Unit 8: The Use of Power Series

2.8.1(L)

a. Since two of the three summations involve x^n and only one involves x^{n-1}, it seems logical to "jack up" the index in x^{n-1} by 1. That is, we replace n by n + 1 inside the Σ-sign and lower the starting point of the summation by 1. We obtain

$$\sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n,$$

so that

$$\sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n - \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^n. \qquad (2)$$

We then "split off" the first term in each of the summation in (2) which begin with n = 0 (in this way, each summation begins with n = 1). We therefore rewrite (2) as:

$$[a_1 + \sum_{n=1}^{\infty} (n+1)a_{n+1}x^n] - \sum_{n=1}^{\infty} na_n x^n + [a_o + \sum_{n=1}^{\infty} a_n x^n]$$

$$= a_o + a_1 + \sum_{n=1}^{\infty} [(n+1)a_{n+1}x^n - na_n x^n + a_n]x^n$$

$$= a_o + a_1 + \sum_{n=1}^{\infty} [(n+1)a_{n+1} - (n-1)a_n]x^n. \qquad (3)$$

Note:
Especially for those who may still be "edgy" about extensive use of the Σ -notation, it may be worthwhile to show the equivalence of (1) and (3) by long hand techniques. In this way, one does not review what we've done without recourse to the Σ -notation, and one may also learn to appreciate better the compactness of the Σ -notation.

To begin with, (1) may be expanded as:

2.8.1(L) continued

$$\begin{array}{l} (a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots) \\ -(\qquad a_1x + 2a_2x^2 + 3a_3x^3 + \ldots) \\ \underline{+(a_o + \ a_1x + \ a_2x^2 + \ a_3x^3 + \ldots)} \\ (a_o + a_1) + 2a_2x + (3a_3 - a_2)x^2 + (4a_4 - 2a_3)x^3 + \ldots; \end{array}$$

and this agrees with (3) since the expansion of (3) yields

$$(a_o + a_1) + [(2a_2 - 0a_1)x + (3a_3 - a_2)x^2 + (4a_4 - 2a_3)x^3 + \ldots]$$

equals

$$(a_o + a_1) + 2a_2x + (3a_3 - a_2)x^2 + (4a_4 - 2a_3)x^3 + \ldots$$

It should also be noted that in either approach, we used the idea of absolute convergence to justify our taking the liberty of re-arranging and combining terms in the way we did.

b. Replacing (1) by (3), we see that

$$\sum_{n=1}^{\infty} na_nx^{n-1} - \sum_{n=1}^{\infty} na_nx^n + \sum_{n=0}^{\infty} a_nx^n \equiv 0 \qquad (4)$$

implies that

$$(a_o + a_1) + \sum_{n=1}^{\infty} [(n+1)a_{n+1} - (n-1)a_n]x^n \equiv 0 \ . \qquad (5)$$

Now, since the only way that two convergent power series can be identical is for their coefficients to be equal term-by-term and since

$$0 \equiv \sum_{n=0}^{\infty} 0x^n,$$

we see from (5) that

$$a_o + a_1 = 0 \qquad (6)$$

and

2.8.1(L) continued

for $n \geq 1$; $(n + 1)a_{n+1} - (n - 1)a_n = 0$. That is, $n \geq 1$ implies that

$$a_{n+1} = \frac{(n - 1)}{n + 1} a_n. \tag{7}$$

From (6) we see that a_o may be chosen at random, whereupon

$$a_1 = -a_o \tag{8}$$

and we may now use (7) to compute a_2 from a_1, a_3 from a_2, a_4 from a_3, etc.

For example with n = 1, (7) becomes

$$a_2 = 0 \quad a_1 = 0$$

but once $a_2 = 0$, (7) tells us that for $n > 2$, $a_n = 0$. That is, from (7) a_{n+1} is a multiple of a_n so that a_{n+1} must be 0 once $a_n = 0$.

Thus, we have that a_o is arbitrary, $a_1 = -a_o$, and $a_n = 0$ for $n \geq 2$. Since

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_o + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots,$$

we have that

$$f(x) = a_o - a_o x \ (+ 0)$$

$$= a_o(1 - x). \tag{9}$$

c. If

$$(1 - x)\frac{dy}{dx} + y = 0 \tag{10}$$

has a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n,$$

2.8.1(L) continued

then

$$y' = \sum_{n=1}^{\infty} na_n x^{n-1}$$

so that (10) becomes

$$\frac{dy}{dx} - x\frac{dy}{dx} + y = 0$$

or

$$\sum_{n=1}^{\infty} na_n x^{n-1} - x\sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

or

$$\sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0. \quad (11)$$

Notice that equation (11) is precisely equation (4) of part (b) [where the equality in (11) is understood to be our identity. We used the identity symbol in part (b) for emphasis, but other than that, we revert to the usual notation that $L(y) = 0$ means $L(y) \equiv 0$].

Hence, from part (b) we deduce from (11) that $y = a_o(1 - x)$.

Note #1:

As usual we have elected to start with an example that could be solved more easily by a more familiar technique. In equation (10) we could separate variables to obtain that for $x \neq 1$ and $y \neq 0$, $dy/y = dx/x - 1$; or $\ln|y| = \ln|x - 1| + c_1$; or

$$|y| = e^{c_1}|x - 1|, \; e^{c_1} > 0;$$

or

$$y = \pm e^{c_1}(x - 1) = c(x - 1), \text{ where } c = \pm e^{c_1} \neq 0. \quad (12)$$

If we let $c = 0$ in (12) then we obtain the "forbidden" case, $y = 0$. But $y = 0$ satisfies (10) trivially. Thus, we may conclude

2.8.1(L) continued

that (12) may be extended to cover the case $y = 0$ by letting $c = 0$. In other words, we see that (9) may be verified by our earlier technique of variables separable.

Note #2:

Without the main theorem discussed in our lecture, all we have proven is that if the equation

$$(1 - x)\frac{dy}{dx} + y = 0$$

has a solution which is analytic at $x = 0$, then the solution is given by (9).

What the theorem tells us is that the equation does have a solution which is analytic at $x = 0$. In fact, if we write the equation in standard form, we obtain

$$\frac{dy}{dx} + \frac{y}{1 - x} = 0,$$

whereby we see that the coefficient of y is $1/1 - x$ which is analytic for $|x| < 1$. We may therefore use the theorem to conclude that the solution given by (9) is valid at least for $|x| < 1$.

Note #3:

In this particular exercise we were able to check directly that (9) was the solution of the given equation for all x, except $x = 1$. Thus, we see that the theorem actually tells us the smallest region which we can be sure that we have a solution. It does not say that the solution cannot extend beyond this region. In other words, with respect to the present exercise, we see that by the theorem, (9) is the solution of our equation at least if $|x| < 1$. It also happens here that (9) is also the solution if $|x| > 1$; but we cannot conclude this simply from the theorem.

2.8.2(L)

Given that

$$(1 - x)\frac{dy}{dx} - y = 0\ , \tag{1}$$

we know that the general solution may be expressed in the form

$$y = \sum_{n=0}^{\infty} a_n x^n,\ a_o \text{ arbitrary;} \tag{2}$$

valid at least for $|x| < 1$.*

From (2), we have that

$$\frac{dy}{dx} = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

so that (1) becomes

$$\frac{dy}{dx} - x\frac{dy}{dx} - y = 0$$

or

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0;$$

or

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Hence,

*This was discussed in Exercise 2.8.1. Namely, written in standard form, equation (1) becomes

$\frac{dy}{dx} + \left(\frac{1}{1 - x}\right)y = 0$; and $\frac{1}{1 - x}$ is analytic for $|x| < 1$.

2.8.2(L) continued

$$\sum_{n=0}^{\infty} (n+1)a_{n+1}x^n - \sum_{n=1}^{\infty} na_nx^n - \sum_{n=0}^{\infty} a_nx^n = 0.*$$

Therefore,

$$a_1 + \sum_{n=1}^{\infty} (n+1)a_{n+1}x^n - \sum_{n=1}^{\infty} na_nx^n - a_o - \sum_{n=1}^{\infty} a_nx^n = 0.$$

Consequently,

$$(a_1 - a_o) + \sum_{n=1}^{\infty} [(n+1)a_{n+1} - (n+1)a_n]x^n = 0 = \sum_{n=0}^{\infty} 0x^n. \quad (3)$$

Thus, by equating coefficients of like terms, we see from (3) that

$$a_1 - a_o = 0, \quad (4)$$

or

$$a_1 = a_o; \quad (4')$$

and for $n \geq 1$;

$$(n+1)a_{n+1} - (n+1)a_n = 0 \quad (5)$$

or

$$a_{n+1} = a_n, \text{ for } n \geq 1.** \quad (5')$$

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*Hopefully what we are doing seems familiar. We have copied the equation of the previous exercise, except for a sign change, in order to give you a chance to rederive the sequence of steps which were explained then.

**Since $n \geq 1$, $n + 1 \neq 0$. Hence we may cancel $(n + 1)$ in (5).

2.8.2(L) continued

From (4') we see that a_o may be chosen at random whence $a_1 = a_o$. Then, from (5'), we see that $a_2 = a_1$, $a_3 = a_2$, etc. In other words, (4') and (5') combine to tell us that a_o is an arbitrary constant and that for $n > 0$, $a_n = a_o$.

Putting this information into (2) we obtain

$$\begin{aligned} y = \sum_{n=0}^{\infty} a_n x^n &= a_o + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots \\ &= a_o + a_o x + a_o x^2 + \ldots + a_o x^n + \ldots \\ &= a_o \, (1 + x + x^2 + \ldots + x^n + \ldots) \\ &= a_o \sum_{n=0}^{\infty} x^n. \end{aligned} \tag{6}$$

Since we know that (1) has a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

valid at least for $|x| < 1$, we see from (6) that this solution is given unambiguously by

$$y = a_o \sum_{n=0}^{\infty} x^n.$$

Note #1:
The sign change here (versus the previous exercise) changes the "complexity" of our series solution. Yet

$$\sum_{n=0}^{\infty} x^n$$

should suggest to you 1/1-x (i.e., the geometric series

$$\sum_{n=0}^{\infty} x^n$$

converges to 1/1 - x for $|x| < 1$). Once this observation is made, we see that (6) becomes

$$y = \frac{a_o}{1 - x}, \quad |x| < 1. \tag{7}$$

2.8.2(L) continued

This can be checked directly by observing that

$$(1 - x)\frac{dy}{dx} - y$$

is equal to

$$\frac{d}{dx}[(1 - x)y]$$

so that (1) is

$$\frac{d}{dx}[(1 - x)y] = 0.$$

Hence, $(1 - x)y = c$; so that for $x \neq 1$

$$y = \frac{c}{1 - x}$$

which agrees with (7), except the restriction $|x| < 1$ may be replaced by $x \neq 1$, a fact that cannot be deduced from the key theorem.

Note #2:

In this exercise the series solution (6) could be written in a convenient closed form (which is not always the case). The key point is that even if we could not "simplify"

$$\sum_{n=0}^{\infty} a_o x^n,$$

letting $f(x) = \sum_{n=0}^{\infty} a_o x^n$,

we can still compute $f'(x)$, $f''(x)$, etc. by term-by-term differentiation, at least for $|x| < 1$. In other words, the series solution is as exact as any closed form expression would be, but we may not feel as "at home" with it.

2.8.3(L)

Given

$$\frac{dy}{dx} - 2xy = 0 \tag{1}$$

and

$$y = \sum_{n=0}^{\infty} a_n x^n, \tag{2}$$

we have that

$$\frac{dy}{dx} = \sum_{n=1}^{\infty} na_n x^{n-1}. \tag{3}$$

Putting (2) and (3) into (1) yields

$$\sum_{n=1}^{\infty} na_n x^{n-1} - 2x \sum_{n=0}^{\infty} a_n x^n = 0,$$

or

$$\sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0. \tag{4}$$

Replacing n by n - 2 in the second summation on the left side of (4) yields

$$\sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=2}^{\infty} 2a_{n-2} x^{n-1} = 0$$

or

$$a_1 + \sum_{n=2}^{\infty} na_n x^{n-1} - \sum_{n=2}^{\infty} 2a_{n-2} x^{n-1} = 0$$

or

$$a_1 + \sum_{n=2}^{\infty} (na_n - 2a_{n-2}) x^{n-1} = 0 = \sum_{n=0}^{\infty} 0x^n. \tag{5}$$

Equating coefficients in (5) we see that a_1 (the constant term on the left side) must equal 0 (the constant term on the right side). That is,

2.8.3(L) continued

$a_1 = 0;$ (6)

and that for $n \geq 2$

$na_n - 2a_{n-2} = 0.$

That is,

$n \geq 2 \rightarrow a_n = \dfrac{2a_{n-2}}{n}.$ (7)

Equation (7) tells us that each a_n (for $n \geq 2$) is a multiple of the coefficient which comes two earlier. In particular, since $a_1 = 0$, we deduce from (7) that $a_3, a_5, a_7, \ldots a_{2n+1}, \ldots$ are all equal to 0.

Moreover, the first n for which (7) applies, namely n = 2, implies that

$a_2 = \dfrac{2a_o}{2} = a_o.$ (8)

Since no condition is imposed on a_o, we see from (8) that a_o may be chosen at random, whereupon $a_2 = a_o$.

We then use (7) with n = 4 (we omit odd values of n since we have just seen that $a_n = 0$ for all odd values of n) to obtain

$a_4 = \dfrac{2a_2}{4} = \frac{1}{2} a_2,$

so by (8)

$a_4 = \frac{1}{2} a_o \ \left(= \dfrac{a_o}{2!}\right).$ (9)

We then use (7) with n = 6 to obtain

$a_6 = \dfrac{2a_4}{6} = \dfrac{a_4}{3},$

so by (9),

2.8.3(L) continued

$$a_6 = \frac{a_o}{6}\left(= \frac{a_o}{3!}\right). \tag{10}$$

Next, we use (7) with n = 8 to obtain

$$a_8 = \frac{2a_6}{8} = \frac{a_6}{4},$$

so by (10),

$$a_8 = \frac{a_o}{24} = \frac{a_o}{4!}. \tag{11}$$

Letting n = 10 in (7) yields

$$a_{10} = \frac{2a_8}{10} = \frac{a_8}{5}$$

so by (11)

$$a_{10} = \frac{a_o}{5!}. \tag{12}$$

Note #1:

Looking at (8) through (12) we might venture the guess that

$$a_{2n} = \frac{a_o}{n!}.$$

This may be verified by induction but the important point is that we may use (7) plus previous values of a_n to find each new a_n; knowing by our general theory that the series which is thus generated yields the solution of the equation.

Returning to the mainstream of the exercise, we have that

$$a_{2n+1} = 0 \quad \text{for } n = 0,1,2,3,... \tag{13}$$

$$a_{2n} = \frac{a_o}{n!} \quad \text{for } n = 0,1,2,3,... \tag{14}$$

and using this information in (2) yields

2.8.3(L) continued

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n \text{ even}}^{\infty} a_n x^n + \sum_{n \text{ odd}}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1*},$$

so that by (13) and (14),

$$y = \sum_{n=0}^{\infty} \frac{a_o}{n!} x^{2n} + \sum_{n=0}^{\infty} 0\ x^{2n+1}$$

or

$$y = a_o \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} . \qquad (15)$$

Note #2:

The solution given by (15) is valid for all real x since the coefficients in (1) are everywhere analytic.

Note #3:

Notice that e^{-x^2} is an integrating factor of (1). In fact, if we multiply both sides of (1) by e^{-x^2} we obtain

$$\frac{d}{dx}\left(ye^{-x^2}\right) = 0$$

or

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*Omitting the Σ-notation, all we are saying is that by the properties of absolute convergence,

$$(a_o + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots)$$

$$=$$

$$(a_o + a_2 x^2 + a_4 x^4 + \ldots + a_{2n} x^{2n} + \ldots) + (a_1 x + a_3 x^3 + a_5 x^5 + \ldots + a_{2n+1} x^{2n+1} + \ldots)$$

2.8.3(L) continued

$$ye^{-x^2} = a_o;*$$

hence,

$$y = a_o e^{x^2}.$$

It is easily checked that

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

is the power series expansion of e^{x^2} about $x = 0$.

Note #4:
Had we not known a closed form expression [such as (16)] for (15), we could still have let

$$f(x) = a_o \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

whereupon

$$f'(x) = a_o \sum_{n=1}^{\infty} \frac{2nx^{2n-1}}{n!},$$

etc. For a less contrived example, see Exercise 2.8.7(L).

2.8.4(L)

With

$$y = \sum_{n=0}^{\infty} a_n x^n, \tag{1}$$

we have that

$$y' = \sum_{n=1}^{\infty} na_n x^{n-1}. \tag{2}$$

*Since (15) uses a_o as the arbitrary constant, we have elected to use a_o to denote the arbitrary constant here also. In this way, we can make a more direct comparison of (15) and (16).

2.8.4(L) continued

Putting (1) and (2) into

$$x \frac{dy}{dx} + y = 0 \tag{3}$$

yields

$$x \sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0,$$

or

$$\sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0,$$

or

$$\sum_{n=1}^{\infty} na_n x^n + a_o + \sum_{n=1}^{\infty} a_n x^n = 0,$$

or

$$a_o + \sum_{n=1}^{\infty} (n + 1)a_n x^n = 0 = \sum_{n=0}^{\infty} 0x^n. \tag{4}$$

Equating "like" coefficients in (4), we obtain

$$a_o = 0 \tag{5}$$

and

$$(n + 1)a_n = 0 \quad \text{for all } n \geq 1. \tag{6}$$

Since $n \geq 1 \rightarrow n + 1 \neq 0$, we see from (6) that $a_n = 0$ for $n \geq 1$. This coupled with (5) tells us that every a_n in (1) must equal 0.

In other words, we have shown that only the trivial solution of (3), namely $y = 0$, is analytic at $x = 0$. That is, except for $y = 0$, equation (3) possesses no solution which is analytic at $x = 0$.

This does not contradict any results discussed in our lecture. Namely, the fundamental theorem requires that all our coefficients

2.8.4(L) continued

be analytic at x = 0 when the equation is written in standard form. When equation (3) is written in standard form, it becomes

$$\frac{dy}{dx} + \frac{y}{x} = 0.$$

In this form, the coefficient of y is $\frac{1}{x}$, which is not analytic at x = 0.

The key point is that our fundamental theorem only quarantees the general solution in the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

when all of the coefficients are analytic at x = 0, when the equation is written in standard form.

Again, we have deliberately chosen an "easy" example so that we may see what went wrong here. Namely,

$$\frac{d(xy)}{dx} = x\frac{dy}{dx} + y,$$

so that (3) is equivalent to

$$\frac{d}{dx}(xy) = 0,$$

from which we conclude that xy = c or y = c/x, provided $x \neq 0$.

In other words, the general solution of (3) is not analytic at x = 0.

Optional Note:
Up to now, we have been using examples in which our solutions have been of the form

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

More generally, we strive for solutions of the form

2.8.4(L) continued

$$y = \sum_{n=0}^{\infty} a_n (x - x_o)^n.$$

In the present exercise, equation (3) may be written in standard form as

$$\frac{dy}{dx} + \frac{y}{x} = 0$$

and the coefficient of y, namely $\frac{1}{x}$, is analytic except when $x = 0$.

Thus, according to the general theory, if we pick any $x_o \neq 0$, equation (3) has a solution in the form

$$y = \sum_{n=0}^{\infty} a_n (x - x_o)^n$$

valid for all x such that $|x - x_o| < R$ where R is chosen so that $x = 0$ is not included in the interval.

For example, we could choose $x_o = -1$ and then obtain the general solution of (3) in the form

$$y = \sum_{n=0}^{\infty} a_n (x + 1)^n \tag{8}$$

valid at least for $|x + 1| < 1$ since $0 \notin \{ x: |x + 1| < 1\}$.

From (8) we have

$$y' = \sum_{n=1}^{\infty} n a_n (x + 1)^{n-1}. \tag{9}$$

We then rewrite (3) as

$$(x + 1) \frac{dy}{dx} - \frac{dy}{dx} + y = 0*$$

*Writing (3) in this way permits us to keep working with powers of x + 1. Otherwise, we would have to work with expressions like

$$x \sum_{n=1}^{\infty} n a_n (x + 1)^{n-1}$$

and these are not too pleasant.

2.8.4(L) continued

whereupon (8) and (9) yield

$$\sum_{n=1}^{\infty} na_n(x+1)^n - \sum_{n=1}^{\infty} na_n(x+1)^{n-1} + \sum_{n=0}^{\infty} a_n(x+1)^n = 0. \qquad (10)$$

We may now rewrite

$$\sum_{n=1}^{\infty} na_n(x+1)^{n-1}$$

as

$$\sum_{n=0}^{\infty} (n+1)a_{n+1}(x+1)^n$$

so that (10) becomes

$$\sum_{n=1}^{\infty} na_n(x+1)^n - \sum_{n=0}^{\infty} (n+1)a_{n+1}(x+1)^n + \sum_{n=0}^{\infty} a_n(x+1)^n = 0. \qquad (11)$$

We may then split off the first term in each of the last two summations on the left side of (11) to obtain:

$$\sum_{n=1}^{\infty} na_n(x+1)^n - a_1 - \sum_{n=1}^{\infty}(n+1)a_{n+1}(x+1)^n + a_o + \sum_{n=1}^{\infty} a_n(x+1)^n = 0$$

or

$$(a_o - a_1) + \sum_{n=1}^{\infty} [(n+1)a_n - (n+1)a_{n+1}](x+1)^n = 0 = \sum_{n=0}^{\infty} 0(x-1)^n. \qquad (12)$$

Equating "like coefficients" in (12) we obtain

$$a_o - a_1 = 0$$

or

2.8.4(L) continued

$a_1 = a_o;$

and

$(n + 1)a_n - (n + 1)a_{n+1} = 0$ for $n \geq 1$.

That is, for $n \geq 1$

$$a_{n+1} = a_n. \tag{14}$$

Hence, from (13) and (14) we see that we may choose a_o arbitrarily afterwhich $a_o = a_1 = a_2 = \ldots = a_n = \ldots$

In other words, (8) yields

$$y = \sum_{n=0}^{\infty} a_o(x + 1)^n \quad |x + 1| < 1 \tag{15}$$

as the general solution of (3).

As a check, we already know that $y = c/x$ is the solution of (3) provided $x \neq 0$. Moreover,

$$\frac{c}{x} = \frac{c}{(x + 1) - 1}$$

$$= \frac{-c}{1 - (x + 1)}$$

Letting $u = x + 1$, we have that

$$\frac{-1}{x} = \frac{1}{1 - (x + 1)}$$

$$= \frac{1}{1 - u}$$

$$= \sum_{n=0}^{\infty} u^n, \quad |u| < 1$$

$$= \sum_{n=0}^{\infty} (x + 1)^n, \quad |x + 1| < 1.$$

2.8.4(L) continued

Hence,

$$\frac{c}{x} = -c\sum_{n=0}^{\infty}(x+1)^n, \quad |x+1| < 1$$

or letting $a_o = -c$, we obtain that

$$y = a_o\sum_{n=0}^{\infty}(x+1)^n, \quad |x+1| < 1$$

is the general solution of (3), which checks with (15).

2.8.5(L)

With

$$y = \sum_{n=0}^{\infty} a_n x^n;\ y' = \sum_{n=1}^{\infty} na_n x^{n-1} \text{ and } y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Hence

$$(1-x^2)y'' - xy' + y = 0$$

implies

$$y'' - x^2y'' - xy' + y = 0;$$

or

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Therefore,

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0,$$

or

$$2a_2 + 6a_3x + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n$$

2.8.5(L) continued

$$-a_1x - \sum_{n=2}^{\infty} na_nx^n + a_o + a_1x + \sum_{n=2}^{\infty} a_nx^n = 0.$$

Consequently,

$$(2a_2 + a_o) + 6a_3x + \sum_{n=2}^{\infty} \{(n + 2)(n + 1)a_{n+2} - [n(n - 1) + n - 1]a_n\} x^n = 0. \quad (1)$$

Since all the coefficients on the right side of (1) must equal 0, we conclude that

$$2a_2 + a_o = 0; \text{ or } a_2 = -\tfrac{1}{2} a_o, \quad (2)$$

$$6a_3 = 0; \text{ or } a_3 = 0 \quad (3)$$

and for $n \geq 2$,

$$(n + 2)(n + 1)a_{n+2} - [n(n - 1) + n - 1]a_n = 0$$

or

$$(n + 2)(n + 1)a_{n+2} - [n^2 - 1]a_n = 0$$

or

$$a_{n+2} = \frac{(n - 1)}{n + 2} a_n. \quad (4)$$

Thus, we may pick a_o and a_1 at random whereupon (3) and (4) show us that all other coefficients are then uniquely determined.

In fact, since $a_3 = 0$, (4) shows us that $a_5 = a_7 = a_9 = a_{11} = \ldots = a_{2n+1} = \ldots = 0$.

Moreover, if n is even, (4) reveals that

$$a_4 = \tfrac{1}{4} a_2 = -\tfrac{1}{8} a_o$$

$$a_6 = \tfrac{1}{2} a_4 = -\tfrac{1}{16} a_o$$

2.8.5(L) continued

$a_8 = \frac{5}{8} a_6 = - \frac{5}{128} a_o$

$a_{10} = \frac{7}{10} a_8 = - \frac{35}{1280} a_o.$

Regrouping the terms in

$$\sum_{n=0}^{\infty} a_n x^n$$

we obtain

$$y = (a_o + a_2x^2 + a_4x^4 + a_6x^6 + a_8x^8 + 9_{10}x^{10} + \ldots)$$

$$+ a_1x + (\underset{=0}{a_3}x^3 + \underset{=0}{a_5}x^5 + \underset{=0}{a_7}x^7 + \ldots)$$

$$= a_o(1 - \frac{1}{2} x^2 - \frac{1}{8} x^4 - \frac{1}{16} x^6 - \frac{5}{128} x^8 - \frac{35}{1280} x^{10} + \ldots)$$

$$+ a_1x + 0.$$

Thus, the general solution of $(1 - x^2)y'' - xy' + y = 0$ $(|x|<1)$ is given by

$$y = a_o(1 - \frac{1}{2} x^2 - \frac{1}{8} x^4 - \frac{1}{16} x^4 - \ldots) + a_1x.$$

In still other words, two linearly independent solutions of the equation are

$$y = u_1(x) = x \qquad (1)$$

and

$$y = u_2(x) = 1 - \frac{1}{2} x^2 - \frac{1}{8} x^4 - \frac{1}{16} x^6 - \frac{5}{128} x^8 - \frac{35}{1280} x^{10} - \ldots \qquad (2)$$

Note:
If we compare this with Exercise 2.7.6, we notice that equation $y = u_1(x) = x$ is the solution $y = x$ which we assumed was found by inspection in Exercise 2.7.6. On the other hand, the second equation should correspond to $y = \sqrt{1 - x^2}$ which was the other solution found in Exercise 2.7.6.

2.8.5(L) continued

As a partial check, notice that for $|x| < 1$

$$\left(1 - x^2\right)^n = 1 - nx^2 + \frac{n(n-1)}{2!}x^4 - \frac{n(n-1)(n-2)}{3!}x^6 + \ldots \quad (5)$$

[i.e., the binomial expansion is valid for $|x| < 1$ even when n is not a positive integer].

Letting $n = 1/2$ in (5) yields

$$\left(1 - x^2\right)^{\frac{1}{2}} = 1 - \frac{1}{2}x^2 + \frac{\frac{1}{2}\left(1 - \frac{1}{2}\right)}{2}x^4 - \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)}{3!}x^6 + \ldots$$

$$= 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 + \ldots$$

which appears to agree with (2).

Again, the key point is that it's "frosting on the cake" that allows us to simplify (2) as $y = \sqrt{1 - x^2}$. The crucial point is that

$$u_2(x) = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8 - \frac{35}{1280}x^{10} + \ldots$$

is a well-defined function of x for $|x| < 1$ and for a given x in this interval, we can compute $u_2(x)$ as accurately as we wish simply by considering enough terms.

2.8.6(L)

This looks a lot like the previous exercise since all we have done is changed the coefficient of y" from $1 - x^2$ to $1 + x^2$ (and this makes the coefficients everywhere analytic when the equation is written in standard form).

We may mimic (indeed, <u>copy</u>) the steps of the previous exercise, remembering only to replace the term $-x^2y"$ by $x^2y"$ to obtain

$$(2a_2 + a_o) + 6a_3x + \sum_{n=2}^{\infty} \{(n+2)(n+1)a_{n+2} + [n(n-1) - n + 1]a_n\} x^n = 0. \quad (2)$$

2.8.6(L) continued

Equating coefficients in (2), we conclude that

1. $2a_2 + a_o = 0$

or

$$a_2 = -\frac{a_o}{2}, \tag{3}$$

2. $6a_3x = 0$

or

$$a_3 = 0, \tag{4}$$

and

3. For $n \geq 2$

$$a_{n+2} = -\frac{(n-1)^2 a_n}{(n+2)(n+1)}. \tag{5}$$

Equations (4) and (5) together tell us that

$a_3 = a_5 = a_7 = \ldots = a_{2n+1} = \ldots = 0.$

We may then pick a_o and a_1 as arbitrary constants; whereupon we view

$$\sum_{n=0}^{\infty} a_n x^n$$

as $(a_o + a_2x^2 + a_4x^4 + a_6x^6 + \ldots) + (a_1x + a_3x^3 + a_5x^5 + a_7x^7 + \ldots)$,

and since $a_3 = a_5 = a_7 = \ldots + 0$, the second term becomes simply a_1x so that our solution has the form

$y = a_1x + (a_o + a_2x^2 + a_4x^4 + a_6x^6 + \ldots)$.

We then use (3) and (5) to obtain a_2, a_x, $a_6, \ldots, a_{2n} \ldots$ in terms of a_o.

2.8.6(L) continued

Thus, using (5) with n = 2 yields

$$a_4 = -\frac{1}{(4)(3)} a_2,$$

so by (3)

$$a_4 = \frac{a_o}{4!} . \qquad (6)$$

Then letting n = 4 in (5), we obtain

$$a_6 = - \frac{9\ a_4}{(6)(5)}$$

so that by (6),

$$a_6 = - \frac{9a_o}{6!} . \qquad (7)$$

Next we let n = 6 in (5) to obtain

$$a_8 = -\frac{25\ a_6}{(8)(7)}$$

so that by (7),

$$a_8 = \frac{225\ a_o}{8!} .$$

Hence, our general solution is now given by

$$y = a_o(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{9x^6}{6!} + \frac{225}{8!} x^8 + \ldots) + a_1x.$$

2.8.7(L)

Up to now, most of our exercises were of the type where the series solution could be easily identified with a "well-known" function of x (the previous exercise gave a hint of what's to come). In this exercise, we try to give a better illustration of how the solution is well-defined even when we cannot translate it into a familiar function. Part (a) of this exercise shows how the series solution is still obtained in the same way as usual. Part (b) illustrates how we can still find a particular solution once we specify y and y' at a given

2.8.7(L) continued

value of x (in this case x = 0). Part (c) illustrates how we can compute this particular solution y = f(x) for a given value of x. In other words, f(x) is well-defined as an infinite series to the extent that we can approximate, say, f(1) to as great a degree of accuracy as we wish simply by taking sufficiently many terms of the power series.

a. Letting

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{1}$$

in

$$y'' - xy = 0 \tag{2}$$

we obtain

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0.$$

Therefore,

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=3}^{\infty} a_{n-3} x^{n-2} = 0;$$

or,

$$2a_2 + \sum_{n=3}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=3}^{\infty} a_{n-3} x^{n-2} = 0;$$

or

$$2a_2 + \sum_{n=3}^{\infty} [n(n-1)a_n - a_{n-3}]x^{n-2} = \sum_{n=0}^{\infty} 0x^n. \tag{3}$$

Equating like coefficients in (3) we conclude that

$$a_2 = 0 \tag{4}$$

and for $n \geq 3$,

$$a_n = \frac{a_{n-3}}{n(n-1)}. \tag{5}$$

2.8.7(L) continued

Letting n = 3, (5) becomes

$$a_3 = \frac{a_o}{(3)(2)} = \frac{a_o}{6}. \qquad (6)$$

Letting n = 4, (5) becomes

$$a_4 = \frac{a_1}{(4)(3)} = \frac{a_1}{12}. \qquad (7)$$

Letting n = 5, (5) becomes

$$a_5 = \frac{a_2}{(5)(4)},$$

or by (4),

$$a_5 = 0.$$

In fact, since (5) tells us that each a_n is a multiple of the one that comes three before it, we may conclude that $0 = a_2 = a_5 = a_8 = a_{3n+2}$*$= \ldots$

We also conclude that the other coefficients may all be expressed in terms of a_o and a_1, where a_o and a_1 may be selected arbitrarily (since neither (4) nor (5) imposes any restrictions on a_o or a_1).

For example, letting n = 6 in (5), we obtain

$$a_6 = \frac{a_3}{(6)(5)}$$

or, by (6),

*The subscript 3n + 2 is simply a compact way of expressing all whole numbers which leave a remainder of 2 when divided by 3. This is precisely the set {2,5,8,11,14,...} .

2.8.7(L) continued

$$a_6 = \frac{a_o\,^*}{(6)(5)(3)(2)} = \frac{a_o}{180} . \qquad (8)$$

Similarly, letting n = 7 in (5), we obtain

$$a_7 = \frac{a_4}{(7)(6)} ,$$

or by (7),

$$a_7 = \frac{a_1}{(7)(6)(4)(3)} = \frac{a_1}{504} . \qquad (9)$$

Continuing in this way, we obtain

$$a_9 = \frac{a_6}{(9)(8)}$$

or,

$$a_9 = \frac{a_o}{(9)(8)(6)(5)(3)(2)} = \frac{a_o}{12,960} ;$$

and

$$a_{10} = \frac{a_7}{(10)(9)}$$

$$= \frac{a_1}{(10)(9)(7)(6)(4)(3)}$$

$$= \frac{a_1}{45,360} .$$

What is interesting in this example is that our coefficients do not suggest a "well-known" series. Yet, this is irrelevant. The key point is that the resulting series represents the general solution and that we can compute as many coefficients as we wish to obtain any desired accuracy (as we shall indicate in part (c) of this exercise).

At any rate, we now rewrite

$$\sum_{n=0}^{\infty} a_n x^n$$

*We write (6)(5)(3)(2) rather than 180 in order to emphasize the structure of the coefficient in terms of n.

2.8.7(L) continued

by rearranging the terms so that all terms whose exponent belongs to the family {0,3,6,9,...,3n,...} are grouped together; all terms whose exponent belongs to {1,4,7,10,...3n + 1,...} are grouped together; and all terms whose exponent belongs to {2,5,8,...,3n + 2,... } are grouped together.

This leads to

$$y = (a_o + a_3x^3 + a_6x^6 + a_9x^6 + \ldots)$$
$$+ (a_1x + a_4x^4 + a_7x^7 + a_{10}x^{10} + \ldots)$$
$$+ (a_2x^2 + a_5x^5 + a_8x^8 + \ldots);$$

or

$$\left.\begin{aligned} y &= a_o\left(1 + \frac{x^3}{6} + \frac{x^6}{180} + \frac{x^9}{12{,}960} + \ldots\right) \\ &+ a_1\left(x + \frac{x^4}{12} + \frac{x^7}{504} + \frac{x^{10}}{45{,}360} + \ldots\right) \\ &+ 0\,[x^2 + x^5 + x^8 + \ldots] \end{aligned}\right\} \quad (10)$$

In summary, the general solution of (2) is

$$y = c_1u_1(x) + c_2u_2(x)$$

where $u_1(x) = \left(1 + \frac{x^3}{6} + \frac{x^6}{180} + \frac{x^9}{12{,}960} + \ldots\right)$

and

$$u_2(x) = \left(x + \frac{x^4}{12} + \frac{x^7}{504} + \frac{x^{10}}{45{,}360} + \ldots\right)$$

Optional Note:
The advantage of writing our coefficients in the form, say,

$$a_9 = \frac{a_o}{(9)(8)(6)(5)(3)(2)}$$

rather than

2.8.7(L) continued

$$a_9 = \frac{a_o}{12,960}$$

is that it is easier to express a_n as a function of n. For example, we notice that (9)(8)(6)(5)(3)(2) would have been 9! had the factors 1, 4, and 7 been included. Thus, we have that

$$a_9 = \frac{(1)(4)(7)a_o}{9!} .$$

It is then an easy step to "guess" and then verify by induction that

$$a_{3n} = \frac{(1)(4)....(3n-2)a_o}{(3n)!} . \qquad (11)$$

In a similar way, one notice that

$$a_{10} = \frac{a_1}{(10)(9)(7)(6)(4)(3)}$$

$$= \frac{(2)(5)(8)a_1}{10!}$$

and then conjectures that

$$a_{3n+1} = \frac{(2)(5)...(3n-1)a_1}{(3n+1)!} . \qquad (12)$$

We may then write our general solution in the form

$$y = \sum_{n=0}^{\infty} a_{3n}x^{3n} + \sum_{n=0}^{\infty} a_{3n+1}\, x^{3n+1} \; [+ \sum_{n=0}^{\infty} \overset{=0}{a_{3n+2}}x^{3n+2}]$$

or

$$y = a_o(1 + \sum_{n=1}^{\infty} \frac{(1)(4)...(3n - 2)}{(3n)!}\, x^{3n})$$

$$+ a_1(x + \sum_{n=1}^{\infty} \frac{(2)(5)...(3n - 1)}{(3n + 1)!}\, x^{3n+1}). \qquad (13)$$

b. When x = 0, we see from (10) that $y[= f(x)] = a_o$ (since all other terms have a factor of x). Hence, the fact that f(0) = 0

2.8.7(L) continued

(i.e., $y = 0$ when $x = 0$) means $a_0 = 0$. Therefore,

$$f(x) = a_1\left(x + \frac{x^4}{12} + \frac{x^7}{504} + \frac{x^{10}}{45{,}360} + \ldots\right). \tag{14}$$

Consequently,

$$f'(x) = a_1\left(1 + \frac{x^3}{3} + \frac{x^6}{72} + \frac{x^9}{4536} + \ldots\right) \tag{15}$$

so that

$$f'(0) = a_1.$$

Since we are told that $f'(0) = 1$, we have for (15) that $a_1 = 1$ and thus from (14) we, therefore, conclude

$$f(x) = x + \frac{x^4}{12} + \frac{x^7}{504} + \frac{x^{10}}{45{,}360} + \ldots \tag{16}$$

Note #1:

Again, notice that in obtaining (15) from (14) we used the fact that

$$\sum_{n=0}^{\infty} a_n x^n$$

converged uniformly to $f(x)$.

Optional Note #2:

If we wanted to use (13), we have that

$$y' = a_o\left(\sum_{n=1}^{\infty} \frac{(1)(4)\ldots(3n-2)x^{3n-1}}{(3n-1)!}\right)$$

$$+ a_1\left(1 + \sum_{n=1}^{\infty} \frac{(2)(3)\ldots(3n-1)x^{3n}}{(3n)!}\right) \tag{17}$$

where we have again differentiated term by term.

The fact that $a_o = 0$ and $a_1 = 1$ reduces (17) to

$$f'(x) = 1 + \sum_{n=1}^{\infty} \frac{(2)(5)\ldots(3n-1)x^{3n}}{(3n)!}.$$

2.8.7(L) continued

c. Letting x = 1 in (16) we obtain

$$f(1) = 1 + \frac{1}{12} + \frac{1}{504} + \frac{1}{45,360} + \cdots$$

$$= 1.0000 + 0.0833 + 0.0019 + 0.0000^{+}.$$

Hence, to the nearest hundredth

$f(1) = 1.09.$

2.8.8(L)

Letting

$$y = \sum_{n=0}^{\infty} a_n x^n ,$$

we obtain as usual

$$y' = \sum_{n=1}^{\infty} na_n x^{n-1}$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Hence,

$$x^3y'' + xy' - y = 0 \quad (1)$$

implies that

$$x^3 \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x \sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0,$$

or

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n+1} + \sum_{n=1}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0. \quad (2)$$

Rewriting

2.8.8(L) continued

$$\sum_{n=3}^{\infty} (n - 1)(n - 2)a_{n-1}\, x^n + (a_1x + 2a_2x^2 + \sum_{n=3}^{\infty} na_nx^n)$$

$$- (a_o + a_1x + a_2x^2 + \sum_{n=3}^{\infty} a_nx^n) = 0,$$

$$- a_o + a_2x^2 + \sum_{n=3}^{\infty} [(n - 1)(n - 2)a_{n-1} + (n + 1)a_n]x^n = 0 = \sum_{n=0}^{\infty} 0x^n.$$

Comparing coefficients of like terms we see that

$- a_o = 0$

$a_2 = 0$

and for $n \geq 3$

$(n - 1)(n - 2)a_{n - 1} + (n + 1)a_n = 0$

or, $n \geq 3 \rightarrow a_n = -(n - 1)(n - 2)/n + 1 \quad a_{n-1}$. (3)

Letting n = 3 in (3) yields

$a_3 = - \frac{(2)(1)}{4}(0) = 0$

and, in fact, since each a_n is a multiple of the preivous one, we see that $n \geq 3 \rightarrow a_n = 0$.

Hence, each $a_n = 0$ except a_1 (since no condition is imposed on a_1).

In other words, if

$$y = \sum_{n=0}^{\infty} a_nx^n$$

is a solution of (1), then a_1 is arbitrary and $a_n = 0$ if $n \neq 1$. Therefore,

$y = a_1x$ (4)

2.8.8(L) continued

is a solution of (1), but not the general solution. To be sure, we can find the general solution from (4) by using the method of variation of parameters (as we did in the previous unit) but in terms of the theory of this unit, the key point is that when it is written in standard form, (1) becomes

$$y'' + \frac{1}{x^2} y' - \frac{1}{x^3} y = 0$$

and our coefficients are not analytic at x = 0. Consequently, there is no guarantee that the general solution of (1) can be put in the form

$$y = \sum_{n=0}^{\infty} a_n x^n .$$

Indeed, we have just shown that only the solution y = cx has this form.

As a final check on this example, recall that in the previous unit we solved this same problem and found that y = x and y = $e^{1/x}$ were a pair of linearly independent solutions. Notice that $e^{1/x}$ "blows up" at x = 0, and consequently is not analytic at x = 0.

We conclude our exercises on this note to indicate why the concept of series solutions must be developed beyond a discussion of ordinary power series. Namely, the usual power series approach requires that our coefficients be analytic everywhere in a particular region and this is not always the case. A further discussion of these other cases is beyond the intent of our course, but it is important for you to see why such additional refinements are necessary.