of $P(x)$, and (iii) $Q(x) = 0$ has no real roots.

$\displaystyle\int_0^{\infty} \frac{dx}{1 + x^2}$ is a special case of the general result with $P(x) = 1$ and $Q(x) = 1 + x^2$. Both 1 and $1 + x^2$ are polynomials; the degree of $1 + x^2$ is 2 while the degree of 1 is 0; and $1 + x^2 = 0$ has no real roots.

What we do is replace x by z and look at $\displaystyle\oint_C \frac{dz}{1 + z^2}$. We observe that $\displaystyle\frac{1}{1 + z^2} = \frac{1}{(z + i)(z - i)}$ is analytic everywhere except for simple poles at $z = i$ and $z = -i$.

We then evaluate $\displaystyle\oint_C \frac{dz}{1 + z^2}$ around the special contour shown below.

1.9.8(L) continued



That is, $c = c_1 \cup c_R$ where c_1 is the portion of the real axis from -R to R and c_R is the upper half of the circle centered at $z = 0$ and going from R to -R.

More mathematically,

$$c_1 = \{z: z = x, -R \leqslant x \leqslant R\} \tag{1}$$

$$c_R = \{z: z = Re^{i\theta}, 0 \leqslant \theta \leqslant \pi\}. \tag{2}$$

In our present example, we observe that once $R > 1$ (since the only pole [singularity] of $\frac{1}{1 + z^2}$ in the enclosed region occurs at $z = i$ and $|i| = 1$), $\oint_c \frac{dz}{z^2 + 1}$ is a constant. In particular, for $R > 1$,

$$\oint_c \frac{dz}{1 + z^2} = \oint_c \frac{dz}{(z + i)(z - i)}$$

$$= \oint_c \frac{g(z)dz}{z - i} \tag{3}$$

where $g(z) = \frac{1}{z + i}$ is analytic in and on c.

Hence,

1.9.8(L) continued

$$\oint_c \frac{g(z)dz}{z - i} = 2\pi i\ g(i)$$

$$= 2\pi i\left[\frac{1}{i + i}\right]$$

$$= \pi. \tag{4}$$

Combining (3) and (4), we conclude that

$$\oint_c \frac{dz}{1 + z^2} = \pi \qquad (R > 1). \tag{5}$$

But there is a second way to compute $\oint_c \frac{dz}{1 + z^2}$ as a function of R. Namely,

$$\oint_c \frac{dz}{1 + z^2} = \int_{c_1} \frac{dz}{1 + z^2} + \int_{c_2} \frac{dz}{1 + z^2}. \tag{6}$$

From (1), dz = dx and we have that

$$\int_{c_1} \frac{dz}{1 + z^2} = \int_{-R}^{R} \frac{dx}{1 + x^2}. \tag{7}$$

Hence, for every R > 1, we see by substituting (7) into (6) that

$$\oint_c \frac{dz}{1 + z^2} = \int_{-R}^{R} \frac{dx}{1 + x^2} + \int_{c_R} \frac{dz}{1 + z^2}. \tag{8}$$

By equating the expressions for $\oint_c \frac{dz}{1 + z^2}$ given in (5) and (9), we conclude that <u>for every R > 1</u>,

$$\int_{-R}^{R} \frac{dx}{1 + x^2} + \int_{c_R} \frac{dz}{1 + z^2} = \pi. \tag{9}$$

1.9.8(L) continued

In particular, then, the equality in (9) should still hold when we let $R\to\infty$.

Since

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{1 + x^2} = \int_{-\infty}^{\infty} \frac{dx}{1 + x^2}$$

and since $\lim\limits_{R\to\infty} \pi = \pi$ (because π is a constant), we see from (9) that

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} + \lim_{R\to\infty} \int_{c_R} \frac{dz}{1 + z^2} = \pi \tag{10}$$

and since we are interested in evaluating $\int_{-\infty}^{\infty} \frac{dx}{1 + x^2}$, we rewrite (10) in the form

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \pi - \lim_{R\to\infty} \int_{c_R} \frac{dz}{1 + z^2}. \tag{11}$$

Our claim now is that $\lim\limits_{R\to\infty} \int_{c_R} \frac{dz}{1 + z^2} = 0$.

This is not a coincidence. In terms of the general result stated previously, if c_R is the upper half of the circle $|z| = R$ from R to -R and c_R encloses all the poles of $\frac{P(z)}{Q(z)}$ which lie in the upper half plane (i.e., those poles whose imaginary parts are positive) then $\lim\limits_{R\to\infty} \int_{c_R} \frac{P(z)dz}{Q(z)} = 0$ provided only that degree of Q exceeds the degree of P by at least 2. However, we shall demonstrate the result only for the given exercise.

We use the fact that $\left|\int_{c_R} f(z)dz\right| \leqslant ML$ where M is the maximum value of $|f(z)|$ on c_R and L is the length of c_R. Since c_R is the upper half of the circle $|z| = 1$, we have that $L = \pi R$. Therefore,

$$\left| \int_{c_R} \frac{dz}{1 + z^2}\right| \leqslant \pi R \left\{\max_{z\varepsilon c_R} \frac{1}{|1 + z^2|}\right\}. \tag{12}$$

1.9.8(L) continued

Now $\frac{1}{|1 + z^2|}$ is greatest when $|1 + z^2|$ is least.

There are several ways of estimating $|1 + z^2|$. One way is to invoke the property of absolute values that $|a - b| \geqslant \left||a| - |b|\right|$, whereupon

$$|1 + z^2| = |1 - (-z^2)| \geqslant \left|1 - |z|^2\right| = |1 - R^2| = R^2 - 1 \quad (\text{if } R > 1)$$

Then, since $|1 + z^2| \geqslant R^2 - 1$,

$$\frac{1}{|1 + z^2|} \leqslant \frac{1}{R^2 - 1}.$$

Hence, $\max\limits_{z \varepsilon C_R} \frac{1}{|1 + z^2|} \leqslant \frac{1}{R^2 - 1}$, so that from (12)

$$\left| \int_{C_R} \frac{dz}{1 + z^2} \right| \leqslant \frac{\pi R}{R^2 - 1}$$

and since $\lim\limits_{R \to \infty} \left[\frac{\pi R}{R^2 - 1}\right] = 0$, it follows that

$$\lim_{R \to \infty} \left| \int_{C_R} \frac{dz}{1 + z^2} \right| = 0. \tag{13}$$

But the only way the magnitude of a complex number can approach 0 is if the number itself approaches 0. Thus, we may conclude from (13) that

$$\lim_{R \to \infty} \int_{C_R} \frac{dz}{1 + z^2} = 0. \tag{14}$$

Putting the result of (14) into (11), we conclude (as we already knew by a more elementary method) that

1.9.8(L) continued

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \pi. \tag{15}$$

Before pursuing (15) further, let us hasten to say that if you feel uncomfortable with inequalities involving absolute values, we could "bludgeon out" the required inequality more basically by observing that with $z = Re^{i\theta}$,

$$\begin{aligned}
|1 + z^2| = |1 + R^2e^{i2\theta}| &= |1 + R^2(\cos 2\theta + i \sin 2\theta)| \\
&= |(1 + R^2\cos 2\theta) + i\, R^2\sin 2\theta| \\
&= \sqrt{(1 + R^2\cos 2\theta)^2 + (R^2\sin 2\theta)^2} \\
&= \sqrt{1 + 2R^2\cos 2\theta + R^4\cos^2 2\theta + R^4\sin^2 2\theta} \\
&= \sqrt{1 + 2R^2\cos 2\theta + R^4}
\end{aligned}$$

and since $2R^2\cos 2\theta \geqslant -2R^2$ (since $\min \cos 2\theta = -1$),

$$|1 + z^2| \geqslant \sqrt{1 - 2R^2 + R^4} = \sqrt{(R^2 - 1)^2} = R^2 - 1 \quad (\text{if } R > 1).$$

The point is that the latter approach can become very messy if our expressions are more involved than $\frac{1}{1 + z^2}$.

Returning to (15) we notice that since $\frac{1}{1 + x^2}$ is an even function,*

$\int_{-a}^{a} \frac{dx}{1 + x^2} = 2 \int_{0}^{a} \frac{dx}{1 + x^2}$. Hence, we may conclude from (15) that

$$\int_{0}^{\infty} \frac{dx}{1 + x^2} = \frac{\pi}{2}.$$

*Recall that f is even means that $f(x) \equiv f(-x)$ [i.e., f is symmetric with respect to the y-axis].

1.9.8(L) continued

<u>However</u>, in general, since $\frac{P(x)}{Q(x)}$ need not be even we <u>cannot</u> use this method to evaluate $\int_0^\infty \frac{P(x)dx}{Q(x)}$ because it need not equal $\frac{1}{2}\int_{-\infty}^\infty \frac{P(x)dx}{Q(x)}$. This leads into why the theory of contour integration requires much experience. Namely, we must be clever enough to choose a contour part of which under suitable limits yields the given real integral and the other part must either by "easy" to evaluate when the limit is taken (such as in this exercise where the limit is 0).

Thus, while the general theory is rather simple, the ability to "invent" the right contour as well as the right form into which we translate $\int_a^b f(x)dx$ requires the kind of experience which one usually can obtain only by taking a complete course in the subject. Our aim is only to open the various avenues of exploration to you.

1.9.9 (Optional)

a. The singularities of $\frac{1}{z^6 + 1}$ occur whenever $z^6 + 1 = 0$. Thus, we are looking for those values of z such that $z^6 = -1$.
Using polar coordinates, we have $z = re^{i\theta}$ while $-1 = e^{i(\pi + 2\pi k)}$, so that $z^6 = -1$ implies that

$$r^6e^{i6\theta} = 1\, e^{i(\pi + 2\pi k)}$$

from which we conclude that

$$r = 1$$

and

$$6\theta = \pi + 2\pi k \left\{ \begin{array}{l} k = 0,1,2,3,4,5 \\ \text{since } 6\theta \text{ ranges from } 0 \text{ to } 12\pi \text{ as } \theta \text{ ranges from} \\ 0 \text{ to } 2\pi \end{array} \right\}$$

or

1.9.9 continued

$$\theta = \frac{\pi}{6} + \frac{\pi k}{3};\ k = 0,1,2,3,4,5.$$

Thus, the roots of $z^6 + 1 = 0$ are given, in polar form, by

$$(1,\tfrac{\pi}{6}),\ (1,\tfrac{\pi}{3}),\ (1,\tfrac{5\pi}{6}),\ (1,\tfrac{7\pi}{6}),\ (1,\tfrac{3\pi}{2}),\ \text{and}\ (1,\tfrac{11\pi}{6})$$

or with $z = re^{i\theta}$, we see that $\dfrac{1}{z^6 + 1}$ has simple poles at

$$z_1 = e^{i\frac{\pi}{6}},\ z_2 = e^{i\frac{\pi}{3}},\ z_3 = e^{i\frac{5\pi}{6}},\ z_4 = e^{i\frac{7\pi}{6}},\ z_5 = e^{i\frac{3\pi}{2}},\ \text{and}\ z_6 = e^{i\frac{11\pi}{6}}$$

In Cartesian form,

$$z_1 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{1}{2}(\sqrt{3} + i)$$

$$z_2 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$$

$$z_3 = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = \frac{1}{2}(-\sqrt{3} + i)$$

$$z_4 = \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6} = -\frac{1}{2}(\sqrt{3} + i)$$

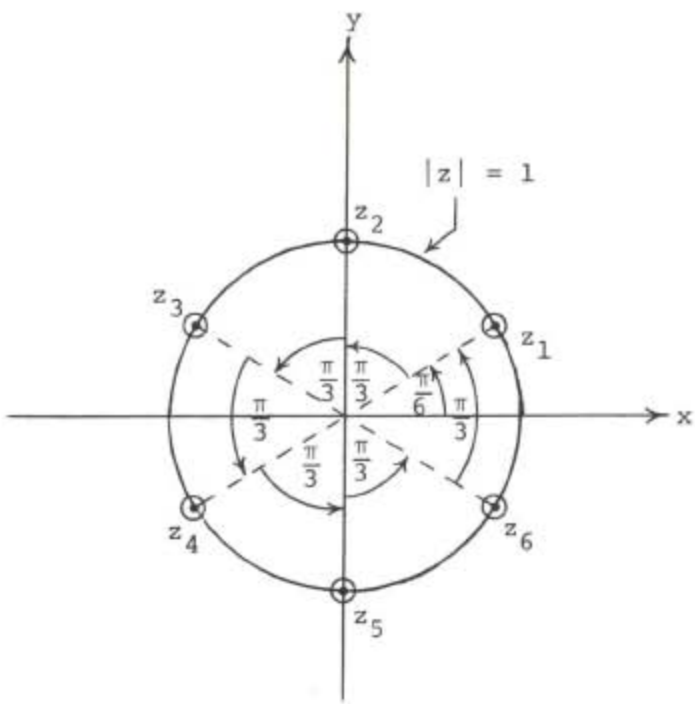
$$z_5 = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i$$

$$z_6 = \cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6} = \frac{1}{2}(\sqrt{3} - i)$$

[By way of review, notice that the roots occur as pairs of complex conjugates; a fact that must be obeyed because $z^6 + 1 = 0$ is a polynomial equation with real coefficients.]

Pictorially,

1.9.9 continued



z-plane

Figure 1

b. With $z_1, \ldots, z_6$ as in part (a), we see that $\overline{z}_1 = z_6$, $\overline{z}_2 = z_5$, and $\overline{z}_3 = z_4$. Thus,

$$\begin{aligned} z^6 + 1 &= (z - z_1)(z - z_2)(z - z_3)(z - z_4)(z - z_5)(z - z_6) \\ &= [(z - z_1)(z - z_6)][(z - z_2)(z - z_5)][(z - z_3)(z - z_4)] \\ &= [(z - z_1)(z - \overline{z}_1)][(z - z_2)(z - \overline{z}_2)][(z - z_3)(z - \overline{z}_3)] \quad (1) \end{aligned}$$

Now for any complex number, c,

$$\begin{aligned} (z - c)(z - \overline{c}) &= z^2 - (c + \overline{c})z + c\,\overline{c} \\ &= z^2 - 2\mathrm{Re}(c)z + |c|^2. \quad (2) \end{aligned}$$

Applying (2) with $c = z_1, z_2, z_3$, we have that

1.9.9 continued

(i) $(z - z_1)(z - \overline{z}_1) = z^2 - 2\text{Re}(z_1)z + |z_1|^2$,

or, since $z_1 = \frac{1}{2}(\sqrt{3} + i)$,

$$(z - z_1)(z - \overline{z}_1) = z^2 - \sqrt{3}z + 1. \tag{3}$$

(ii) $(z - z_2)(z - \overline{z}_2) = z^2 - 2\text{Re}(z_2)z + |z_2|^2$,

or, since $z_2 = i$,

$$(z - z_2)(z - \overline{z}_2) = z^2 + 1. \tag{4}$$

(iii) $(z - z_3)(z - \overline{z}_3) = z^2 - 2\text{Re}(z_3)z + |z_3|^2$,

or, since $z_3 = \frac{1}{2}(-\sqrt{3} + i)$,

$$(z - z_3)(z - \overline{z}_3) = z^2 + \sqrt{3}z + 1. \tag{5}$$

Substituting the results of (3), (4), and (5) into (1), we obtain

$$z^6 + 1 = (z^2 - \sqrt{3}z + 1)(z^2 + 1)(z^2 + \sqrt{3}z + 1). \tag{6}$$

<u>Note</u>

Without a knowledge of complex numbers we could write $z^6 + 1$ as $\left(z^2\right)^3 + 1^3$ to obtain the factorization

$$z^6 + 1 = (z^2 + 1)(z^4 - z^2 + 1). \tag{7}$$

By fairly elementary methods, we see that $z^4 - z^2 + 1$ has no rational factors, but this does not mean that it can't have real but irrational factors. In fact, if we compare (6) and (7), we would conclude that

$$(z^2 - \sqrt{3}z + 1)(z^2 + \sqrt{3}z + 1) = z^4 - z^2 + 1. \tag{8}$$

1.9.9 continued

As an independent check, notice that

$$(z^2 - \sqrt{3}z + 1)(z^2 + \sqrt{3}z + 1) = [(z^2 + 1) - \sqrt{3}z][(z^2 + 1) + \sqrt{3}z]$$
$$= \left(z^2 + 1\right)^2 - [\sqrt{3}z]^2$$
$$= z^4 + 2z^2 + 1 - 3z^2$$
$$= z^4 - z^2 + 1.$$

Yet the factorization in (8) seems far from obvious. Thus, our derivation of (8) shows yet another application of complex numbers to the algebra of real numbers.

c. If we use the technique of the previous exercise, we pick any $R > 1$ [since all the singularities of $\frac{1}{z^6 + 1}$ are simple poles which lie on the circle $|z| = 1$.]

Since z_1, z_2, and z_3 are the only singularities in the given region, we may isolate them by the contours c_1, c_2, and c_3 which lie within c, where

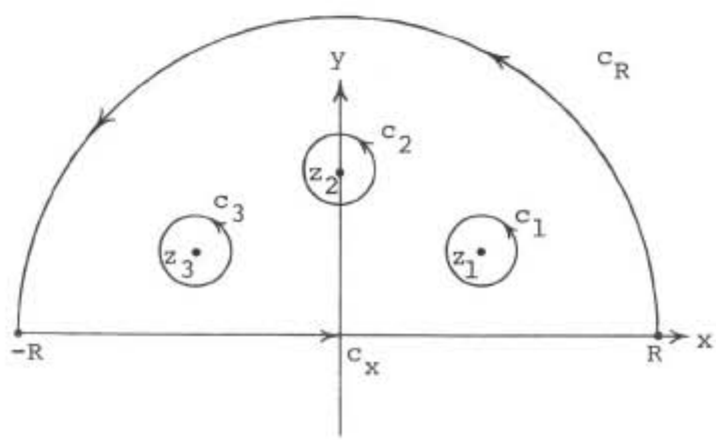


Figure 2

$$\oint_c \frac{dz}{z^6 + 1} = \int_{c_x} \frac{dz}{z^6 + 1} + \int_{c_R} \frac{dz}{z^6 + 1}. \tag{9}$$

1.9.9 continued

Since $c_x = \{z: z = x, -R \leqslant x \leqslant R$, we may rewrite (9) as

$$\oint_c \frac{dz}{z^6 + 1} = \int_{-R}^{R} \frac{dx}{x^6 + 1} + \int_{c_R} \frac{dz}{z^6 + 1}. \tag{10}$$

Since $\oint_c \frac{dz}{z^6 + 1}$ is constant once $R > 1$ and since

$\lim\limits_{R\to\infty} \int_{c_R} \frac{dz}{|z^6 + 1|} = 0$ [i.e., just as in the previous exercise, for

$z \epsilon c_R$,

$$\frac{1}{|z^6 + 1|} = \frac{1}{|z^6 - (-1)|} \leqslant \frac{1}{\left||z|^6 - |-1|\right|} = \frac{1}{|R^6 - 1|}.$$

Thus, for $R > 1$

$$\left|\int_{c_R} \frac{dz}{z^6 + 1}\right| \leqslant \pi R \max_{z \epsilon c_R} \left|\frac{1}{z^6 + 1}\right|$$

$$\leqslant \frac{\pi R}{R^6 - 1}$$

whence

$$\lim_{R\to\infty}\left|\int_{c_R} \frac{dz}{z^6 + 1}\right| = \lim_{R\to\infty} \frac{\pi R}{R^6 - 1} = 0, \text{ etc.]}$$

we may let $R\to\infty$ in (10) to conclude that

$$\oint_c \frac{dz}{z^6 + 1} = \int_{-\infty}^{\infty} \frac{dx}{x^6 + 1} \quad (+\ 0). \tag{11}$$

We may now compute the left side of (11) by observing that

1.9.9 continued

$$\oint_c \frac{dz}{z^6+1} = \oint_{c_1} \frac{dz}{z^6+1} + \oint_{c_2} \frac{dz}{z^6+1} + \oint_{c_3} \frac{dz}{z^6+1} \qquad (12)$$

where c_1, c_2, and c_3 are as shown in Figure 2 (the exact shapes are irrelevant; all that is required is that they separate z_1, z_2, and z_3 and be within c).

All that happens when we try to evaluate the right side of (12) is that the algebra gets a bit messy unless we can find some clever ways of doing the computations. [This quest for clever computational techniques is part of any course in the calculus of complex variables.]

For example, if we try to use brute force in computing, say, $\oint_{c_2} \frac{dz}{z^6+1}$, we find that $\oint_{c_2} \frac{dz}{z^6+1} = \oint_{c_2} \frac{h(z)dz}{z - z_2}$ where

$$h(z) = \frac{1}{(z-z_1)(z-z_3)(z-z_4)(z-z_5)(z-z_6)}$$

is analytic in and on c_2. We would then obtain

$$\oint_{c_2} \frac{dz}{z^6+1} = 2\pi i\ h(z_2) \qquad (13)$$

and while (13) is correct, the right side is difficult to evaluate. Using (6) helps somewhat, but even this is cumbersome.

It turns out that when all we have are simple poles (i.e., poles of order 1) there is a complex-analog of L'Hopital's Rule which we may use. [See the note at the end of this exercise.]

Recognizing that $\frac{1}{z^6+1}$ has a simple pole at $z = z_2 = i$, we write $\frac{1}{z^6+1}$ in the form

$$\frac{1}{z^6+1} = \left(\frac{z-i}{z^6+1}\right)\left(\frac{1}{z-i}\right).$$

1.9.9 continued

Letting $h(z) = \frac{z - i}{z^6 + 1}$, we see that h is analytic in and on c_2 except at $z = z_2 = i$ in which case $h(z_2) = \frac{0}{0}$.

Recalling that by L'Hopital's Rule, that if $\lim_{z \to z_2} \frac{h_1(z)}{h_2(z)} = \frac{0}{0}$ then $\lim_{z \to z_2} \frac{h_1(z)}{h_2(z)} = \lim_{z \to z_2} \frac{h_1'(z_2)}{h_2'(z_2)}$, we see that if we define $h(z_2)$ to be $\lim_{z \to z_2} h(z)$ and recall that $z_2 = i$, then

$$h(z_2) = h(i) = \lim_{z \to i} \frac{z - i}{z^6 + 1}$$

$$= \lim_{z \to i} \left[\frac{1}{6z^5}\right]$$

$$= \frac{1}{6i^5}$$

$$= \frac{1}{6i}$$

$$= \frac{-i}{6}.$$

Hence,

$$\oint_{c_2} \frac{dz}{z^6 + 1} = \oint_{c_2} \frac{h(z)dz}{z - i}$$

$$= 2\pi i\ h(i)$$

$$= 2\pi i \left(\frac{-i}{6}\right)$$

$$= \frac{1}{3}\pi. \qquad (14)$$

1.9.9 continued

In a similar way, since $z_1 = e^{\frac{i\pi}{6}}$,

$$\oint_{c_1} \frac{dz}{z^6 + 1} = \oint_{c_1} \left[\frac{z - e^{\frac{i\pi}{6}}}{z^6 + 1}\right]\left[\frac{1}{z - e^{\frac{i\pi}{6}}}\right] dz$$

$$= 2\pi i\ h_1\left(e^{\frac{i\pi}{6}}\right)$$

where

$$h_1(z) = \frac{z - e^{\frac{i\pi}{6}}}{z^6 + 1}. \tag{15}$$

Since h_1 is analytic in and on c_1 except at $z = e^{\frac{i\pi}{6}}$ where it is $\frac{0}{0}$, we may use L'Hopital's Rule to conclude that

$$h_1\left(e^{\frac{i\pi}{6}}\right) = \lim_{z \to e^{\frac{i\pi}{6}}} \left[\frac{z - e^{\frac{i\pi}{6}}}{z^6 + 1}\right]$$

$$= \lim_{z \to e^{\frac{i\pi}{6}}} \left[\frac{1}{6z^5}\right]$$

$$= \frac{1}{6\ e^{i\frac{5\pi}{6}}}$$

$$= \frac{1}{6}\ e^{-i\frac{5\pi}{6}}$$

$$= \frac{1}{6}\left[\cos\left(-\frac{5\pi}{6}\right) + i\ \sin\left(-\frac{5\pi}{6}\right)\right]$$

$$= \frac{1}{6}\left[-\frac{1}{2}\sqrt{3} - \frac{i}{2}\right]$$

$$= -\frac{1}{12}(\sqrt{3} + i).$$

1.9.9 continued

Hence, from (15),

$$\oint_{c_1} \frac{dz}{z^6 + 1} = 2\pi i\left[-\frac{1}{12}(\sqrt{3} + i)\right]$$

$$= -\frac{1}{6}\pi i(\sqrt{3} + i)$$

$$= \frac{1}{6}\pi - \frac{\sqrt{3}}{6}\pi i. \qquad (16)$$

Finally, we evaluate $\oint_{c_3} \frac{dz}{z^6 + 1}$ by observing that

$$\oint_{c_3} \frac{dz}{z^6 + 1} = \oint_{c_3} \frac{h_2(z)dz}{z - z_3} \text{ where } h_2(z) = \frac{(z - z_3)}{z^6 + 1}$$

or, since $z_3 = e^{i\frac{5\pi}{6}}$,

$$\oint_{c_3} \frac{dz}{z^6 + 1} = 2\pi i\ h_2\left(e^{i\frac{5\pi}{6}}\right)$$

$$= 2\pi i \lim_{z \to i\frac{5\pi}{6}} \left[\frac{z - e^{i\frac{5\pi}{6}}}{z^6 + 1}\right]$$

$$= 2\pi i \lim_{z \to i\frac{5\pi}{6}} \left[\frac{1}{6z^5}\right]$$

$$= 2\pi i \left[\frac{1}{6\ e^{i\frac{25\pi}{6}}}\right]$$

$$= 2\pi i \left[\frac{1}{6\ e^{i\frac{\pi}{6}}}\right]$$

$$= \frac{1}{3}\pi i\ e^{-i\frac{\pi}{6}}$$

$$= \frac{1}{3}\pi i\left[\frac{1}{2}\sqrt{3} - \frac{1}{2}i\right]$$

$$= \frac{\sqrt{3}\pi}{6}i + \frac{1}{6}\pi. \qquad (17)$$

1.9.9 continued

Combining the results of (14), (16), and (17) by substituting them into (12), we obtain

$$\oint_C \frac{dz}{z^6 + 1} = \frac{1}{6}\pi - \frac{\sqrt{3}\pi i}{6} + \frac{1}{3}\pi + \frac{\sqrt{3}\pi i}{6} + \frac{1}{6}\pi$$

$$= \frac{2}{3}\pi . \qquad (18)$$

Comparing the value of $\oint_C \frac{dz}{z^6 + 1}$ in (11) with its value in (18), we conclude that

$$\int_{-\infty}^{\infty} \frac{dx}{x^6 + 1} = \frac{2}{3}\pi$$

and since $\frac{1}{x^6 + 1}$ is an even function

$$\int_0^{\infty} \frac{dx}{x^6 + 1} = \frac{\pi}{3}.$$

<u>A Note on L'Hopital's Rule</u>

Suppose $f(z)$ and $g(z)$ are <u>analytic</u> at $z = a$ and that <u>$f(a) = g(a) = 0$</u>. Our claim is that, just as in the real case,

$$\lim_{z \to a} \frac{f(z)}{g(z)} = \lim_{z \to a} \frac{f'(z)}{g'(z)}.$$

The proof is not difficult. Namely, we know that <u>near</u> $z = a$,

$$f(z) = f(a) + f'(a)(z - a) + \frac{f''(a)}{2!}(z - a)^2 + (z - a)^3 \; [\ldots\ldots]$$

and since $f(a) = 0$,

$$f(z) = f'(z)(z - a) + \frac{f''(a)}{2!}(z - a)^2 + (z - a)^3 \; [\ldots\ldots]$$

$$= (z - a)\left[f'(a) + \frac{f''(a)}{2!}(z - a) + (z - a)^2 \; \{\ldots\ldots\}\right]. \qquad (1)$$

1.9.9 continued

Similarly,

$$g(z) = (z - a)\left[g'(a) + \frac{g''(a)}{2!}(z - a) + (z - a)^2 \{\ldots\ldots\}\right]. \tag{2}$$

With $z \neq a$, we may divide (1) by (2) to obtain

$$\frac{f(z)}{g(z)} = \frac{f'(a) + \frac{f''(a)}{2!}(z - a) + (z - a)^2 \{\ldots\ldots\}}{g'(a) + \frac{g''(a)}{2!}(z - a) + (z - a)^2 \{\ldots\ldots\}}. \tag{3}$$

Letting $z \to a$ [i.e., $(z - a) \to 0$] in (3) yields

$$\lim_{z \to a} \frac{f(z)}{g(z)} = \frac{f'(a)}{g'(a)},$$

unless $f'(a) = g'(a) = 0$, in which case we apply the rule again to

$$\lim_{z \to a} \frac{f'(z)}{g'(z)} \text{ etc.}$$

The most important point is to recognize that while there are very few basic ideas involved in the study of analytic functions, these ideas are used with great subtlety and in considerable depth. Our only hope in this block is that you get enough feeling for the basic ideas to appreciate the great scope and realness of the complex numbers, as well as a feeling for how the structures of the real and complex numbers are in some ways similar and in other ways very different.

When $f(z)$ is analytic in a deleted neighborhood of $z = a$ and $f(a) = \frac{0}{0}$ then we say that f has a removable singularity at $z = a$ if $\lim_{z \to a} f(z)$ exists. In this case, letting $h = \lim_{z \to a} f(z)$, we define a new function g such that

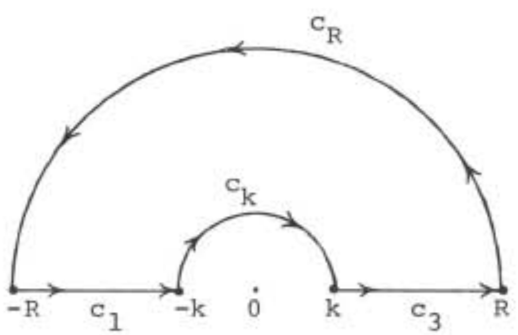
$$g(z) = \begin{cases} f(z), \text{ if } z \neq a \\ L, \text{ if } z = a \end{cases}$$

1.9.9 continued

In this way, $g(z)$ behaves exactly like $f(z)$ when $z \neq a$, but fills in the "missing point" when $z = a$. $g(z)$ is usually obtained by using L'Hopital's Rule to compute $\lim_{z \to a} f(z)$.

1.9.10 (Optional)

Since $\frac{e^{iz}}{z}$ is analytic everywhere except at $z = 0$, $\oint_c \frac{e^{iz}}{z}\,dz = 0$ around any simple closed path which excludes $z = 0$.* The given path c fulfills this requirement. Namely,



Since $c = c_1 \cup c_k \cup c_3 \cup c_R$ and since $\oint_c \frac{e^{iz}}{z}\,dz = 0$ [because $\frac{e^{iz}}{z}$ is analytic in any region which excludes $z = 0$], we may conclude that

$$\int_{c_1} \frac{e^{iz}dz}{z} - \int_{c_k} \frac{e^{iz}dz}{z} + \int_{c_3} \frac{e^{iz}dz}{z} + \int_{c_R} \frac{e^{iz}dz}{z} = 0 \qquad (1)$$

(we use the minus sign, as shown below, to reverse the sense of c_k to agree with our diagram) where

---

*Notice, therefore, that unlike in the previous exercise, our contour must exclude part of the real axis (i.e., $z = 0$). This was why it was crucial in the previous that $Q(x)$ have no real zeroes - otherwise our integral $\int_{c_1} \frac{P(z)dz}{Q(z)}$ would be improper once R was large enough to include a real singular point of $\frac{P(z)}{Q(z)}$.

1.9.10 continued

$c_1 = \{z: z = x, -R \leqslant x \leqslant -k\}$

$c_k = \{z: z = ke^{i\theta}, 0 \leqslant \theta \leqslant \pi\}$ $\left[\text{so } \frac{dz}{z} = \frac{ike^{i\theta}d\theta}{ke^{i\theta}} = id\theta\right]$

$c_3 = \{z: z = x, k \leqslant x \leqslant R\}$

$c_R = \{z: z = Re^{i\theta}, 0 \leqslant \theta \leqslant \pi\}.$

Putting this information into (1) yields

$$\int_{-R}^{-k} \frac{e^{ix}dx}{x} - \int_0^{\pi} e^{ike^{i\theta}} id\theta + \int_k^R \frac{e^{ix}dx}{x} + \int_0^{\pi} e^{iRe^{i\theta}} id\theta = 0. \qquad (2)$$

Now it seems like a good hunch that $\int_0^{\infty} \frac{\sin x\, dx}{x}$ $\left[= \text{Im} \int_0^{\infty} \frac{e^{ix}dx}{x}\right]$ is going to be identified with $\int_k^R \frac{\sin x\, dx}{x}$, taken in the limit as $k \to 0$ and $R \to \infty$. With this in mind, we rewrite $\int_{-R}^{-k} \frac{e^{ix}dx}{x}$ by first letting $u = -x$ to obtain

$$\int_R^k \frac{e^{-iu}\ (-du)}{-u} = \int_R^k \frac{e^{-iu}du}{u} = -\int_k^R \frac{e^{-iu}du}{u}$$

and this in turn, since u is simply a dummy variable, is $-\int_k^R \frac{e^{-ix}dx}{x}$. That is,

$$\int_{-R}^{-k} \frac{e^{ix}dx}{x} = -\int_k^R \frac{e^{-ix}dx}{x}.$$

Hence

1.9.10 continued

$$\int_{-R}^{-k} \frac{e^{ix}dx}{x} + \int_{k}^{R} \frac{e^{ix}dx}{x} = -\int_{k}^{R} \frac{e^{-ix}dx}{x} + \int_{k}^{R} \frac{e^{ix}dx}{x}$$

$$= \int_{k}^{R} \frac{e^{ix} - e^{-ix}}{x}\, dx.$$

Then, since $e^{ix} - e^{-ix} = 2i \sin x$, we see that (2) may be written as

$$2i\int_{k}^{R} \frac{\sin x\, dx}{x} - \int_{0}^{\pi} e^{ike^{i\theta}} i d\theta + \int_{0}^{\pi} e^{iRe^{i\theta}} i d\theta = 0. \tag{3}$$

If we now let $k \to 0$, we see that $e^{ike^{i\theta}} \to e^{i0} = 1$ so that

$$\lim_{k \to 0} \int_{0}^{\pi} e^{ike^{i\theta}} i d\theta = \int_{0}^{\pi} i d\theta = \pi i. \tag{4}$$

Moreover, since R and k are independent (except that $k < R$) $\int_{0}^{\pi} e^{iRe^{i\theta}} i d\theta$ is independent of k.

However, we do know that

$$\left| \int_{0}^{\pi} e^{iRe^{i\theta}} i d\theta \right| \leqslant \pi \max_{0 < \theta < \pi} \left| e^{iRe^{i\theta}} i \right|,$$

and since $|i| = 1$, we may say that

$$\left| \int_{0}^{\pi} e^{iRe^{i\theta}} i d\theta \right| \leqslant \pi \max_{0 \leqslant \theta \leqslant \pi} \left| e^{iRe^{i\theta}} \right|.$$

Now

1.9.10 continued

$e^{iRe^{i\theta}}$ is quite a mouthful, but recalling that $e^{i\theta} = \cos\theta + i\sin\theta$, we see that

$$e^{iRe^{i\theta}} = e^{iR(\cos\theta + i\sin\theta)}$$

$$= e^{iR\cos\theta - R\sin\theta}$$

$$= e^{iR\cos\theta}\, e^{-R\sin\theta}.$$

Hence

$$\left|e^{iRe^{i\theta}}\right| = |e^{iR\cos\theta}\, e^{-R\sin\theta}|$$

$$= |e^{iR\cos\theta}|\; |e^{-R\sin\theta}|, \tag{6}$$

but $|e^{ix}| = 1$ for all real x, so since R cos θ is real

$$|e^{iR\cos\theta}| = 1. \tag{7}$$

Moreover, $e^{\pm x}$ is positive for all real x, so since $-R\sin\theta$ is real, $e^{-R\sin\theta}$ is positive,

$$|e^{-R\sin\theta}| = e^{-R\sin\theta}. \tag{8}$$

Putting (7) and (8) into (6) yields

$$\left|e^{iRe^{i\theta}}\right| = e^{-R\sin\theta}. \tag{9}$$

Since $z = Re^{i\theta}$ covers just one special contour, it might be well to generalize (9) by observing that $e^{iz} = e^{i(x + iy)} = e^{ix - y} = e^{ix}\, e^{-y}$. Hence

$$|e^{iz}| = |e^{ix}|\; |e^{-y}|$$

$$= e^{-y}$$

$$= e^{-\text{Im } z} \tag{10}$$

1.9.10 continued

and we see that (9) is a special case of (10).

Now on the interval $0 \leqslant \theta \leqslant \pi$, $e^{-R \sin \theta}$ has an interesting behavior. Namely, if we pick $\theta_o \in (0,\pi)$ [i.e., $\theta_o \neq 0$, $\theta_o \neq \pi$], we see that

$$e^{-R \sin \theta_o} = \left[e^{-\sin \theta_o}\right]^R = \left[\frac{1}{e^{\sin \theta_o}}\right]^R. \tag{11}$$

Since $0 < \theta_o < \pi$, we know that $\sin \theta_o > 0$ and hence that

$e^{\sin \theta_o} > 1$. Therefore, $\frac{1}{e^{\sin \theta_o}} < 1$ (and non-negative). Hence,

$\left[\frac{1}{e^{\sin \theta_o}}\right]^R \to 0$ as $R \to \infty$.

In other words, we conclude from (11) that if $0 < \theta < \pi$, $\lim_{R\to\infty} e^{-R \sin \theta} = 0$, or in terms of (9),

$$\lim_{R\to\infty} \left|e^{iRe^{i\theta}}\right| = 0 \text{ if } 0 < \theta < \pi. \tag{12}$$

However, if $\theta = 0$ or $\theta = \pi$, $e^{-R \sin \theta} = e^{-0} = 1$ in which case

$$\lim_{R\to\infty} \left|e^{iRe^{i\theta}}\right| = 1.$$

Technically speaking then, all we can conclude from (3) is that

$$\left|\int_0^{\pi} e^{iRe^{i\theta}} i d\theta\right| \leqslant \pi.$$

However, if we can exclude the exceptional behavior at $\theta = 0$ and $\theta = \pi$, we could obtain from (3) that

1.9.10 continued

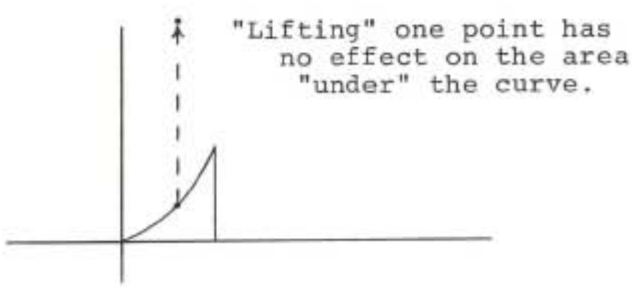
$$\lim_{R\to 0}\left|\int_0^{\pi} e^{iRe^{i\theta}}\, i\,d\theta\right| \leqslant \lim_{R\to\infty}\left[\max_{0<\theta<\pi}\left|e^{iRe^{i\theta}}\right|\right] = 0. \tag{13}$$

If (13) is valid, it means that we may neglect $\int_R \frac{e^{iz}dz}{z}$ when we let $R\to\infty$ in (3).

The validity of (13) stems from the refinement that $\left|\int_{c_R} f(z)dz\right| \leqslant ML$ where M is the maximum value of $|f(z)|$ on c_R except possibly at a finite number of points on c_R. That is, changing the value of the integrand at a finite number of points does not affect the value of the integral.

This was also true in the real case. For example, suppose $f(x) = x^2$, $0 \leqslant x \leqslant 1$, then $\int_0^1 f(x)dx = \frac{1}{3}$. On the other hand, if

$$g(x) = \begin{cases} f(x), & x \neq \frac{1}{2} \\ 100, & x = \frac{1}{2} \end{cases}$$ then $\int_0^1 g(x)dx = \frac{1}{3}$ also. Pictorially,



At any rate, using the results of (4) and (13) in (3), letting $k\to 0$, and $R\to\infty$, we obtain

$$2i \lim_{\substack{k\to 0\\ R\to\infty}} \int_k^R \frac{\sin x\, dx}{x} - \pi i = 0$$

so that

1.9.10 continued

$$\lim_{\substack{k\to 0\\ R\to\infty}} \int_k^R \frac{\sin x\, dx}{x} = \frac{\pi}{2}. \tag{14}$$

We may write the left side of (14) as

$$\int_0^\infty \frac{\sin x\, dx}{x}$$

provided we remember our discussion of the previous unit and recall that we mean the Cauchy Principal Value, $P\int_0^\infty \frac{\sin x\, dx}{x}$, when we write $\int_0^\infty \frac{\sin x\, dx}{x}$.

However, since the improper integral $\int_0^\infty \frac{\sin x\, dx}{x}$ is convergent, its value must agree with the Cauchy Principal Value, so in this case, we may, without ambiguity, write (14) as

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}. \tag{15}$$

Again, since $\frac{\sin(-x)}{-x} = \frac{\sin x}{x}$, $\frac{\sin x}{x}$ is an even function and we may conclude from (15) that

$$\int_{-\infty}^\infty \frac{\sin x\, dx}{x}\, dx = \pi.$$

Quiz

1. By DeMoivre's Theorem, we have that for each whole number n,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \tag{1}$$

Letting $n = 7$ in (1), we obtain

$$(\cos\theta + i\sin\theta)^7 = \cos 7\theta + i\sin 7\theta. \tag{2}$$

Expanding the left side of (2) by the binomial theorem, we have

$$\begin{aligned}(\cos\theta + i\sin\theta)^7 &= \cos^7\theta + 7\cos^6\theta(i\sin\theta) + 21\cos^5\theta(i\sin\theta)^2 \\ &\quad + 35\cos^4\theta(i\sin\theta)^3 + 35\cos^3\theta\,(i\sin\theta)^4 \\ &\quad + 21\cos^2\theta(i\sin\theta)^5 + 7\cos\theta(i\sin\theta)^6 \\ &\quad + (i\sin\theta)^7 \\ &= \cos^7\theta + i\,7\cos^6\theta\,\sin\theta - 21\cos^5\theta\sin^2\theta \\ &\quad -i\,35\cos^4\theta\sin^3\theta + 35\cos^3\theta\sin^4\theta \\ &\quad +i\,21\cos^2\theta\sin^5\theta - 7\cos\theta\sin^6\theta - i\sin^7\theta \\ &= \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta \\ &\quad - 7\cos\theta\sin^6\theta + i\,(7\cos^6\theta\sin\theta \\ &\quad - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta).\end{aligned} \tag{3}$$

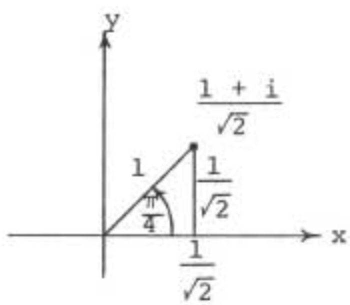
Comparing the right sides of (2) and (3) and recalling that two complex numbers are equal if and only if their real and imaginary parts are equal, we see by equating the imaginary parts that

$$\sin 7\theta = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta.$$

2. (a) Perhaps the simplest approach here is to observe that

$$\frac{1 + i}{\sqrt{2}} = e^{i\frac{\pi}{4}}.$$

Pictorially



Noticing that

$$e^{i\frac{\pi}{4}},\ e^{i(\frac{\pi}{4}+2\pi)},\ e^{i(\frac{\pi}{4}+4\pi)},\ \text{and } e^{i(\frac{\pi}{4}+6\pi)}$$

all name the same number, and letting $z = re^{i\theta}$, we conclude that $z^4 = \frac{1 + i}{\sqrt{2}}$ implies that

$$r^4e^{i4\theta} = e^{i(\frac{\pi}{4}+2\pi n)},\ \text{where } n = 0,1,2,3.$$

Hence,

$$r = 1 \tag{1}$$

and

$$4\theta = \frac{\pi}{4} + 2\pi n,$$

so that

$$\theta = \frac{\pi}{16} + \frac{\pi n}{2},\ n = 0,1,2,3.$$

2. continued

That is,

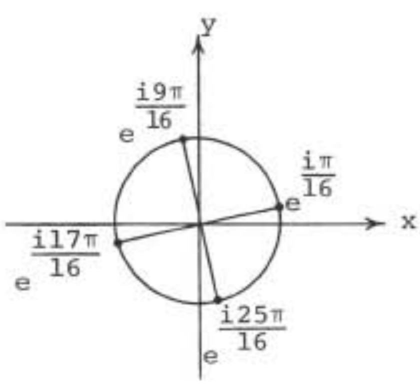
$$\theta = \frac{\pi}{16}, \frac{9\pi}{16}, \frac{17\pi}{16}, \frac{25\pi}{16}. \tag{2}$$

Combining (1) and (2) and recalling that $(r,\theta) = re^{i\theta}$, we conclude

$$\left\{z\colon z^4 = \frac{1 + i}{\sqrt{2}}\right\} = \left\{e^{\frac{i\pi}{16}}, e^{\frac{i9\pi}{16}}, e^{\frac{i17\pi}{16}}, e^{\frac{i25\pi}{16}}\right\}.$$

Note #1

Since $\left|\frac{1 + i}{\sqrt{2}}\right| = 1$, the roots all lie on the unit circle centered at the origin. The "primitive" root has its argument equal to one-fourth that of $\frac{1 + i}{\sqrt{2}}$ (i.e. $\frac{\pi}{16}$) and the four roots are then distributed at 90° intervals. Pictorially



Note #2

If we prefer the a + bi form then, for example,

$$\begin{aligned} e^{\frac{i\pi}{16}} &= \cos \frac{\pi}{16} + i \sin \frac{\pi}{16} \\ &= \cos 11.25° + i \sin 11.25° \\ &\approx .9808 + .1951i. \end{aligned}$$

2. continued

In fact, the four roots are then given by (approximately)

$$.9808 + .1951i = \cos \frac{\pi}{16} + i \sin \frac{\pi}{16}$$

$$-.1951 + .9808i = \cos(\frac{\pi}{16} + \frac{\pi}{2}) + i \sin(\frac{\pi}{16} + \frac{\pi}{2})$$

$$-.9808 - .1951i = \cos(\frac{\pi}{16} + \pi) + i \sin(\frac{\pi}{16} + \pi)$$

$$.1951 - .9808i = \cos(\frac{\pi}{16} + \frac{3\pi}{2}) + i \sin(\frac{\pi}{16} + \frac{3\pi}{2})$$

(b) If

$$z^4 = \frac{1 + i}{\sqrt{2}}, \tag{3}$$

we may square both sides to obtain

$$z^8 = \frac{(1 + i)^2}{2} = \frac{1 + 2i - 1}{2}$$

or

$$z^8 = i. \tag{4}$$

Squaring both sides of (4), we obtain

$$z^{16} = i^2$$

or

$$z^{16} + 1 = 0.$$

(In other words, the four 4th roots of $\frac{1 + i}{\sqrt{2}}$ are a subset of the sixteen 16th roots of -1. As a partial check, if $z = e^{\frac{i\pi}{16}}$, then $z^{16} = e^{i\pi} = -1$.)

3. (a) Perhaps the most systematic way to proceed is by using the fact that $z\,\bar{z} = |z|^2$. We could then multiply both numerator denominator of $\frac{1}{z^2}$ by $\bar{z}^2$ to obtain

$$\frac{1}{z^2} = \frac{\bar{z}^2}{z^2\,\bar{z}^2}$$

$$= \frac{\bar{z}^2}{(z\,\bar{z})^2}$$

$$= \frac{\bar{z}^2}{\left(|z|^2\right)^2}$$

$$= \frac{\bar{z}^2}{|z|^4}. \tag{1}$$

If we now let $z = x + iy$, we have that $\bar{z} = x - iy$ and $|z|^2 = x^2 + y^2$.

Hence, equation (1) becomes

$$\frac{1}{z^2} = \frac{(x - iy)^2}{\left(x^2 + y^2\right)^2}$$

$$= \frac{x^2 - i2xy + i^2y^2}{\left(x^2 + y^2\right)^2}$$

$$= \frac{(x^2 - y^2) - 2xyi}{\left(x^2 + y^2\right)^2},$$

or

$$\frac{1}{z^2} = \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2} + i\left[\frac{-2xy}{\left(x^2 + y^2\right)^2}\right]. \tag{2}$$

3. continued

From (2), we have that

$$u = \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2} \tag{3}$$

and

$$v = \frac{-2xy}{\left(x^2 + y^2\right)^2}. \tag{4}$$

<u>Note</u>

The identity $z\,\bar{z} = |z|^2$ often helps us avoid much "dog work," but, in relatively non-complicated cases, "brute force" works as well. In terms of the present exercise, we could have converted at once to x and y, and obtained

$$\frac{1}{z^2} = \frac{1}{(x + iy)^2}$$

$$= \frac{1}{(x^2 - y^2) + i2xy}$$

$$= \frac{(x^2 - y^2) - i2xy}{[(x^2 - y^2) + i2xy][(x^2 - y^2) - i2xy]}$$

$$= \frac{x^2 - y^2 - i2xy}{\left(x^2 - y^2\right)^2 + 4x^2y^2} \tag{5}$$

and since $\left(x^2 - y^2\right)^2 + 4x^2y^2 = x^4 + 2x^2y^2 + y^4 = \left(x^2 + y^2\right)^2$, we see that (5) agrees with (2). The major point is that this latter approach becomes more tedious in computing $\frac{1}{z^n}$ for large (integral) values of n. More generally, by our first approach

3. continued

$$\frac{1}{z^n} = \frac{\bar{z}^n}{z^n\ \bar{z}^n}$$

$$= \frac{\bar{z}^n}{(z\ \bar{z})^n}$$

$$= \frac{\bar{z}^n}{\left(|z|^2\right)^n}$$

$$= \frac{(x + iy)^n}{\left(x^2 + y^2\right)^n}.$$

(b) From (3) and (4), we see that u_x, u_y, v_x, v_y exist and are continuous except when the denominators are zero. But

$$\left(x^2 + y^2\right)^2 = 0 \leftrightarrow x^2 + y^2 = 0$$

$$\leftrightarrow x = y = 0$$

$$\leftrightarrow z = x + iy = 0.$$

Hence, the Cauchy-Riemann conditions, virtually by default, are not obeyed when $z = 0$.

As long as $z \neq 0$, we have from (3) that

$$u_x = \frac{\left(x^2 + y^2\right)^2(2x) - (x^2 - y^2)\ 2(x^2 + y^2)2x}{\left(x^2 + y^2\right)^4}$$

$$= \frac{2x(x^2 + y^2)[(x^2 + y^2) - 2(x^2 - y^2)]}{\left(x^2 + y^2\right)^4}$$

$$= \frac{2x(x^2 + y^2)(3y^2 - x^2)}{\left(x^2 + y^2\right)^4},$$

3. continued

or, since $x^2 + y^2 \neq 0$,

$$u_x = \frac{2x(3y^2 - x^2)}{\left(x^2 + y^2\right)^3}. \tag{6}$$

Similarly,

$$u_y = \frac{\left(x^2 + y^2\right)^2(-2y) - (x^2 - y^2)\ 2(x^2 + y^2)2y}{\left(x^2 + y^2\right)^4}$$

$$= \frac{(x^2 + y^2)(-2y)[(x^2 + y^2) + 2(x^2 - y^2)]}{\left(x^2 + y^2\right)^4}$$

$$= \frac{-2y(3x^2 - y^2)}{\left(x^2 + y^2\right)^3}. \tag{7}$$

Now from (4), we conclude that

$$v_x = \frac{\left(x^2 + y^2\right)^2(-2y) + 2xy[2(x^2 + y^2)2x]}{\left(x^2 + y^2\right)^4}$$

$$= \frac{2y(x^2 + y^2)[-(x^2 + y^2) + 4x^2]}{\left(x^2 + y^2\right)^4}$$

$$= \frac{2x(3x^2 - y^2)}{\left(x^2 + y^2\right)^3} \tag{8}$$

and

3. continued

$$v_y = \frac{\left(x^2 + y^2\right)^2 (-2x) + 2xy[2(x^2 + y^2)2y]}{\left(x^2 + y^2\right)^4}$$

$$= \frac{2x(x^2 + y^2)[-(x^2 + y^2) + 4y^2]}{\left(x^2 + y^2\right)^4}$$

$$= \frac{2x(3y^2 - x^2)}{\left(x^2 + y^2\right)^3}. \tag{9}$$

Comparing (7) with (8) and (6) with (9), we have that for $z \neq 0$,

$$\begin{cases} u_y = -v_x \\ u_x = \ \ v_y \end{cases}$$

which are precisely the Cauchy-Riemann conditions.

Hence, $\frac{1}{z^2}$ is analytic except at $z = 0$.

(c) In the study of real variables, we learned that

$$\int_a^b f(x)dx = F(b) - F(a) \text{ where } F'(x) = f(x),$$

provided f is piecewise continuous on [a,b]. In the complex case, the equivalent result is that

$$\int_a^b f(x)dx = F(b) - F(a) \text{ where } F'(x) = f(x),$$

provided f is analytic in a sufficiently small neighborhood of the curve which joins a to b.

Since $z = 0$ is not on our curve C, we have

3. continued

$$\int_C \frac{dz}{z^2} = \int_i^{2i} \frac{dz}{z^2} = -\frac{1}{z}\bigg]_{z=i}^{2i}$$

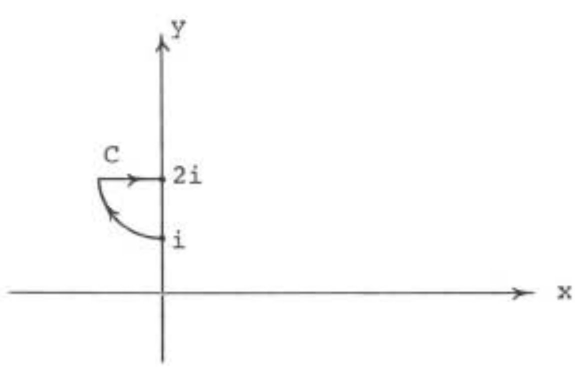
$$= -\frac{1}{2i} - \left(-\frac{1}{i}\right)$$

$$= \frac{1}{2i}$$

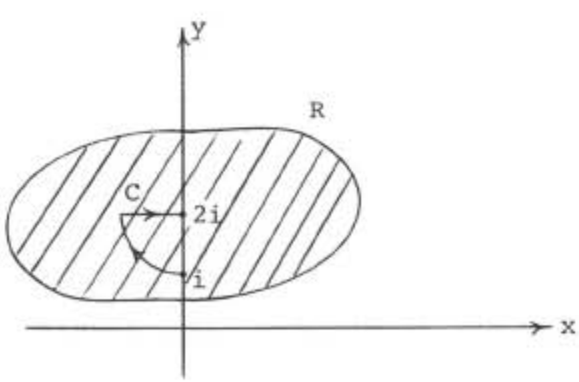
$$= \frac{i}{2i^2}$$

$$= -\frac{i}{2}.$$

Pictorially, if C is any curve which does not pass through $z = 0$, we may choose a region R which contains C, in which $\frac{1}{z^2}$ is analytic. For example, if we are given



we may view R as



3. continued

Since C excludes $z = 0$, we may always choose R so that it, too, excludes $z = 0$. Then, since is analytic in R and $z = i$ and $z = 2i$ are in R, $\int_i^{2i} f(z)dz = \int_i^{2i} \frac{dz}{z^2}$ is determined to be $-\frac{1}{z}\Big|_{z=i}^{2i}$ along <u>any</u> simple curve (of which the given curve C is but one) lying in R and which originates at $z = i$ and terminates at $z = 2i$.

4. For $u + iv$ to be analytic, we must have that

$$v_y = u_x \tag{1}$$

and

$$v_x = -u_y. \tag{2}$$

Given that

$$u = x^4 - 6x^2y^2 + y^4, \tag{3}$$

we have

$$u_x = 4x^3 - 12xy^2 \tag{4}$$

and

$$u_y = -12x^2y + 4y^3. \tag{5}$$

Putting (4) into (1), we see that

$$v_y = 4x^3 - 12xy^2, \tag{6}$$

whereupon

$$v = 4x^3y - 4xy^3 + g(x). \tag{7}$$

Notice that v is determined from (7) once we know $g(x)$. To find $g(x)$, we compute v_x from (7) to obtain

4. continued

$$v_x = 12x^2y - 4y^3 + g'(x). \tag{8}$$

Since, by (2), v_x must equal $-u_y$, we may use equations (5) and (8) to conclude that

$$12x^2y - 4y^3 + g'(x) = -(-12x^2y + 4y^3)$$

or

$$g'(x) = 0,$$

whence

$g(x) = C$, where C is any (real) constant.

Putting this result into (7) yields

$$v(x,y) = 4x^3y - 4xy^3 + C. \tag{9}$$

Combining (3) and (9), we have that

$$(x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3 + C) \tag{9'}$$

yields a family (the particular member of which depends on the choice of C) of analytic functions which have $u = x^4 - 6x^2y^2 + y^4$ as the real part.

Note

This exercise might have seemed more intuitive in the form $f(z) = z^4$. Clearly, $f(z)$ is then analytic; in fact, $f'(z) = 4z^3$. Moreover, written in the form $u + iv$, we have

$$\begin{aligned} z^4 &= (x + iy)^4 \\ &= x^4 + 4x^3(iy) + 6x^2(iy)^2 + 4x(iy)^3 + (iy)^4 \\ &= \underbrace{(x^4 - 6x^2y^2 + y^4)}_{u} + i\underbrace{(4x^3y - 4xy^3)}_{v}. \end{aligned}$$

4. continued

More generally, $f(z) = z^4 + C$, for any complex constant C is analytic. However, the real part of $z^4 + C$ is

$$x^4 - 6x^2y^2 + y^4 + (\text{real part of } C). \tag{10}$$

From (10), we see that

$$u = x^4 - 6x^2y^2 + y^4 \leftrightarrow \text{real part of } C = 0.$$

Thus, in this example, C must be a purely imaginary constant which accounts for C being real in (9'); noticing in (9') that C is multiplied by i.

5. Suppose we follow the procedure of the previous problem. Since

$$u = x^5y + y^5, \tag{1}$$

we have

$$u_x = 5x^4y \tag{2}$$

and

$$u_y = x^5 + 5y^4. \tag{3}$$

By the Cauchy-Riemann conditions, we must have, by (2), that

$$5x^4y = v_y$$

whence

$$v = \frac{5}{2}x^4y^2 + g(x). \tag{4}$$

To determine g(x), we differentiate (4) with respect to x to obtain

$$v_x = 10x^3y^2 + g'(x). \tag{5}$$

5. continued

Again, by the Cauchy-Riemann conditions, we must have that $v_x = -u_y$, so from (5) and (3) we conclude that

$$10x^3y^2 + g'(x) = -(x^5 + 5y^4).$$

Hence,

$$g'(x) = -(x^5 + 5y^4 + 10x^3y^2). \tag{6}$$

But equation (6) is a contradiction since $g'(x)$ is a function of x alone while the right side of (6) depends on both x and y. Hence, there is no $g(x)$ with the required properties. In other words, $x^5y + y^5$ cannot be the real part of an analytic function.

Note

$u(x,y)$ is the real part of an analytic function $\leftrightarrow u_{xx} + u_{yy} \equiv 0$. In the present example,

$$u_x = 5x^4y \text{ and } u_y = x^5 + 5y^4.$$

Hence,

$$u_{xx} = 20x^3y \text{ and } u_{yy} = 20y^3.$$

Therefore,

$$u_{xx} + u_{yy} = 20x^3y + 20y^3 \not\equiv 0.$$

On the other hand, in the previous exercise

$$u_x = 4x^3 - 12xy^2 \text{ and } u_y = -12x^2y + 4y^3.$$

Hence,

$$u_{xx} = 12x^2 - 12y^2 \text{ and } u_{yy} = -12x^2 + 12y^2.$$

5. continued

Therefore,

$$u_{xx} + u_{yy} \equiv 0.$$

The success of the method used in this problem and the previous one requires that $u(x,y)$ satisfies

$$u_{xx} + u_{yy} = 0.$$

6. Since sin z is the limit of the convergent series

$$z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots + \frac{(-1)^n z^{2n+1}}{(2n+1)!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!},$$

we have that

$$\frac{\sin z}{z^2} = \frac{1}{z^2} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)! z^2}$$

$$= \sum_{n=0}^{\infty} \frac{(-1) z^{2n-1}}{(2n+1)!}. \tag{1}$$

<u>Note</u>

More intuitively

$$\frac{\sin z}{z^2} = \frac{1}{z^2}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots\right)$$

$$= \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \frac{z^5}{7!} + \ldots ,$$

6. continued

where the term-by-term division by z^2 is justified by the absolute convergence of the series.

We then obtain

$$\oint_C \frac{\sin z}{z^2}\,dz = \oint_C \left(\frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \frac{z^5}{7!} + \dots\right) dz \tag{2}$$

$$= \oint_C \frac{dz}{z} - \oint_C \frac{z\,dz}{3!} + \oint_C \frac{z^3dz}{5!} - \oint_C \frac{z^5dz}{7!} + \dots , \tag{3}$$

where the term-by-term integration in (3) is justified by the fact that the series in (2) is <u>uniformly</u> convergent.

Now, since $\frac{z}{3!}, \frac{z^3}{5!}, \frac{z^5}{7!}, \dots, \frac{z^{2n-1}}{(2n+1)!}$ $(n = 1,2,3,\dots)$ are each analytic in the entire plane, we have that

$$\oint_C \frac{z^{2n-1}dz}{(2n+1)!} = 0, \text{ for } n = 1,2,3,\dots .$$

Consequently, we conclude from (3) that

$$\oint_C \frac{\sin z\,dz}{z^2} = \oint_C \frac{dz}{z}. \tag{4}$$

We have already seen as an exercise in this block that if $z = 0$ lies in C, then

$$\oint_C \frac{dz}{z} = 2\pi i.$$

Hence, we see from (4) that

$$\oint_C \frac{\sin z\,dz}{z^2} = 2\pi i. \tag{5}$$

6. continued

<u>Note #1</u>

Had $z = 0$ been outside the region enclosed by C, then $\frac{1}{z}$ would be analytic in a region containing C. In this case,

$$\oint \frac{dz}{z} = 0.$$

In summary, if C is any simple closed curve, then

$$\oint_C \frac{\sin z \, dz}{z^2} = \begin{cases} 2\pi i, \text{ if C encloses } z = 0 \\ 0, \text{ if } z = 0 \text{ lies outside of C} \end{cases}$$

(The integral is "improper" if $z = 0$ lies on C.)

<u>Note #2</u>

Had we remembered the formula given in Exercise 1.9.3; namely,

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)dz}{(z-a)^{n+1}}, \tag{6}$$

then we could let $a = 0$, $n = 1$ and $f(z) = \sin z$ to obtain

$$\cos z \Big|_{z=0} = \frac{1!}{2\pi i} \oint_C \frac{\sin z \, dz}{(z-0)^{1+1}},$$

or

$$\underbrace{\cos 0}_{=1} = \frac{1}{2\pi i} \oint_C \frac{\sin z \, dz}{z^2},$$

whence $\oint_C \frac{\sin z \, dz}{z^2} = 2\pi i$, which agrees with (5).

6. continued

Note #3

In case we have forgotten how to compute directly $\oint_C \frac{dz}{z}$ where $z = 0$ is inside C, recall that since $\frac{1}{z}$ is analytic except at $z = 0$ then $\oint_{C_1} \frac{dz}{z} = \oint_C \frac{dz}{z}$ where C_1 is any simple closed curve which contains $z = 0$ in its interior. In particular, we may pick C_1 to be the unit circle centered at $z = 0$. That is

C_1: $z = e^{i\theta}$ $0 \leqslant \theta \leqslant 2\pi$.

Hence, on C_1,

$dz = ie^{i\theta}d\theta$.

Therefore,

$$\oint_{C_1} \frac{dz}{z} = \int_0^{2\pi} \frac{ie^{i\theta}d\theta}{e^{i\theta}}$$

$$= \int_0^{2\pi} id\theta$$

$$= 2\pi i.$$