Unit 4: Complex Functions of a Complex Variable

1.4.1(L)

Clearly each element of S is a complex number since both cos t and sin t are real for all $0 \leq t \leq \pi$. Thus, S exists without any reference to a picture. The point is, however, that if we use the Argand diagram, we view x + iy as the point z[= (x,y)] in the xy-plane.

In other words if we identify the position vector R with the complex number z, we see that the "graph" of S (by which we mean the set of points in the Argand diagram which represents S) is the curve whose vector equation is
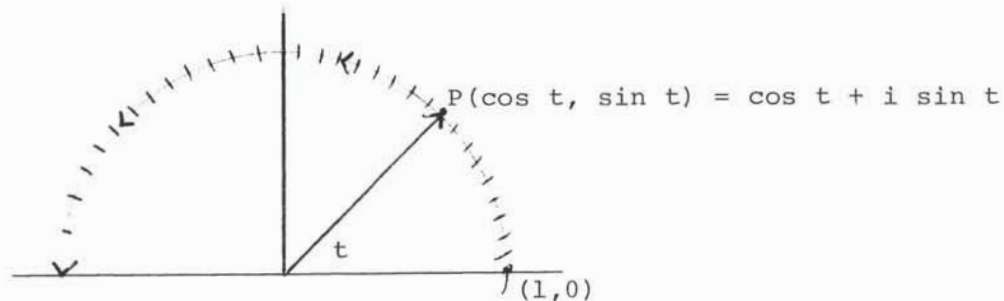
$$R(t) = \cos t \, i + \sin t \, j, \quad 0 \leq t \leq \pi. \tag{1}$$

This, as we already know from our study of vectors, is the curve whose parametric form is

$$\left. \begin{array}{l} x = \cos t \\ y = \sin t \end{array} \right\} \quad 0 \leq t \leq \pi \tag{1'}$$

which we recognize as the upper half of the unit circle centered at the origin.

Pictorially,

$P(\cos t, \sin t) = \cos t + i \sin t$

$(1,0)$

There is a 1-1 correspondence between complex numbers in S and points on the above semi-circle. The correspondence is defined by $(\cos t, \sin t) \leftrightarrow \cos t + i \sin t$.

Solutions
Block 1: An Introduction to Functions of a Complex Variable
Unit 4: Complex Functions of a Complex Variable

1.4.1(L) continued

More generally, every curve in the z-plane has the equation of
the *form*

$$z = x(t) + y(t)i \qquad (2)$$

(we say more about this in Exercise 1.4.9) and because the
Argand diagram has the structure of a 2-dimensional vector space,
we see that equation (2) is equivalent to the vector function
of a scalar variable, defined by

$$\vec{R}(t) = x(t)\ \vec{i} + Y(t)\ \vec{j}.$$
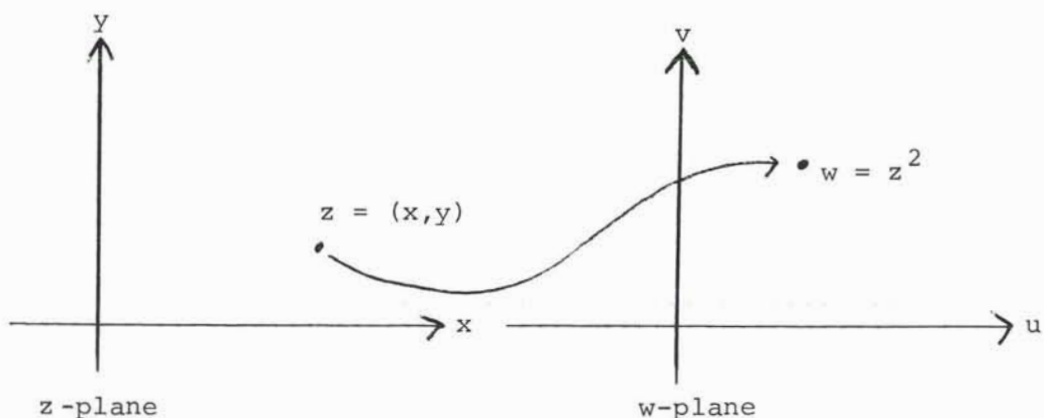
Summarized pictorially
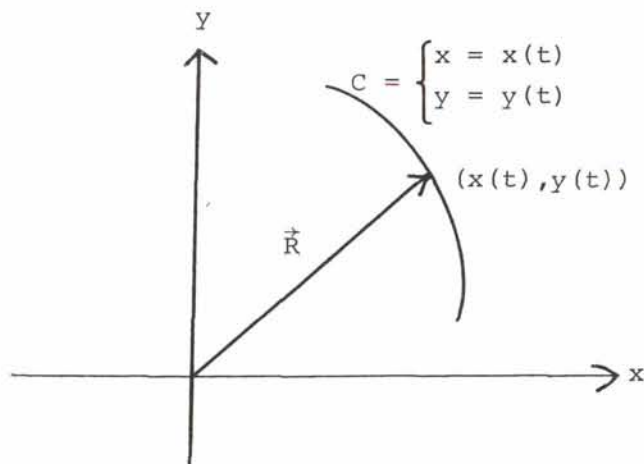
1. In vector form, C is given by $\vec{R}(t) = x(t)\vec{i} + y(t)\vec{j}$.
2. In the Argand diagram $\vec{R}$ represents $\vec{z}$, and C is then the
set of complex numbers, $\{z: z = x(t) + iy(t)\}$.

b. Let w denote the image of z with respect to f. In this case $w = z^2$.
   Since both z and w are complex, f is actually a mapping of a
   2-dimensional vector space (the z-plane) into a 2-dimensional
   vector space (the w-plane).

   If we now identify the z-plane with the xy-plane and the w-plane
   with the uv-plane, we see that $w = z^2$ actually is equivalent
   to mapping the xy-plane into the uv-plane (a topic we have al-
   ready studied fairly thoroughly).

Solutions
Block 1: An Introduction to Functions of a Complex Variable
Unit 4: Complex Functions of a Complex Variable

1.4.1(L) continued

Pictorially

z-plane   w-plane

More specifically, if $z = x + iy$ then $z^2 = (x^2 - y^2) + i\ 2xy$;
so that

$$w = u + iv, \text{ with } \begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases} \tag{3}$$

Notice that we have already discussed the mapping given by (3)
in Blocks 3 and 4 of Part 2.

Of course, we have something "going for us" now that we didn't
have then. Namely, we are now able to view mappings of the
xy-plane into the uv-plane (a concept which certainly exists
independently of the invention of complex numbers) as complex
valued functions of a complex variable which map the z-plane
into the w-plane.

With this interpretation, we are now able to discuss a vector
product that was undefined before (although with hindsight we
could have gone back to Blocks 2, 3, and 4 of Part 2 and invented
the vector product which corresponds to the product of two com-
plex numbers) and we may conclude that z^2 is the complex number
whose magnitude is the square of the magnitude of z and whose
argument is twice the argument of z.

In particular, then, since each point in S has unit magnitude,
its image under the squaring function also has unit magnitude.

Solutions
Block 1: An Introduction to Functions of a Complex Variable
Unit 4: Complex Functions of a Complex Variable

1.4.1(L) continued

Moreover, since the argument of the image is twice the argument
of the point, we see that since the arguments of the points in
S vary from 0 to 180°, the arguments of the images range from
0° to 360°. In summary, then, the mapping $w = z^2$ carries the
set S into the whole unit circle centered at the origin. In
particular, the point (r,θ) maps onto $(r,2\theta)$.

Here we see, as an important aside, how the theory of mapping
the complex plane into the complex plane gives us new insight to
real mappings. In particular, with respect to equation (3)
we now have that this mapping, in terms of what it means to
multiply complex numbers, is easy to explain pictorially,
Specifically, the image of a given point in the xy-plane is
found by doubling the argument of the point (vector) and
squaring its magnitude.

Again we hasten to point out that we could have invented the
product of two vectors to be the vector in the same plane
equivalent to the product of the two given vectors as complex
numbers. That is,

$$(a\ \vec{i} + b\ \vec{j})(c\ \vec{i} + d\ \vec{j}) = (ac - bd)\ \vec{i} + (bc + ad)\ \vec{j}.$$

But notice how much more natural this definition becomes in
terms of the language of complex numbers.

In other words, one major real application of the theory of
complex functions of a complex variable is to the real problem
of mapping the xy-plane into the uv-plane. These problems can
be tackled without reference to the complex numbers, but a
knowledge of the complex numbers gives us a considerable amount
of "neat" notation which is helpful in obtaining results
fairly quickly.

As a final observation, let us observe that as a function f
has the same structure (but a different domain) whether we
write $f(x) = x^2$ or $f(z) = z^2$. In either case we have a function
machine in which the output is the square of the input. The
big difference is from the geometrical point of view. In the

Solutions
Block 1: An Introduction to Functions of a Complex Variable
Unit 4: Complex Functions of a Complex Variable

1.4.1(L) continued

expression $f(x) = x^2$ se may view both the domain and the image of f as being 1-dimensional (since x is assumed to be real). Accordingly, we may graph the function in the 2-dimensional xy-plane in terms of the curve $y = x^2$.

On the other hand, in the expression $w = f(z) = z^2$, the domain and the image of f must be 2-dimensional since neither z nor z^2 is required to be real. Thus, we would require a 4-dimensional space to graph this function if we wanted a graph which was the analog of the graph $y = f(x)$. Since we cannot, in the usual geometric sense, draw a 4-dimensional space, our geometric interpretation must involve viewing the z-plane (the domain of f) as being mapped into the w-plane (the range of f).

1.4.2

a. If we look at z as being the point (r, θ) in the z-plane, then $z^3 = (r, \theta)^3 = (r^3, 3\theta)$. Thus, under f each point in the z-plane is mapped into the point (ρ, ϕ) in the w-plane where $\rho = r^3$ and $\phi = 3\theta$ [i.e., the mapping cubes the magnitude and triples the argument].

In particular the point $(1, \theta)$ where $0 \leq \theta \leq 90°$ is mapped onto $(1^3, 3\theta) = (1, 3\theta)$ and since $0° \leq \theta \leq 90°$, $0° \leq \theta \leq 270°$. Thus, the first quadrant S of the unit circle is mapped onto the first three quadrants of the unit circle.

Again pictorially,
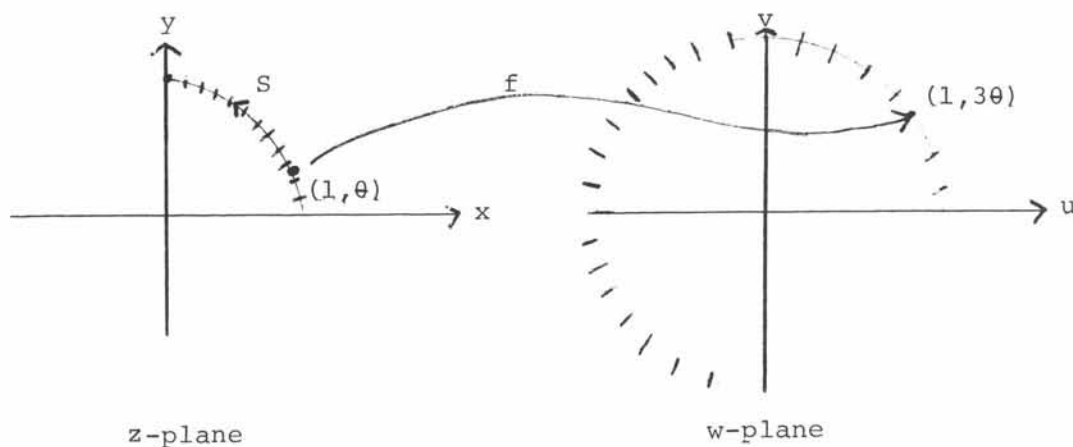

z-plane   w-plane

Solutions
Block 1: An Introduction to Functions of a Complex Variable
Unit 4: Complex Functions of a Complex Variable

1.4.2 continued

By the same token, each point in T, written in polar coordinates, has the form $(r, 45°)$. [If the line extended into the third quadrant, the points on this part would be represented as $(r, 225°)$.] Hence "cubing" such a point yields $(r^3, 135°)$. In other words, the mapping defined by $f(z) = z^3$ maps the ray $\theta = 45°$ onto the ray $\theta = 135°$ in such a way that the image of each point has the cube of the magnitude of the point.

Pictorially,

z-plane   w-plane
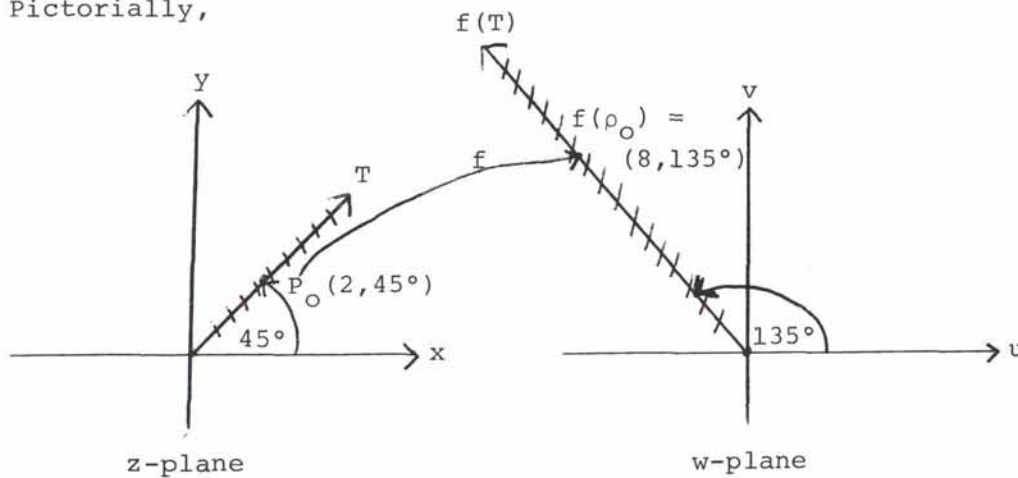
b. $f(z) = z^3$
$$= (x + iy)^3$$
$$= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3$$
$$= x^3 + 3x^2yi - 3xy^2 - iy^3$$
$$= (x^3 - 3xy^2) + i(3x^2y - y^3).$$

Hence,

$$\left.\begin{aligned} u &= x^3 - 3xy^2 \\ v &= 3x^2y - y^3 \end{aligned}\right\} \quad . \tag{1}$$

Again, by way of review, equation (1) defines a real mapping of 2-space into 2-space, but from our knowledge of complex variables, we know that the rather cumbersome system (1) is equivalent to mapping each point (vector) in the xy-plane into the point whose magnitude is the cube of the given magnitude and whose argument is triple that of the given argument.

Solutions
Block 1: An Introduction to Functions of a Complex Variable
Unit 4: Complex Functions of a Complex Variable

1.4.3

a. $z = x + iy \rightarrow 2z = 2x + i2y$. Therefore, $w = 2z = 2x + i2y$. Letting u denote the real part of w and v the imaginary part we have

$$\left.\begin{array}{l} u = 2x \\ v = 2y \end{array}\right\} \; .$$

b. $w = f(z) = \bar{z} \rightarrow w = x - iy$. Hence,

$$\left.\begin{array}{l} u = x \\ v = -y \end{array}\right\} \; .$$

c. $f(z) = |z| \rightarrow$
    $f(z) = |x + iy| \rightarrow$
    $f(z) = \sqrt{x^2 + y^2} \rightarrow$
    $f(z) = \sqrt{x^2 + y^2} + 0i \rightarrow$

$$u = \sqrt{x^2 + y^2}$$
$$v = 0$$

d. $f(z) = z^2 + 2z + i$

$$= (x + iy)^2 + 2(x + iy) + i$$

$$= x^2 - y^2 + i2xy + 2x + i2y + i$$

$$= (x^2 - y^2 + 2x) + (2xy + 2y + 1)i.$$

Hence,

$$\left.\begin{array}{l} u = x^2 - y^2 + 2x \\ v = 2xy + 2y + 1 \end{array}\right\}$$

e. $f(z) = \dfrac{1}{z}$

$$= \frac{1}{x + iy}$$

$$= \frac{x - iy}{(x + iy)(x - iy)} = \frac{x}{x^2 + y^2} + i\left[\frac{-y}{x^2 + y^2}\right].$$

Solutions
Block 1: An Introduction to Functions of a Complex Variable
Unit 4: Complex Functions of a Complex Variable

1.4.3 continued

Hence,

$$u = \frac{x}{x^2 + y^2}$$
$$y = \frac{-y}{x^2 + y^2}$$
$\left.\right\}$ $x^2 + y^2 \neq 0$
(since $z \neq 0$)

Again, as a reminder, this problem shows us that we may view the mapping

$$u = 2x$$
$$v = 2y$$
$\left.\right\}$

as $f(z) = 2z$

$$u = x$$
$$v = -y$$
$\left.\right\}$

as $f(z) = \bar{z}$;

$$u = x^2 - y^2 + 2x$$
$$v = 2xy + 2y + 1$$
$\left.\right\}$

as $f(z) = z^2 + 2z + i$; and

$$u = \frac{x}{x^2 + y^2}$$

$$v = \frac{-y}{x^2 + y^2}$$

as $f(z) = \frac{1}{z}$.

1.4.4(L)

Our main aim in this exercise is to get a better feeling for the "reality" of complex functions of a complex variable. Parts (b) and (c) are concerned with extending the analogs of $f(x) = x + c$ and $f(x) = cx$ where c and x are real numbers to $f(z) = z + c$ and $f(z) = cz$ where c and z are now complex numbers. As we shall

Solutions
Block 1: An Introduction to Functions of a Complex Variable
Unit 4: Complex Functions of a Complex Variable

1.4.4(L) continued

see, the algebra of these functions is the same as that of their real analogs, but the geometric interpretation is a bit more sophisticated (the result of both our domain and image space being 2-dimensional rather than 1-dimensional). In part (a) we want to emphasize the fact that what looks like a new function to us is really an old function that we handled in a very real situation. In particular,

a. Recall in our treatment of the double integral that when we wanted to reverse the order of integration, the technique was geometrically expressed by the mapping of the xy-plane into the uv-plane given by

$$\left. \begin{array}{l} u = y \\ v = x \end{array} \right\}. \tag{1}$$

It should be clear that we do not have to know anything about complex numbers to talk about the mapping defined by equation (1). If, however, we want to view the mapping as being from the z-plane into the w-plane, our procedure is to write (1) in the form u + iv, which in this case means that we study the complex function of a complex variable defined by

$$\begin{aligned} f(z) &= x + i(-y) \\ &= x - iy. \end{aligned} \tag{2}$$

If we now recall that z is x + iy, we see that x - iy is by definition $\overline{z}$. Thus, (2) becomes

$$f(z) = \overline{z}. \tag{3}$$

Of course we arrived at (3) rather inversely to the wording of the exercise in which we were to begin with (3) and derive (1). Our purpose for doing this was simply to start the exercise emphasizing the relationship between complex functions of a complex variable and real mappings. Had we begun with (3), we would have merely reversed our steps to obtain:
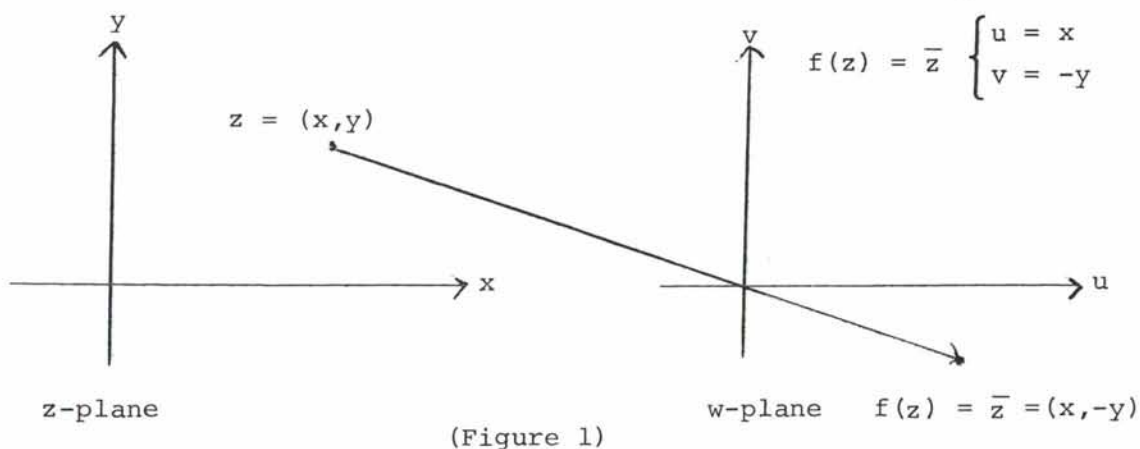
Solutions
Block 1: An Introduction to Functions of a Complex Variable
Unit 4: Complex Functions of a Complex Variable

1.4.4(L) continued

$$f(z) = \bar{z}$$

$$= \overline{x + iy}$$

$$= x - iy \tag{4}$$

and from the real and imaginary parts of $f(z)$, we would have
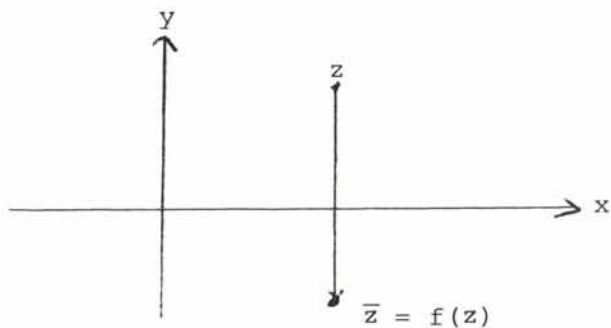concluded that the graph of f was equivalent to the mapping de-
fined by

$$\begin{cases} u = x \\ v = -y. \end{cases}$$

This mapping is equivalent to reflecting the xy-plane about the
x-axis (i.e., we leave x alone and change the sign of y.

Pictorially,

z-plane   w-plane $f(z) = \bar{z} = (x,-y)$

(Figure 1)

But since the w-plane is a replica of the z-plane we may super-
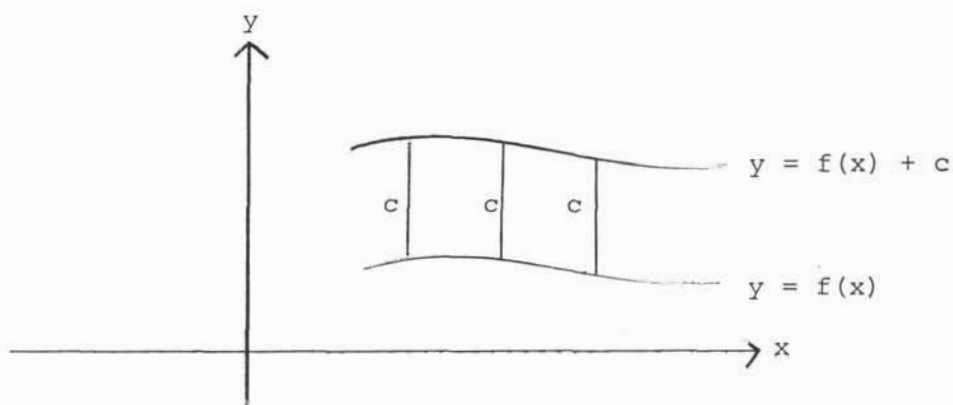impose the two planes in Figure 1 to obtain

Solutions
Block 1: An Introduction to Functions of a Complex Variable
Unit 4: Complex Functions of a Complex Variable

1.4.4(L) continued

Thus, the effect of f on set S in the Argand diagram is to pro-
duce the mirror image of S with respect to the x-axis. In
particular if S is any closed region, $f(S) \overset{\sim}{=} S$ (i.e., S and its
image have the same size and shape).

b. In the real case, we saw that the graph of $f(x) + c$ just "raised"
each point of the curve $y = f(x)$ by an amount c. In particular
the graph of $f(x) = x + c$, was obtained by lifting each point on
the line $y = x$ by c units. Pictorially,

Now, given $f(z) = z + c$, we see that in the Argand diagram this
sum must be interpreted as a vector sum. As a vector the complex
number c is written as $c_1 \vec{i} + c_2 \vec{j}$ (where we are assuming that
$c = c_1 + c_2 i$). Letting c denote $c_1 i + c_2 j$, we see that adding
c to z is equivalent to displacing z by an amount equal to the
magnitude of c in the direction of c.

For example, the mapping defined by $f(z) = z + 3 + 4i$ maps the
point z into the point 5 units from z in the direction
$\frac{3}{5} \vec{i} + \frac{4}{5} \vec{j}$.

Pictorially,
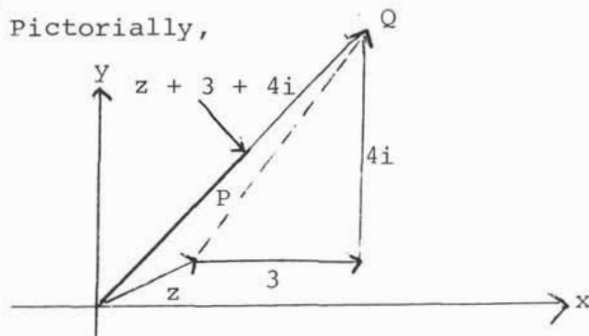
Solutions
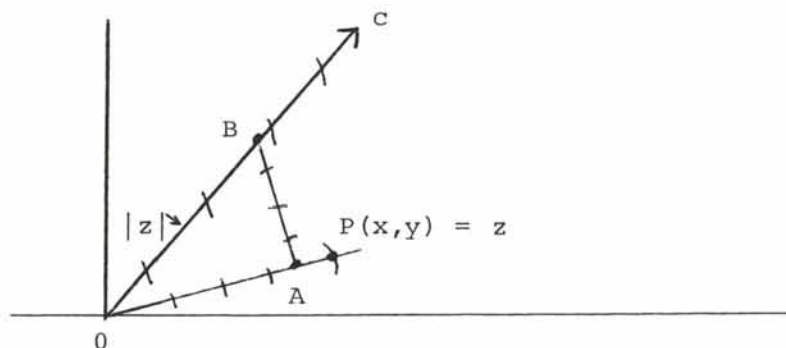Block 1: An Introduction to Functions of a Complex Varib
Unit 4: Complex Functions of a Complex Variable

1.4.4(L) continued

Geometrically, adding 3 + 4i onto z shifts (translates) P to Q.
That is, P is translated 5 units in the direction $3\vec{i} + 4\vec{j}$.

c. Here we invoke the fact that we have a very convenient way of
multiplying complex numbers using polar coordinates. In parti-
cular if $c = (r_o, \theta_o)$ then cz has as its magnitude r_o times the
magnitude of z and as its argument θ_o plus the argument of z.
In other words we obtain the image of z by rotating the vector
z through θ_o degrees and increasing its magnitude by a factor of
r_o.

By way of an example, if f(z) = (3 + 4i)z, then the image of a
given number z is obtained by rotating z through an angle equal
to arc tan $\frac{4}{3}$ and replacing the magnitude of z by 5 times its
value. Pictorially,

1. We pick any point on OP.
2. We erect a perpendicular to OA and locate B on OA such that
$\overline{AB} = \frac{4}{3} \vec{OA}$. Therefore, tan ∢ AOB = $\frac{4}{3}$.
3. We mark off the length $\overline{OP}$ (i.e., $|z|$) 5 times along OB.
4. $\vec{OC}$ then denotes (3 + 4i) $\vec{OP}$ = (3 + 4i)z.

Solutions
Block 1: An Introduction to Functions of a Complex Variable
Unit 4: Complex Functions of a Complex Variable

1.4.4(L) continued

As a very interesting special case, notice that if the magnitude
of c is 1 then the mapping $f(z) = cz$ simply rotates z through an
angle equal to the argument of c(i.e., the magnitude is pre-
served because c has unit magnitude).

If we let θ denote the argument of c, the fact that c is of unit
magnitude means that $c = \cos \theta + i \sin \theta$.

Hence,

$$cz = (\cos \theta + i \sin \theta)(x + iy)$$
$$= x \cos \theta - y \sin \theta + i (x \sin \theta + y \cos \theta),$$

and as discussed in our earlier exercises, this is equivalent
to the real mapping

$$u = x \cos \theta - y \sin \theta$$
$$v = x \sin \theta + y \cos \theta . \tag{1}$$

Thus, comparing this result with our polar coordinate interpre-
tation, we see that the mapping defined by equation (1) is equi-
valent to rotating the xy-plane through $\theta°$.

Hopefully, this shows us still another way in which complex
numbers have a real interpretation. By the way, in the special
case that c is real, the argument of c is either $0°$ or $180°$,
depending upon whether c is positive or negative. Notice then
that in this case the result checks with the usual result in
the real case; i.e., multiplying by (real) c leaves the
direction alone, changes the magnitude by a factor of $|c|$ and
preserves the sense if $c > 0$, reverses the sense if $c < 0$.

As a final note on this exercise notice that the linear mapping
defined by $f(z) = c_1 z + c_2$ where both c_1 and c_2 are complex-
valued constants maps lines through the origin into lines
through the origin; and circles centered at the origin into
circles centered at the origin. Namely, the mappping $f(z)$ is
a rotation (accompanied by a uniform magnification factor equal
to c_1) followed by a translation. Under a rotation, lines

Solutions
Block 1: An Introduction to Functions of a Complex Variable
Unit 4: Complex Functions of a Complex Variable

1.4.4(L) continued

through the origin remain lines through the origin, and circles
centered at the origin remain circles centered at the origin.

Notice also that the algebra of inverting this type of function
is word-for-word the same as in the real case since the structural
rules are the same. Namely, if $w = c_1 z + c_2$ $(c_1 \neq 0)$ then

$$z = \frac{w - c_2}{c_1}$$

etc.

In other words, the algebra remains the same, but the geometric
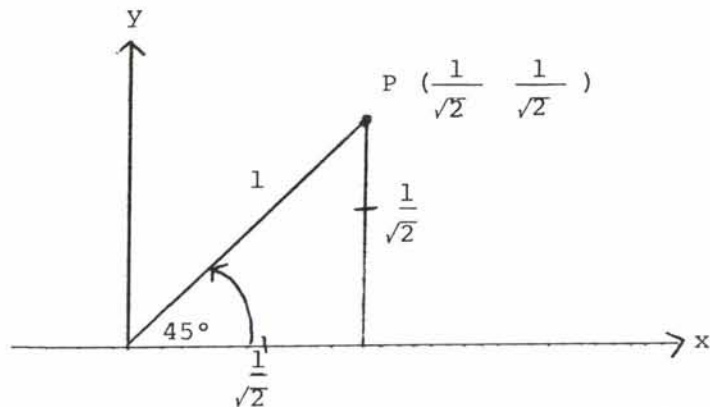interpretation is elevated by a dimension of sophistication
(so to speak).

1.4.5

a. Here we have $f(z) = c_1 z + c_2$ where $c_1 = \dfrac{1 + i}{\sqrt{2}}$ and $c_2 = i$.

By the result of the previous exercise $c_1 z$ rotates z through
an angle equal to the argument of c_1 and multiples the magnitude
of z by $|c_1|$. In our case, $|c_1| = 1$ [i.e.,

$$\sqrt{(\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2}]$$

while the argument of c_1 is 45°.

Hence, $c_1 z$ is a 45° rotation of the z-plane. Then since

Solutions
Block 1: An Introduction to Functions of a Complex Variable
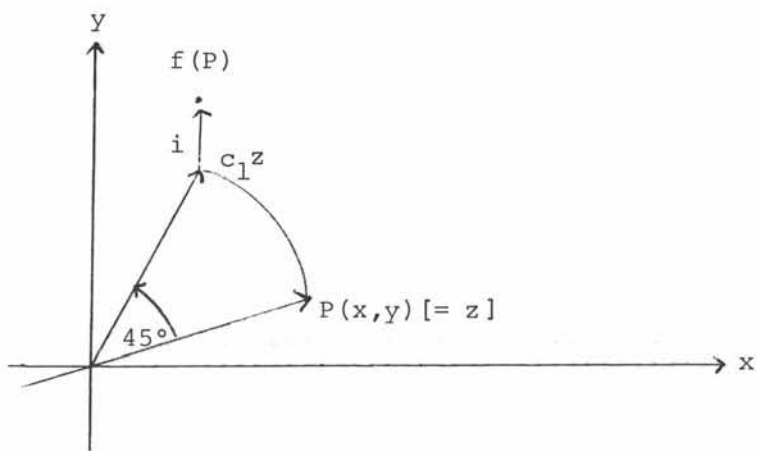Unit 4:   Complex Functions of a Complex Variable

1.4.5 continued

$c_1 z + i$ "translates" $c_1 z$ an amount equivalent to the vector $\vec{j}$
(i.e., adding i raised the point by 1 unit; we see that

$$f(z) = (\frac{1 + i}{\sqrt{2}})z + i$$

is equivalent to rotating each point in the plane through 45°
and then raising it 1 unit).

Pictorially,

Rotate P through 45° and then lift it (i.e., move it parallel to
the y-axis) one unit.

b. $(\frac{1 + i}{\sqrt{2}})z + i$

$$= \frac{(1 + i)(x + iy)}{\sqrt{2}} + i$$

$$= \frac{(x - y) + i(x + y)}{\sqrt{2}} + i$$

$$= \frac{(x - y) + i(x + y) + \sqrt{2}\, i}{\sqrt{2}}$$

$$= (\frac{x - y}{\sqrt{2}}) + i\,(\frac{x + y + \sqrt{2}}{\sqrt{2}}).$$

Solutions
Block 1: An Introduction to Functions of a Complex Variable
Unit 4: Complex Functions of a Complex Variable

1.4.5 continued

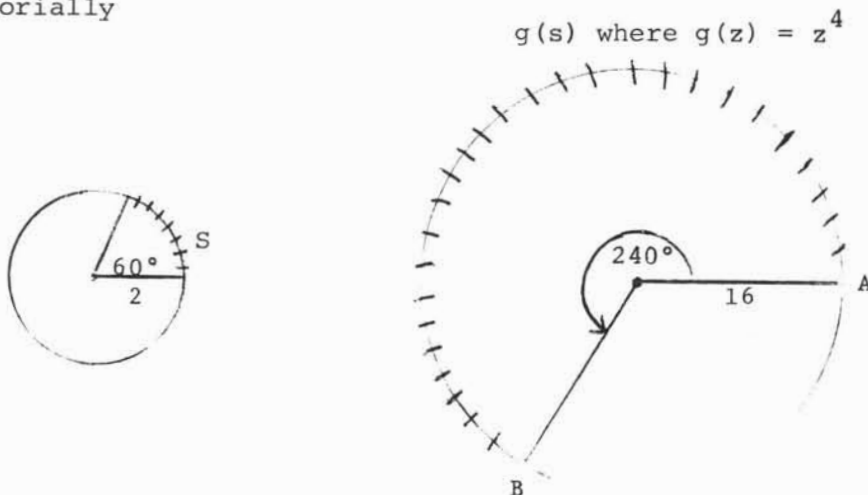Hence in Cartesian form, the mapping is given by

$$u = \frac{1}{\sqrt{2}} (x - y)$$

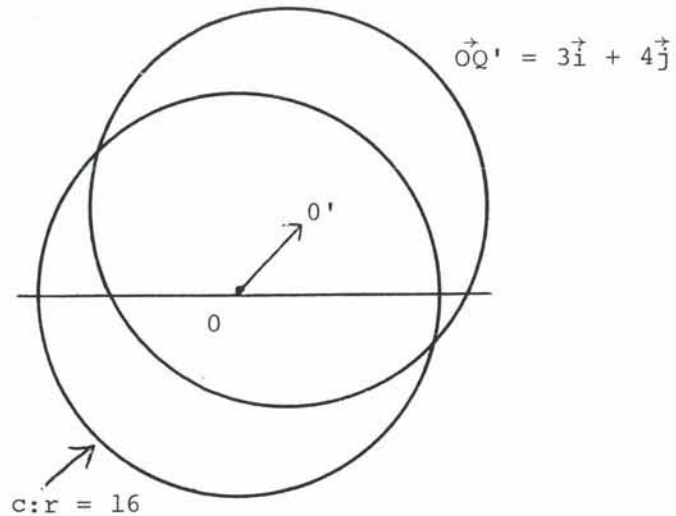$$v = \frac{1}{\sqrt{2}} (x + y + \sqrt{2})$$

1.4.6

z^4 has magnitude 16 if z has magnitude 2, and the argument of z^4 is four times the argument of z. Hence as z traces the portion of the circle r = 2 between $\theta = 0°$ and $\theta = 60°$, f(z) traces the portion of the circle r = 16 between $\theta = 0°$ and $\theta = 240°$.
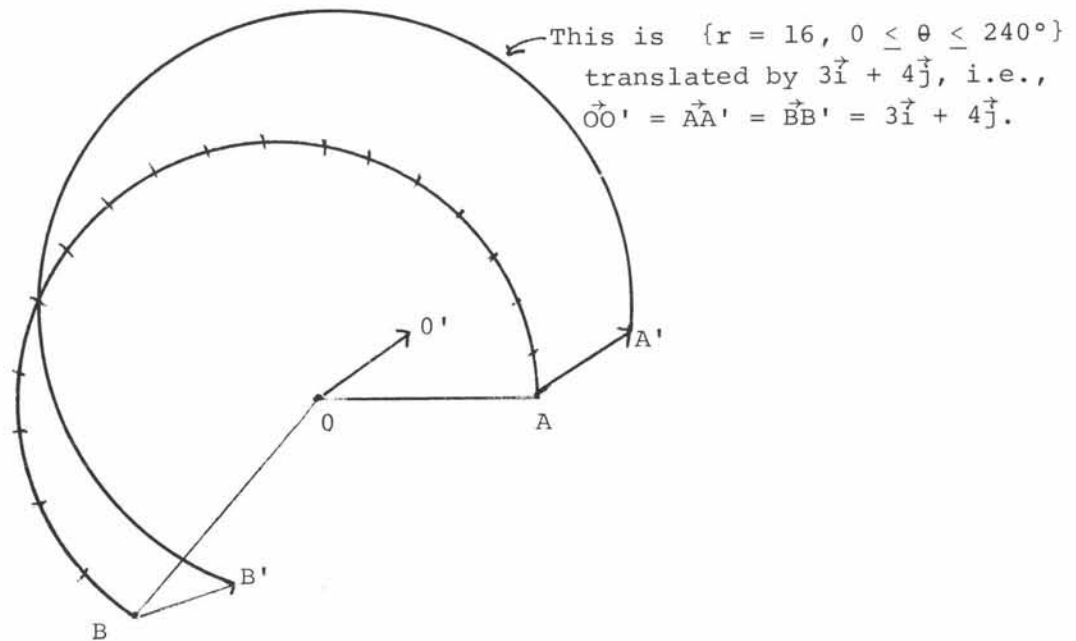
Pictorially

Finally, adding 3 + 4i translates each point 5 units in the direction $\frac{3}{5} + \frac{4}{5} i$.

Solutions
Block 1: An Introduction to Functions of a Complex Variable
Unit 4: Complex Functions of a Complex Variable

1.4.6 continued

$\vec{OQ}' = 3\vec{i} + 4\vec{j}$

0'

0

c:r = 16

Adding 3 + 4i to each point on the circle r = 16 translates the
circle from center at 0 to center at 0'.

This is $\{r = 16, 0 \le \theta \le 240°\}$
translated by $3\vec{i} + 4\vec{j}$, i.e.,
$\vec{OO}' = \vec{AA}' = \vec{BB}' = 3\vec{i} + 4\vec{j}$.

0'

A'

0   A

B'

B

Solutions
Block 1: An Introduction to Functions of a Complex Variable
Unit 4: Complex Functions of a Complex Variable

1.4.7(L)

Here again we see how our knowledge of vector calculus helps us here. Namely, it is natural, if only by mimicking, to define

$$\lim_{z \to c} f(z) = L$$

which means that, given $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < |z - c| < \delta \to |f(z) - L| < \varepsilon$.

The above definition makes sense even though z, c, and L need not be real since we are dealing only with absolute values - which are (non-negative) real numbers.

Moreover, from a pictorial point of view (i.e., in terms of the Argand diagram) the above definition is precisely the same as our limit definition when we dealt with vector functions of a vector variable.

Recall in that case we showed that the definition was equivalent to saying that if $\vec{f}(\vec{R}) = u(x,y)\vec{i} + v(x,y)\vec{j}$ and if $\vec{L} = L_1\vec{i} + L_2\vec{j}$, $\vec{c} = c_1\vec{i} + c_2\vec{j}$; then

$$\lim_{\vec{R} \to \vec{c}} \vec{f}(\vec{R}) = \vec{L}$$

was equivalent to

$$\lim_{(x,y) \to (c_1,c_2)} u(x,y) = L_1$$

and

$$\lim_{(x,y) \to (c_1,c_2)} v(x,y) = L_2.$$

Translated into the Argand diagram this says that if $c = c_1 + c_2 i$ then

$$\lim_{z \to c} f(z) = L$$

S.1.4.18

Solutions
Block 1: An Introduction to Functions of a Complex Variable
Unit 4: Complex Functions of a Complex Variable

1.4.7(L) continued

means

$$\lim_{(x,y) \to (c_1, c_2)} \text{Re}[f(z)] = \text{Re}(L)$$

and

$$\lim_{(x,y) \to (c_1, c_2)} \text{Im}[f(z)] = \text{Im}(L).$$

For example, in the given exercise

$$f(z) = z^3$$

$$= (x + iy)^3$$

$$= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3$$

$$= (x^3 - 3xy) + (3x^2y - y^3)i.$$

Hence,

$$\lim_{z \to (1 + i)} f(z) = \lim_{(x,y) \to (1,1)} (x^3 - 3xy) + i \lim_{(x,y) \to (1,1)} (3x^2 y - y^3)$$

$$= -2 + 2i.$$

The key point is that using the Argand diagram model for the
complex numbers we need not invent any new ideas to handle
$\lim_{z \to c} f(z)$ if c is complex and f is complex-valued.

In particular, every limit theorem that was true in our study of
vector functions of a vector variable remains true in our study
of complex functions of a complex variable. More specifically,
we may continue to use such results as the limit of a sum is
the sum of the limits, the limit of a product is the product of
the limits, etc. Again, the main idea is that once we view

Solutions
Block 1: An Introduction to Functions of a Complex Variable
Unit 4: Complex Functions of a Complex Variable

1.4.7(L) continued

complex numbers in the Argand diagram we cannot distinguish
between complex numbers and planar vectors structurally. Thus,
theorems for one model remain theorems for the other.

1.4.8

$(z + h)^2 = z^2 + 2zh + h^2$ (just as in the real case).

Hence, $(z + h)^2 - z^2 = 2zh + h^2$. Hence,

$$\frac{(z + h)^2 - z^2}{h} = \frac{2zh + h^2}{h} = \frac{h}{h}(2z + h)$$

$$= 2z + h, \text{ provided } h \neq 0.$$

Hence,

$$\lim_{h \to 0} [\frac{(z + h)^2 - z^2}{h}] = \lim_{h \to 0} [2z + h]$$

$$= \lim_{h \to 0} 2z + \lim_{h \to 0} h$$

$$= 2z.$$

Notice that this exercise seems to be the complex equivalent
of finding $f'(x)$ when $f(x) = x^2$. This idea is the topic of the
next unit.

1.4.9

Our main aim in this exercise is to show that the study of complex-
valued functions of a single real variable was made when we
studied the planar problem of vector functions of a scalar
variable.

Namely, if we view z as x + iy, then the fact that z is a function
of the scalar (real) variable t means that we may write

$$z(t) = x(t) + iy(t). \tag{1}$$

Solutions
Block 1: An Introduction to Functions of a Complex Variable
Unit 4: Complex Functions of a Complex Variable

1.4.9 continued

The critical point is that if we elect to use the Argand diagram as a geometric model, we see at once that equation (1) is structurally equivalent to the vector equation:

$$\vec{R}(t) = x(t)\vec{i} + y(t)\vec{j}. \tag{2}$$

In summary, the curve in the xy-plane defined by equation (2) is the "graph" of the complex numbers defined by equation (1). In other words, one way of visualizing a (continuous) complex function of a real variable is as a curve in the z-plane.

The main point is that since we may identify a complex function of a real variable with a vector function of a scalar variable, we may also assume that the calculus structure of vector functions of scalar variables is inherited by complex functions of real variables; and both parts (a) and (b) of this exercise are designed to illustrate this.

a. We assume here that f'(t) has the usual meaning, except that f is now a vector function rather than a scalar function. The point is that had we been given the problem

$$\vec{R}(t) = t\vec{i} + t^2\vec{j} \tag{3}$$

we would have been able to conclude that

$$\vec{R}'(t) = \vec{i} + 2t\vec{j}. \tag{4}$$

Since equation (3) translates, in the Argand diagram, into

$$z = t + t^2 i \quad [= f(t)],$$

it follows that f'(t) must be the analog of equation (4), namely,

$$f'(t) = 1 + 2ti. \tag{5}$$

Solutions
Block 1: An Introduction to Functions of a Complex Variable
Unit 4: Complex Functions of a Complex Variable

1.4.9 continued

More generally, then, in terms of the Argand diagram if $z = f(t)$
where f is a differentiable complex function of a real variable,
we may view $z = f(t)$ as the <u>curve</u> $z = g(t) + h(t)$ i where g is
the real part of f and h is the imaginary part of f. In this
event, $f'(t)$ is a vector tangent to this curve with magnitude equal
to

$$\overline{g'^2(t) + h'^2(t)} \ .$$

The key point is that the calculus here is a "carbon copy" of the
calculus of vector functions of a scalar function.

b. If $R'(t) = t^2 i + e^{3t} j$, then we already know that

$$R(t) = \frac{1}{3} t^3 i + \frac{1}{3} e^{3t} j + c. \tag{6}$$

Translating the result (6) into the language of complex numbers
we have that, if $f'(t) = t^2 + e^{3t} i$, then

$$f(t) = \frac{1}{3} t^3 + \frac{1}{3} e^{3t} i + c, \text{ where c is an arbitrary } \underline{\text{complex}} \text{ constant.} \tag{7}$$

If we now use the fact that $f(0) = 1 + i$, equation (7) becomes
$1 + i = \frac{1}{3} i + c$ so that $c = 1 + \frac{2}{3} i$. Putting this result into (7),
we have that $f(t) = \frac{1}{3} t^3 + \frac{1}{3} e^{3t} i + 1 + \frac{2}{3} i$, or

$$f(t) = (\frac{1}{3} t^3 + 1) + \frac{1}{3}(e^{3t} + 2) i.$$

In summary, we already know how to differentiate and integrate
complex functions of a real variable because our previous knowledge
of vector functions of scalar variables. In particular

1. If $z = x(t) + y(t)$ i, then $\frac{dz}{dt} = \frac{dx}{dt} + \frac{dy}{dt}$ i; and

2. If $z = x'(t) + y'(t)$ i, then $\int z dt = x(t) + y(t)$ i + c; where
$x'(t) = \frac{dx(t)}{dt}$ and $y'(t) = \frac{dy(t)}{dt}$ and c is an arbitrary complex
constant.

Thus, while complex functions of a real variable are important in
our study of complex variables (e.g., as mentioned in Exercise
1.4.1, the "graph" of a set of complex numbers in the Argand

Solutions
Block 1: An Introduction to Functions of a Complex Variable
Unit 4: Complex Functions of a Complex Variable

1.4.9 continued

diagram has this form), we do not devote much time to such a
study since the main results are already available to us from our
study of planar vectors.

1.4.10

The result of this exercise justifies why the study of real-
valued functions of a complex variable is usually ignored from a
calculus point of view*. Namely, assuming that the result of
this exercise holds, we have that if $y = f(z)$ and if $\frac{dy}{dz}$ exists,
then $\frac{dy}{dz} = 0$. This, in turn, implies that $f(z)$ is constant.
Thus, if $f: c \rightarrow R$ such that f' exists, then $f(z)$ must be constant.
In other words, unless $f(z) = $ constant, $\frac{dy}{dz}$ ($= \frac{df}{dz}$) fails to
exist. Thus, the study of differentiable real functions of a
complex variable is "short and sweet".

Now, turning to the specifics of this exercise, we must first
define what we mean by f' in the case that f is a real-valued
function of a complex variable. In terms of our usual approach
in terms of structure, we define $\frac{dy}{dz} = f'(z)$ by

$$f'(z_0) = \lim_{\Delta z \to 0} \left[\frac{f(z_0 + \Delta z) - f(z_0)}{z} \right] \tag{1}$$

provided that the limit exists. Since z, and hence Δz, is complex,
it means that there are many paths by which Δz may approach 0.
One such path is the one defined by the change in the imaginary
part of Δz being 0; and another, by the change of the real part
of Δz being 0.

---

*We hasten to stress "calculus" lest you erroneously be led to
believe that such functions are unimportant in all respects.
For example, the absolute value of a complex variable is
extremely important and this is an example of a real-valued function
of a complex variable. That is, if z is complex and $f(z) = |z|$
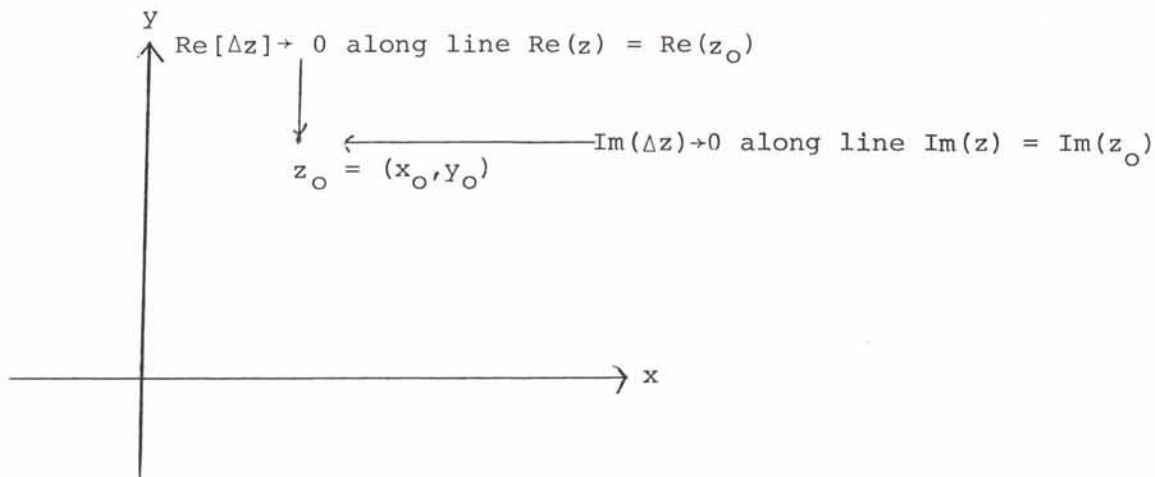then the range of f is the non-negative real numbers.

Solutions
Block 1: An Introduction to Functions of a Complex Variable
Unit 4: Complex Functions of a Complex Variable

1.4.10 continued

In terms of the Argand diagram we have,

Re$[\Delta z] \to 0$ along line Re(z) = Re(z_0)

Im$(\Delta z) \to 0$ along line Im(z) = Im(z_0)

$z_0 = (x_0, y_0)$

Algebraically speaking, we are saying that if $z = x + iy$ then
$\Delta z = \Delta x + i\Delta y$; and we are looking at Δz in one case with $\Delta y = 0$
and in the other with $\Delta x = 0$.

The key point is that numerator in the bracketed expression in
equation (1) must be <u>real</u> since f is given to be real valued.

Thus, with $\Delta y = 0$,

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

is equal to

$$\frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

in which case, f', if it exists must be given by

$$f'(z_0) = \lim_{\Delta x \to 0} \left[\frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} \right] = \left. \frac{\partial f}{\partial x} \right|_{(x_0, y_0)} \qquad (2)$$

Solutions
Block 1: An Introduction to Functions of a Complex Variable
Unit 4: Complex Functions of a Complex Variable

1.4.10 continued

Similarly with $\Delta x = 0$, equation (1) becomes

$$f'(z_o) = \lim_{\Delta y \to 0} \left[\frac{f(z_o + i\Delta y) - f(z_o)}{i\Delta y}\right]$$

$$= \lim_{\Delta y \to 0} \left[\frac{f(x_o, y_o + \Delta y) - f(x_o, y_o)}{i\Delta y}\right]$$

$$= \frac{1}{i} \lim_{\Delta y \to 0} \left[\frac{f(x_o, y_o + \Delta y) - f(x_o, y_o)}{\Delta y}\right]$$

$$= \frac{1}{i} \left.\frac{\partial f}{\partial y}\right|_{(x_o, y_o)}$$

$$= -i \left.\frac{\partial f}{\partial y}\right|_{(x_o, y_o)}. \tag{3}$$

Since the existence of the limit in (1) means that the value of $f'(z_o)$ must be independent of the direction in which $z \to 0$, we may equate the values of $f'(z_o)$ found in (2) and (3) to conclude

$$\left.\frac{\partial f}{\partial x}\right|_{(x_o, y_o)} = -i \left.\frac{\partial f}{\partial y}\right|_{(x_o, y_o)}$$

or

$$f_x(x_o, y_o) + 0i = 0 + i [-f_y(x_o, y_o)]. \tag{4}$$

Solutions
Block 1: An Introduction to Functions of a Complex Variable
Unit 4: Complex Functions of a Complex Variable

1.4.10 continued

Equating the real and imaginary parts in the equality given by (4), we conclude that

$$f_x(x_o,y_o) = 0 \text{ and } f_y(x_o,y_o) = 0. \tag{5}$$

Finally, since $(x_o,y_o) = z_o$ was an arbitrary point (number) in the domain of f we may conclude from equation (5) that

$$f_x(x,y) \equiv f_y(x,y) \equiv 0 \tag{6}$$

and from our knowledge of real-valued functions of several (two) real variables*, we may conclude that

$$f(x,y) = \text{constant}. \qquad (i.e., \ df \equiv 0) \tag{7}$$

Then since $f(x,y)$ is simply the geometric equivalent of $f(z)$, we conclude that $f(z) = $ (real) constant.

---

* Notice that we have identified $f(z)$ with $f(x,y)$ by viewing z as the point (x,y) in the Argand diagram. Since f is real-valued it follows that $f(x,y)$ is a real function of the real variables x and y. Consequently the statement given in (6) is independent of our knowing anything about complex numbers (although the derivation of (6) came from our treatment of the complex numbers). Accordingly (7) is merely a reaffirmation that if $df \equiv 0\,dx + 0\,dy$ then $f(x,y)$ is constant.

Solutions
Block 1: An Introduction to Functions of a Complex Variable

Solutions
Block 1: An Introduction to Functions of a Complex Variable

Unit 5: Differentiating Complex-Valued Functions

1.5.1(L)

---

a. "Mechanically" we obviously expect $f'(z) = 3z^2$ just as in the real
case. Our faith in the mechanical method is justified by the fact
that the rules of arithmetic, needed to justify $f'(x) = 3x^2$ if
$f(x) = x^3$, are obeyed when we turn to complex numbers. More
specifically,

$$f'(z) = \lim_{h \to 0} \left[\frac{f(z + h) - f(z)}{h} \right] \tag{1}$$

$$= \lim_{h \to 0} \left[\frac{(z + h)^3 - z^3}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{z^3 + 3z^2 h + 3zh^2 + h^3 - z^3}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{h}{h} (3z^2 + 3zh + h^2) \right]$$

$$= \lim_{h \to 0} (3z^2 + 3zh + h^2)$$

$$= 3z^2 + \lim_{h \to 0} 3zh + \lim_{h \to 0} h^2$$

$$= 3z^2. \tag{2}$$

Notice that in going from (1) to (2) every step was justified by
the structure of complex arithmetic - but we also see that the
sequence of steps which takes us from (1) to (2) is precisely the
same sequence of steps, only with z replacing x, by which we
deduced that $f'(x) = 3x^2$ if $f(x) = x^3$.

b. $z^3 = (x + iy)^3$

$$= x^3 + 3x^2 (iy) + 3x(iy)^2 + (iy)^3$$

$$= (x^3 - 3xy^2) + i(3x^2 y - y^3). \tag{3}$$

1.5.1(L) continued

From (3) we see that

$$u = x^3 - 3xy^2$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (4)

$$v = 3x^2y - y^3$$

By direct calculation in (4), we have

$$u_x = 3x^2 - 3y^2, \quad u_y = -6xy, \quad v_x = 6xy, \quad \text{and} \quad v_y = 3x^2 - 3y^2.$$

Hence,

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

As an aside to this part of the exercise, notice that in this exercise we knew explicitly how f(z) was related to z without reference to real or imaginary parts. There are cases, however, in which we know u and v as functions of x and y but not explicitly in terms of z. For example, in terms of real mappings, whenever we have the real transformation defined by u = u(x,y) and v = v(x,y), we may invoke complex variables by writing this transformation in the form f(z) = u(x,y) + iv(x,y), where z = x + iy. In this form it may not be easy to see how f(z) is directly related to z. It is then that we have no recourse other than to use the Cauchy-Riemann conditions as a check to see whether f(z) is analytic (differentiable).

[Actually, we want the converse of the Cauchy-Riemann conditions here. Namely, what we have seen is that if f(z) = u + iv is analytic then its real and imaginary parts, u and v, satisfy the Cauchy-Riemann conditions. We have not shown that if the Cauchy-Riemann conditions are satisfied then u + iv is analytic. This is done in Exercise 1.5.12.]

With respect to this exercise, what we have shown is that we may apply the Cauchy-Riemann conditions to $(x^3 - 3x^2y) + i(3x^2y - y^3)$ even if we did not know that this was the Cartesian form of z^3.

1.5.1(L) continued

c. Here our main aim is to stress how one deduces $f'(z)$ from
    $f(z) = u + iv$. What we showed in the lecture was that if $f'(z)$
    existed it had to have the form

$$u_x + iv_x \qquad (5)$$

where equation (5) was obtained from equation (1) by letting $h \to 0$
through real numbers; and it also had to have the form

$$v_y - iu_y \qquad (6)$$

where equation (6) is obtained from equation (1) by letting $h \to 0$
through purely imaginary values.

Thus, equations (5) and (6) give us two answers to part (c).
Moreover, by the Cauchy-Riemann conditions, we may replace v_x in
(5) by $-u_y$, and we may replace u_x in (6) by v_y. This leads us to
the fact that if $f(z) = u + iv$ is an analytic function then $f'(z)$
can be written in any one of the following equivalent form:

$$
\left.
\begin{aligned}
f'(z) &= u_x + iv_x \\
      &= v_y - iu_y \\
      &= u_x - iu_y \\
      &= v_y + iv_x
\end{aligned}
\right\} \qquad (7)
$$

(These results may seem partly self-evident. The non-obvious part
comes from letting $h \to 0$ through the purely imaginary numbers since
in that case our denominator is not Δy but rather $i\Delta y$.)

All we wish to do in this specific exercise is check these results
in the case $f(z) = z^3$.

To this end

1.5.1(L) continued

$$f'(z) = 3z^2$$

$$= 3(x + iy)^2$$

$$= 3(x^2 - y^2) + i6xy. \tag{8}$$

From our results in part (b), we know that $u_x = 3(x^2 - y^2)$, $u_y = -6xy$, $v_x = 6xy$, and $v_y = 3(x^2 - y^2)$. With these values, we see that equation (8) is obeyed by each of the expressions in (7).

1.5.2

a. $f(z) = z^3 + z^2 + z + 1 \rightarrow$

$$f'(z) = 3z^2 + 2z + 1. \tag{1}$$

b. Letting $z = x + iy$, we have

$$f(z) = (x + iy)^3 + (x + iy)^2 + (x + iy) + 1$$

$$= [x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3] + [x^2 - y^2 + i2xy] +$$

$$+ x + iy + 1$$

$$= x^3 + i3x^2y - 3xy^2 - iy^3 + x^2 - y^2 + i2xy + x + iy + 1$$

$$= (x^3 - 3xy^2 + x^2 - y^2 + x + 1) + i(3x^2y - y^3 + 2xy + y). \tag{2}$$

Hence,

$$\left.\begin{aligned}u(x,y) &= x^3 - 3xy^2 + x^2 - y^2 + x + 1\\[2mm]v(x,y) &= 3x^2y - y^3 + 2xy + y\end{aligned}\right\} \tag{3}$$

From (3) it follows that

$$\left.\begin{aligned}u_x &= 3x^2 - 3y^2 + 2x + 1, \quad u_y = -6xy - 2y\\[2mm]v_x &= 6xy + 2y, \quad v_y = 3x^2 - 3y^2 + 2x + 1\end{aligned}\right\} \tag{4}$$

1.5.2 continued

Hence,

$$u_x = v_y$$

and

$$u_y = -v_x.$$

c. $f'(z) = u_x + iv_x.$

Hence, by (4),

$$f'(z) = (3x^2 - 3y^2 + 2x + 1) + i(6xy + 2y). \tag{5}$$

As a check on (5), we may return to equation (1) and write $f'(z)$
in the form $u + iv$.   Namely, from (1),

$$f'(z) = 3(x + iy)^2 + 2(x + iy) + 1$$

$$= 3x^2 - 3y^2 + i6xy + 2x + i2y + 1$$

$$= (3x^2 - 3y^2 + 2x + 1) + i(6xy + 2y)$$

which checks with (5).

1.5.3

a. $f(z) = z^{-2} = \dfrac{1}{z^2} \ (z \neq 0).$

Hence, just as in real-variable calculus,

$$f'(z) = -2z^{-3} = -\frac{2}{z^3}. \tag{1}$$

For the more rigorously-oriented student,

1.5.3 continued

$$f'(z) = \lim_{h \to 0} \left[\frac{\frac{1}{(z + h)^2} - \frac{1}{z^2}}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{z^2 - (z + h)^2}{hz^2 (z + h)^2} \right]$$

$$= \lim_{h \to 0} \left[\frac{-2zh - h^2}{hz^2 (z + h)^2} \right]$$

$$= \lim_{h \to 0} \left[\frac{-2z - h}{z^2 (z + h)^2} \right]$$

$$= \frac{-2z}{z^2 (z)^2}$$

$$= -\frac{2}{z^3},$$

provided $z \neq 0$.

b. $f(z) = \dfrac{1}{z^2}$

$$= \frac{1}{(x + iy)^2}$$

$$= \frac{1}{(x^2 - y^2) + i2xy}$$

$$= \frac{(x^2 - y^2) - i2xy}{[(x^2 - y^2) + i2xy][(x^2 - y^2) - i2xy]}$$

$$= \frac{x^2 - y^2 - i2xy}{\left(x^2 - y^2\right)^2 + 4x^2y^2}$$

and since

$$\left(x^2 - y^2\right)^2 + 4x^2y^2 = x^4 - 2x^2y^2 + y^4 + 4x^2y^2 = \left(x^2 + y^2\right)^2,$$

1.5.3 continued

$$f(z) = \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2} + i\left[\frac{-2xy}{\left(x^2 + y^2\right)^2}\right]. \tag{2}$$

From (2) we conclude that

$$u = \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2} \quad \text{and} \quad v = \frac{-2xy}{\left(x^2 + y^2\right)^2} \tag{3}$$

and both u and v are well defined provided $x^2 + y^2 \neq 0$ and this is guaranteed since $z \neq 0$ (i.e., $0 \notin$ dom f).

From (3) we have

$$u_x = \frac{\left(x^2 + y^2\right)^2 (2x) - \left(x^2 - y^2\right) 2\left(x^2 + y^2\right) 2x}{\left(x^2 + y^2\right)^4}$$

$$= \frac{2x\left(x^2 + y^2\right)\left[\left(x^2 + y^2\right) - 2\left(x^2 - y^2\right)\right]}{\left(x^2 + y^2\right)^4}$$

$$= \frac{2x\left[-x^2 + 3y^2\right]}{\left(x^2 + y^2\right)^3} \quad \text{(since } x^2 + y^2 \neq 0) \tag{4}$$

$$v_y = \frac{\left(x^2 + y^2\right)^2 (-2x) + (2xy) 2\left(x^2 + y^2\right) 2y}{\left(x^2 + y^2\right)^4}$$

$$= \frac{2x\left(x^2 + y^2\right)\left[-\left(x^2 + y^2\right) + 4y^2\right]}{\left(x^2 + y^2\right)^4}$$

$$= \frac{2x\left[-x^2 + 3y^2\right]}{\left(x^2 + y^2\right)^3} \quad (x^2 + y^2 \neq 0). \tag{5}$$

Thus, comparing (4) and (5), we see that $u_x = v_y$ unless $x^2 + y^2 = 0$ and since $0 \notin$ dom f, $u_x = v_y$ for all $z \in$ dom f.

1.5.3 continued

Similarly,

$$u_y = \frac{\left(x^2 + y^2\right)^2 (-2y) - (x^2 - y^2) \, 2(x^2 + y^2) 2y}{\left(x^2 + y^2\right)^4}$$

$$= \frac{2y(x^2 + y^2)[-(x^2 + y^2) - 2(x^2 - y^2)]}{\left(x^2 + y^2\right)^4}$$

$$= \frac{2y[-3x^2 + y^2]}{\left(x^2 + y^2\right)^3}, \quad (x,y) \neq (0,0) \tag{6}$$

$$v_x = \frac{\left(x^2 + y^2\right)^2 (-2y) + (2xy) \, 2(x^2 + y^2) 2x}{\left(x^2 + y^2\right)^4}$$

$$= \frac{-2y(x^2 + y^2)[(x^2 + y^2) - 4x^2]}{\left(x^2 + y^2\right)^4}$$

$$= \frac{-2y[-3x^2 + y^2]}{\left(x^2 + y^2\right)^3}, \quad (x,y) \neq (0,0). \tag{7}$$

Comparing (6) with (7) shows that

$$u_y = -v_x.$$

c. If $f(z) = u + iv$ is analytic, then (among other expressions)

$$f'(z) = u_x + iv_x.$$

Thus, had we been given only equation (2) and we couldn't guess that this was z^{-2}, we could have used equations (4) and (7) [once we knew that $f(z)$ was analytic] to deduce that

1.5.3 continued

$$f'(z) = \frac{2x(-x^2 + 3y^2)}{\left(x^2 + y^2\right)^3} + i\left[\frac{-2y(-3x^2 + y^2)}{\left(x^2 + y^2\right)^3}\right].$$

The validity of this result could be checked by expressing $-\frac{2}{z^3}$ [which we know is $f'(z)$ by part (a)] in terms of real and imaginary parts. In fact, from the previous exercise,

$$-\frac{2}{z^3} = \frac{-2}{(x^3 - 3xy^2) + i(3x^2y - y^3)},$$

$$= \frac{-2[(x^3 - 3xy^2) - i(3x^2y - y^3)]}{\left(x^3 - 3xy^2\right)^2 + \left(3x^2y - y^3\right)^2}$$

$$= \frac{2x(-x^2 + 3y^2) + i[-2y(-3x^2 + y^2)]}{x^6 - 6x^4y^2 + 9x^2y^4 + 9x^4y^2 - 6x^2y^4 + y^6}$$

$$= \frac{2x(-x^2 + 3y^2) + i[-2y(-3x^2 + y^2)]}{\underbrace{x^6 + 3x^4y^2 + 3x^2y^4 + y^6}_{\left(x^2 + y^2\right)^3}}$$

1.5.4(L)

The actual computation involved in this exercise is quite simple. The more interesting part of the result is its relevance to real-valued functions of two real variables.

We have $u + iv$ is analytic. Hence, by the Cauchy-Riemann conditions, we may conclude that

$$u_x = v_y \tag{1}$$

and

$$u_y = -v_x. \tag{2}$$

1.5.4(L) continued

If we now differentiate both sides of (1) with respect to x, we obtain,

$$\left(u_x\right)_x = \left(v_y\right)_x$$

or

$$u_{xx} = v_{yx}. \tag{3}$$

Similarly, if we differentiate both sides of (2) with respect to y, we obtain

$$u_{yy} = -v_{xy}. \tag{4}$$

Under the assumption that v_{xy} is continuous, we know that $v_{xy} = v_{yx}$; and putting this result into (3) and equations (3) and (4), we obtain the required result that

$$u_{xx} + u_{yy} = 0.$$

To prove that $v_{xx} + v_{yy} = 0$, we copy the above procedure, only now we differentiate both sides of (1) with respect to y and both sides of (2) with respect to x.

What this tells us is that the real and imaginary parts of an analytic function satisfy <u>Laplace's equation</u>. In other words, given the <u>real</u> partial differential equation

$$\phi_{xx} + \phi_{yy} = 0$$

then the real and imaginary parts of <u>every</u> analytic function satisfy this equation.

This also tells us quite a bit about the remarkable behavior of an analytic function. Among other things, the stringency of the Cauchy-Riemann conditions seems to indicate that the real and imaginary parts of a complex function of a complex variable must be rather strongly inter-related if the function is to be analytic.

1.5.4(L) continued

The reason for this is that the definition of derivative requires
that a certain limit exist <u>and be the same in infinitely many
directions</u>.*

<u>Note</u>

From this exercise, we see that <u>unless</u> both u and v satisfy
Laplace's equation u + iv cannot be analytic. It turns out that
the converse is also true. For example, if u satisfies Laplace's
equation, we may solve the Cauchy-Riemann equations to find a
function v such that u + iv is analytic. (Notice that we are not
saying that simply because both u and v satisfy Laplace's equation
that u + iv is analytic. Rather u and v must satisfy Laplace's
equation <u>and the Cauchy-Riemann conditions</u> if u + iv is to be
analytic.)

This idea will be stressed more in the next few exercises, but
what we want to emphasize before we end our discussion of this
exercise, is that the study of complex variables provides us a
rather powerful tool in our study of Laplace's equation. Namely,
in looking for solutions of Laplace's equation, we may bring to
bear all our knowledge about analytic functions and then use the
fact that the real and imaginary parts of such functions are solu-
tions of Laplace's equation. (In fact, the role played by u and v
if u + iv is analytic is even more profound than simply that they
are solutions of Laplace's equation, but we shall talk more about
this in later exercises as well as in the next unit.)

---

*That is, $\lim\limits_{h \to 0} \left[\dfrac{f(z + h) - f(z)}{h} \right]$ must not depend on how h approaches
0. Since h is complex, h may approach 0 in infinitely many ways.
Thus, the fact that f(z) is analytic means that the derivative
exists regardless of how h→0. This is a more powerful condition
than the corresponding "real" statement that the directional deri-
vative exists in each direction. Rather this says that the direc-
tional derivative exists in each direction <u>but</u> its value is the
<u>same</u> in <u>each</u> direction.

1.5.5(L)

In the previous exercise, we showed that if u + iv was analytic, then u satisfied Laplace's equation. The converse of this, which we haven't proved, would be that if u satisfied Laplace's equation then u was the real part of an analytic function. We shall not prove the converse but it does happen to be true. What we are doing in this exercise is illustrating the converse in a concrete example. The procedure is to first show that u satisfies Laplace's equation (since from the last exercise, we know that if it doesn't, u + iv cannot be analytic) and we then construct v by solving the Cauchy-Riemann conditions (equations).

a. To this end, given that $u = x^4 - 6x^2y^2 + y^4$, we have that

$$u_x = 4x^3 - 12xy^2, \quad u_{xx} = 12x^2 - 12y^2 \tag{1}$$

and

$$u_y = -12x^2y + 4y^3, \quad u_{yy} = -12x^2 + 12y^2. \tag{2}$$

From (1) and (2), it follows that

$$u_{xx} + u_{yy} = 0$$

and consequently, u is at least eligible to be the real part of an analytic function.

b. To find the imaginary part v of such an analytic function, we utilize the fact that we must have

$$v_x = -u_y \tag{3}$$

and

$$v_y = u_x. \tag{4}$$

By (2), $u_y = -12x^2y + 4y^3$; so (3) implies

$$v_x = 12x^2y - 4y^3.$$

1.5.5(L) continued

Hence,

$$v = 4x^3y - 4xy^3 + g(y). \tag{5}$$

Now from (5)

$$v_y = 4x^3 - 12xy^2 + g'(y). \tag{6}$$

But from (4) and (1),

$$v_y = u_x = 4x^3 - 12xy^2. \tag{7}$$

Equating the values of v_y in (6) and (7), we conclude that

$$g'(y) = 0$$

or

$$g(y) = c. \tag{8}$$

Putting the result in (8) into (5), we conclude that

$$v = 4x^3y - 4xy^3 + c. \tag{9}$$

What (9) tells us is that <u>if</u> the function $(x^4 - 6x^2y^2 + y^4) +$ $iv(x,y)$ is to be analytic then $v(x,y)$ must be $4x^3y - 4xy^3 + c$. That is, the function is not analytic if equation (9) isn't obeyed.

In this particular example it is easy to verify that u and v are the real and imaginary parts of the analytic function $f(z) = z^4 + ic,$* where c is a real constant.

Namely,

---

*f is analytic because $f'(z) = 4z^3$.

1.5.5(L) continued

$$z^4 + ic = (x + iy)^4 + ic$$

$$= x^4 + 4x^3(iy) + 6x^2(iy)^2 + 4x(iy)^3 + (iy)^4 + ic$$

$$= x^4 + i(4x^3y) - 6x^2y^2 - i(4xy^3) + y^4 + ic$$

$$= \underbrace{(x^4 - 6x^2y^2 + y^4)}_{u} + \underbrace{i(4x^3y - 4xy^3 + c)}_{v}$$

In Exercise 1.5.9, we shall try to generalize this result to cover
the case of less "familiar" functions.

c. Our aim here is simply to verify that the role of real and imagi-
nary are in a sense interchangeable. That is, once $u_{xx} + u_{yy} = 0$
there is an analytic function which has u as its real part and an
(another) analytic function which has u as its imaginary part.

The key point is that if f(z) is analytic so also is cf(z) where c
is any complex constant. [Namely, just as in the real case,
$\frac{d[cf(z)]}{dz} = cf'(z)$.] In particular, if u + iv is analytic so also
is i(u + iv) = -v + iu.

Using the results of part (b), we see that

$$-(4x^3y - 4xy^3 + c) + i(x^4 - 6x^2y^2 + y^4) \tag{10}$$

is such a function [i.e., $f(z) = i(z^4 + ic) = iz^4 - c$]. [This
same result could have also been obtained, of course, by inter-
changing the roles of u and v, and using the technique of part (b).]

1.5.6

a. Given that $u = x^3y^4$, we have that

$$u_x = 3x^2y^4, \quad u_{xx} = 6xy^4$$

and

$$u_y = 4x^3y^3, \quad u_{yy} = 12x^3y^2.$$

1.5.6 continued

Hence,

$$u_{xx} + u_{yy} = 6xy^4 + 12x^3y^2 \neq 0. \tag{1}$$

By our results in Exercise 1.5.4, u cannot be the real part of any analytic function u + iv because if it were, $u_{xx} + u_{yy} \equiv 0$ which is contrary to what we proved in (1).

b. $v_x = -u_y$ implies $v_x = -4x^3y^3$. Hence,

$$v = -x^4y^3 + g(y). \tag{2}$$

Therefore,

$$v_y = -4x^3y^3 + g'(y) \tag{3}$$

but $v_y = u_x$ implies

$$v_y = 3x^2y^4. \tag{4}$$

Comparing (3) and (4), we conclude that

$$g'(y) = 3x^2y^4 + x^4y^4$$

which contradicts the fact that g'(y) depends only on y. (Notice how this process resembles the construction in our treatment of exact differentials.)

Thus, g(y) doesn't exist, and combining this with equation (1) shows that the required v fails to exist.

In summary, then, there does not exist an analytic function whose real part is x^3y^4.

1.5.7(L)

Aside from giving us a nice review about certain results of real-valued functions of two real variables, this exercise gives us further insight about the real properties of pairs of functions

1.5.7(L) continued

that are the real and imaginary parts of an analytic function.
(Such pairs of functions are called <u>harmonic conjugates</u>.) In
particular, this exercise shows that a harmonic pair do more than
satisfy Laplace's equation. More specifically, we are going to
show that if u + iv is analytic then the family of curves defined
by

$$u(x,y) = \text{constant} \tag{1}$$

and

$$v(x,y) = \text{constant} \tag{2}$$

always intersect at right angles, except possibly when the deriva-
tive of u + iv is zero. This fact plays an important role in why
mappings defined by (1) and (2) where u + iv is analytic are so
important in real applications. This idea will be pursued in even
more detail in our next unit when we discuss conformal mappings.

At any rate, returning to the actual computational details of the
present exercise, the technique is to find $\frac{dy}{dx}$ for each of the
family curves defined by equations (1) and (2).

Differentiating equation (1) implicitly we conclude that

$$u_x dx + u_y dy = 0$$

or

$$\frac{dy}{dx} = \frac{-u_x}{u_y} \quad (u_y \neq 0).* \tag{3}$$

Thus, equation (3) gives us the slope of each member of the family
u(x,y) = constant.

Similarly, from equation (2), we deduce that the slope of each
curve in the family v = v(x,y) is given by

---

*Recall the Implicit Function Theorem discussed in Block 4 of
Part 2.

1.5.7(L) continued

$$\frac{dy}{dx} = \frac{-v_x}{v_y} \quad (v_y \neq 0).$$ (4)

Now equations (3) and (4) apply for any functions u and v — provided only that u and v are <u>continuously differentiable</u>. If, however, we now assume that $u_x = v_y$ and $u_y = -v_x$,* we see that equation (2) becomes

$$\frac{dy}{dx} = \frac{-v_y}{-v_x} = \frac{v_y}{v_x}.$$ (5)

Comparing (4) and (5), we see that $\frac{dy}{dx}$ for the family $u(x,y) =$ constant is the negative reciprocal of $\frac{dy}{dx}$ for the family $v(x,y) =$ constant. This means that wherever a member of $u(x,y) =$ constant intersects a member of $v(x,y) =$ constant the intersection is at right angles — unless possibly when both v_y and u_y equal zero (since then implicit differentiation need not be valid).

We know, however, that if $f(z) = u + iv$ and f is analytic, then $f'(z) = v_y - iu_y$ [see, for example, Exercise 1.5.1, part (c)].

Thus, the only time that $u(x,y) =$ constant and $v(x,y) =$ constant need not meet at right angles is when the derivative of $u + iv$ equals zero (i.e. when u_y and v_y both equal zero). This establishes the result stated in this exercise.

_____

1.5.8

$u = x^2 - y^2$ and $v = 2xy$ imply that $u + iv = x^2 - y^2 + i(2xy) = (x + iy)^2 = z^2$.

Hence, $u + iv$ is analytic and its derivative is

$$2z = 2(x + iy).$$

Therefore,

_____

*Again, notice in this form, no reference to complex numbers is needed. In other words, the Cauchy-Riemann conditions apply to pairs of real valued functions of two real variables, but the language of complex functions is often more convenient.
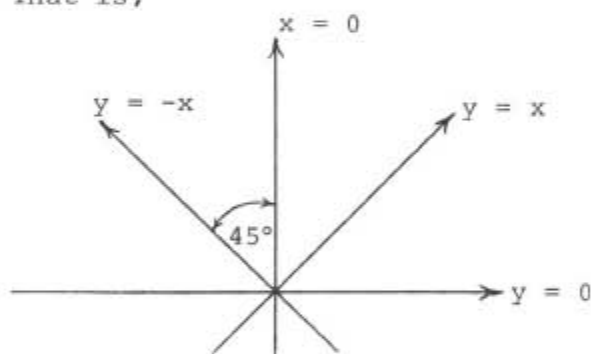
1.5.8 continued

$(u + iv)^1 = 0 \longleftrightarrow x = y = 0.$

So by the result of the previous exercise, the curves $x^2 - y^2 =$ constant and $2xy =$ constant intersect at right angles except at those points for which $x = y = 0$ (i.e., the origin). At the origin the curves in question are

$$\left. \begin{array}{l} x^2 - y^2 = 0 \\ \text{and} \\ 2xy = 0 \end{array} \right\}$$

$x^2 - y^2 = 0$ defines the pair of lines $y = \pm x$ while $2xy = 0$ defines the lines $x = 0$ and $y = 0$.

That is,

In other words, $x^2 - y^2 =$ constant and $2xy =$ constant intersect orthogonally at all points of intersection except the origin.

As a check notice that

$$\left. \begin{array}{l} x^2 - y^2 = c \rightarrow 2x - 2y \frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = \frac{x}{y} \\ \\ 2xy = c \rightarrow 2[x dy + y dx] = 0 \rightarrow \frac{dy}{dx} = -\frac{y}{x} \end{array} \right]$$
negative reciprocals

and the only trouble spot occurs when $x = y = 0$ since then $\frac{dy}{dx}$ has a zero denominator.

1.5.9   (Optional)

We are given that u and v are continuously differentiable and that
(i) $u_x = v_y$ and (ii) $u_y = -v_x$.   We want to prove that

u + iv

is analytic.

Letting $f(z) = u + iv$, we see that

$$f(z + \Delta z) = (u + \Delta u) + i(v + \Delta v)$$

so that

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{\Delta u + i\Delta v}{\Delta z}$$

and since $\Delta z = \Delta x + i\Delta y$, it follows that

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}. \tag{1}$$

Hence, to prove that $f(z)$ is analytic, we see from equation (1)
that we must prove that $\lim\limits_{\substack{\Delta x \to 0 \\ \Delta y}} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}$ exists.

Since u and v are assumed to be continuously differentiable, we
know that

$$\Delta u = u_x \Delta x + u_y \Delta y + k_1 \Delta x + k_2 \Delta y$$

and
$$\left. \right\} \tag{2}$$

$$\Delta v = v_x \Delta x + v_y \Delta y + k_3 \Delta x + k_4 \Delta y$$

where k_1, k_2, k_3, and k_4 all approach zero as Δx and Δy approach
zero.

Using the results of (2) in (1), we have

1.5.9 continued

$$f'(z) = \lim_{\Delta z \to 0} \left[\frac{f(z + \Delta z) - f(z)}{\Delta z} \right]$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y}} \left[\frac{\Delta u + i\Delta v}{\Delta x + i\Delta y} \right]$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y}} \left[\frac{u_x \Delta x + u_y \Delta y + k_1 \Delta x + k_2 \Delta y + i(v_x \Delta x + v_y \Delta y + k_3 \Delta x + k_4 \Delta y)}{\Delta x + i\Delta y} \right] \quad (3)$$

If we now invoke the Cauchy-Riemann conditions, the bracketed
expression in (3) can be greatly simplified. Our strategy is to
use the facts that $u_x = v_y$ and $u_y = -v_x$ to make $\Delta x + i\Delta y$ a factor
of the numerator in (3).* To this end, we write

$$f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y}} \left[\frac{u_x \Delta x - v_x \Delta y + k_1 \Delta x + k_2 \Delta y + i(v_x \Delta x + u_x \Delta y + k_3 \Delta x + k_4 \Delta y)}{\Delta x + i\Delta y} \right]$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y}} \left[\frac{u_x(\Delta x + i\Delta y) + iv_x(\Delta x + i\Delta y) + (k_1 + ik_3)\Delta x + (k_2 + ik_4)\Delta y}{\Delta x + i\Delta y} \right]$$

$$= u_x + iv_x + \lim_{\substack{\Delta x \to 0 \\ \Delta y}} \left[\frac{(k_1 + ik_3)\Delta x + (k_2 + ik_4)\Delta y}{\Delta x + i\Delta y} \right]. \quad (4)$$

Hence, from equation (4) it follows that $f'(z)$ exists provided
only that

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y}} \left[\frac{(k_1 + ik_3)\Delta x + (k_2 + ik_4)\Delta y}{\Delta x + i\Delta y} \right]$$

exists.

---

*For a shorter proof, see note at the end of this exercise.

1.5.9 continued

It is not difficult to show that this limit exists and, in fact,
that it is zero.

Namely,

$$\left| \frac{(k_1 + ik_3)\Delta x + (k_2 + ik_4)\Delta y}{\Delta x + i\Delta y} \right| =$$

$$\frac{|(k_1 + ik_3)\Delta x + (k_2 + ik_4)\Delta y|}{|\Delta x + i\Delta y|} \leqslant$$

$$\frac{|k_1 + ik_3| \; |\Delta x| + |k_2 + ik_4| \; |\Delta y|}{|\Delta x + i\Delta y|} =$$

$$\frac{\sqrt{k_1^2 + k_3^2} \, |\Delta x| + \sqrt{k_2^2 + k_4^2} \; |\Delta y|}{\sqrt{\Delta x^2 + \Delta y^2}} =$$

$$\sqrt{k_1^2 + k_3^2} \left(\frac{|\Delta x|}{\sqrt{\Delta x^2 + \Delta y^2}} \right) + \sqrt{k_2^2 + k_4^2} \left(\frac{|\Delta y|}{\sqrt{\Delta x^2 + \Delta y^2}} \right). \qquad (5)$$

Since $\dfrac{a}{\sqrt{a^2 + b^2}} \leqslant 1$ for all real numbers a and b, we see from (5)
that

$$\left| \frac{(k_1 + ik_3)\Delta x + (k_2 + ik_4)\Delta y}{\Delta x + i\Delta y} \right| \leqslant \sqrt{k_1^2 + k_3^2} + \sqrt{k_2^2 + k_4^2} \qquad (6)$$

and since k_1, k_2, k_3, and $k_4 \to 0$ as Δx and $\Delta y \to 0$, it follows from
(6) that

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y}} \left[\frac{(k_1 + ik_3)\Delta x + (k_2 + ik_4)\Delta y}{\Delta x + i\Delta y} \right] = 0$$

so that from equation (4) we conclude

$$f'(z) = u_x + iv_x \qquad (7)$$

1.5.9 continued

and f'(z), therefore, exists. In fact, equation (7) checks with our earlier result that when u + iv is analytic, its derivative is $u_x + iv_x$.

Note

If we are willing to accept that $\Delta u \approx u_x \Delta x + u_y \Delta y$ and $\Delta v \approx v_x \Delta x + v_y \Delta y$ with a negligible error, then equation (3) becomes

$$f'(z) \approx \lim_{\substack{\Delta x \to 0 \\ \Delta y}} \left[\frac{u_x(\Delta x + i\Delta y) + iv_x(\Delta x + i\Delta y)}{\Delta x + i\Delta y} \right]$$

$$\approx u_x + iv_x.$$

The remainder of the proof was merely(?) verifying that the error was indeed negligible, thus validating the assumption we accepted.

1.5.10   (Optional)

a.   Just as we did in the real case, we let $\frac{\Delta w}{\Delta z} - \frac{dw}{dz} = k$ (where now k is complex since $\frac{\Delta w}{\Delta z}$ and $\frac{dw}{dz}$ are complex). We then have that

$$\lim_{\Delta z \to 0} \left[\frac{\Delta w}{\Delta z} - \frac{dw}{dz} \right] = \lim_{\Delta z \to 0} k$$

or

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} - \lim_{\Delta z \to 0} \frac{dw*}{dz} = \lim_{\Delta z \to 0} k. \tag{1}$$

Now since $\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{dw}{dz}$ (by definition of the fact that w = f(z) is analytic) and since $\lim_{\Delta z \to 0} \frac{dw}{dz} = \frac{dw}{dz}$, we may conclude from equation

---

*Notice structurally that we are using the same limit theorems (such as the limit of a sum equals the sum of the limits) in the complex case as we used in the real case and this is why our step-by-step translation from the real case is valid.

1.5.10 continued

(1) that $\lim\limits_{\Delta z \to 0} k = \dfrac{dw}{dz} - \dfrac{dw}{dz} = 0.$

Hence

$$\frac{\Delta w}{\Delta z} - \frac{dw}{dz} = k$$

implies that

$$\frac{\Delta w}{\Delta z} = \frac{dw}{dz} + k, \quad \lim_{\Delta z \to 0} k = 0.$$

Consequently, since $\Delta z \neq 0$,

$$\Delta w = (\frac{dw}{dz}) \Delta z + k \Delta z, \quad \lim_{\Delta z \to 0} k = 0. \tag{2}$$

b. Since $k \Delta z$ is a higher order infinitesimal, we may assume that

$$\Delta w \approx \frac{dw}{dz} \Delta z$$

for "sufficiently small" values of Δz.

In terms of f, if f is analytic at $z = z_o$, then "near" z_o

$$\Delta w \approx f'(z_o) \Delta z. \tag{3}$$

Recall that $f'(z_o)$ is a complex number, as is Δz. We know that to multiply two complex numbers, we multiply the magnitudes and add the arguments. Now the only complex number whose argument is undefined is 0 (i.e., the complex number 0 has the same problem as the vector $\vec{0}$ has when it comes to defining direction).

Hence, assuming that $f'(z_o) \neq 0$, we have that $f'(z_o)$ has a well-defined argument, say, θ_o.

Thus, the complex number $f'(z_o) \Delta z$ has a magnitude equal to $|f'(z_o)|$ times that of Δz's magnitude and its argument is that of Δz's plus θ_o.
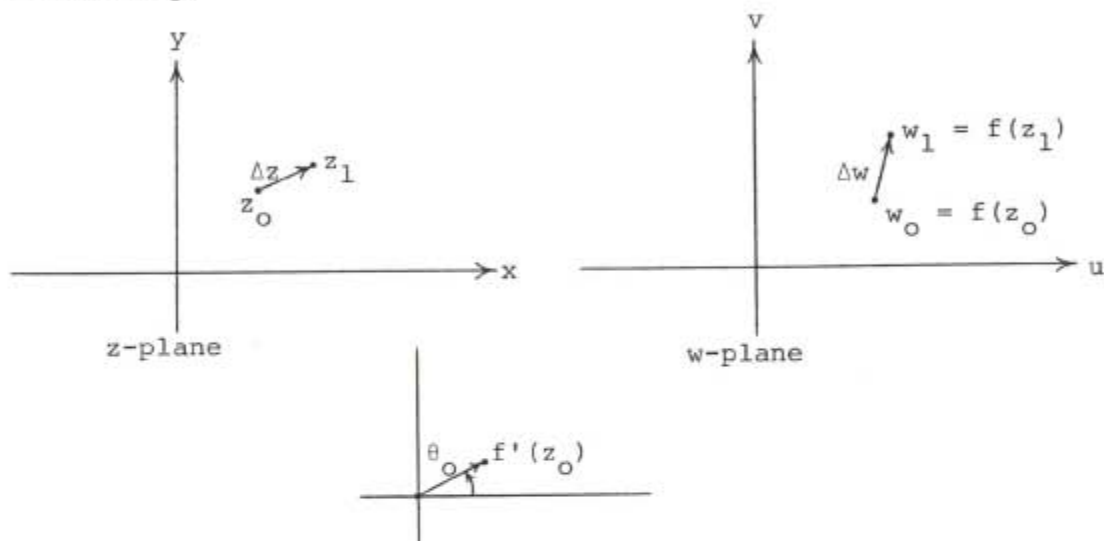
1.5.10 continued

In terms of equation (3), then, near $z = z_o$ Δw has $|f'(z_o)|$ times
the magnitude of Δz and its argument exceeds that of Δz by θ_o. In
other words, the direction of Δw is obtained by rotating Δz
through the angle θ_o.

Pictorially,

We multiply the length of Δz by the length of $f'(z_o)$ to obtain the
length of Δw and we rotate Δz by θ_o to obtain the direction of Δw.

In summary, then, near z_o, f magnifies z by a scaling factor equal
to $|f'(z_o)|$ and rotates z by an angle equal to the argument of
$f'(z_o)$.

We shall discuss this further in the next unit and we shall give a
specific illustration in the next part of this exercise, but for
now we wanted to point out the very important aspect that as long
as $f'(z_o) \neq 0$, the mapping defined by f preserves angles. That
is, if two curves C_1 and C_2 meet at a point P in the xy-plane and
if the mapping $u = u(x,y)$, $v = v(x,y)$ has the property that $u + iv$
is analytic (or without reference to complex numbers, that $u_x = v_y$
and $u_y = -v_x$) then the angle at which the images of C_1 and C_2 meet
in the uv-plane is the same as the angle of intersection in the
xy-plane. The proof of this remark is nothing more than a direct
translation of the discussion of this exercise to viewing complex
numbers as vectors in the plane.

1.5.10 continued

The key idea is that while the scaling factor need not be 1,
unless $f'(z_o) = 0$ each vector emanating from z_o is rotated the same
amount θ_o. It is for this reason that analytic functions are very
important in the study of transformations of the plane. Namely,
these mappings [provided only that $f'(z_o) \neq 0$] yield the <u>conformal</u>
mappings which we shall discuss in the next unit.

c. $u = x^2 - y^2$

$v = 2xy$

corresponds to the analytic function

$$f(z) = z^2 \qquad\qquad (1)$$

[i.e., $f(z) = (x + iy)^2 = \underbrace{x^2 - u^2}_{u} + \underbrace{i2xy}_{v}$].

From (1) it follows that

$f'(z) = 2z$

so that at $(1,1)$ $[= 1 + i]$

$f'(z) = 2(1 + i)$.

Hence,

$|f'(1 + i)| = 2|1 + i| = 2\sqrt{2}$

and

$\arg f(1 + i) = 45°$.

Therefore, in a sufficiently small neighborhood of $(1,1)$ the

mapping $\left.\begin{array}{l} u = x^2 - y^2 \\ v = 2xy \end{array}\right\}$ may be defined as multiplying the distance

of each point from $(1,1)$ by $2\sqrt{2}$ and rotating it by $45°$.

MIT OpenCourseWare
http://ocw.mit.edu

Resource: Calculus Revisited: Complex Variables, Differential Equations, and Linear Algebra
Prof. Herbert Gross

The following may not correspond to a particular course on MIT OpenCourseWare, but has been provided by the author as an individual learning resource.

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.