Unit 2: The Dimension of a Vector Space

3.2.1(L)

a. Since $S(\alpha_1) = \{c\alpha_1 : c \varepsilon R\}$ and since $\alpha_1 = (1,2,3,4)$, we see that $\beta \varepsilon S(\alpha_1)$ if and only if there exists a scalar c such that

$$\beta = c\alpha_1 = (c,2c,3c,4c). \tag{1}$$

Hence, by (1),

$$\alpha_2 = (2,5,7,7) \notin S(\alpha_1).$$

b. $S(\alpha_1,\alpha_2) = \{c_1\alpha_1 + c_2\alpha_2 : c_1,c_2 \varepsilon R\}$.

Hence,

$$\beta \varepsilon S(\alpha_1,\alpha_2)$$

if and only if there exist real numbers c_1 and c_2 such that

$$\begin{aligned}\beta &= c_1\alpha_1 + c_2\alpha_2 \\ &= c_1(1,2,3,4) + c_2(2,5,7,7) \\ &= (c_1,2c_1,3c_1,4c_1) + (2c_2,5c_2,7c_2,7c_2) \\ &= (c_1 + 2c_2,\ 2c_1 + 5c_2,\ 3c_1 + 7c_2,\ 4c_1 + 7c_2). \qquad (2)\end{aligned}$$

The main problem with (2) is that it may not seem apparent how, for example, the last two components depend on the first two. That is, we know that once two of the four components of (2) are given, c_1 and c_2 are determined, whereupon the other two components are uniquely determined.

One thing that we might do is replace the first two components of (2) by the single symbols, say, x_1 and x_2. That is, we could make the substitutions

3.2.1(L) continued

$$\left.\begin{aligned} c_1 + 2c_2 &= x_1 \\ &\text{and} \\ 2c_1 + 5c_2 &= x_2 \end{aligned}\right\} \tag{3}$$

from which it follows that

$$\left.\begin{aligned} c_1 &= 5x_1 - 2x_2 \\ &\text{and} \\ c_2 &= -2x_1 + x_2 \end{aligned}\right\} \tag{4}$$

In terms of row-reduced matrices, we obtain (4) from (3) by

$$\begin{array}{c} \begin{array}{cccc} \underline{c_1} & \underline{c_2} & \underline{x_1} & \underline{x_2} \end{array} \\ \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right] \end{array} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 & -2 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

From (4), it is readily seen that

$$\begin{aligned} 3c_1 + 7c_2 = {} & 15x_1 - 6x_2 \\ & -14x_1 + 7x_2 \\ = {} & x_1 + x_2 \end{aligned}$$

while

$$\begin{aligned} 4c_1 + 7c_2 = {} & 20x_1 - 8x_2 \\ & -14x_1 + 7x_2 \\ = {} & 6x_1 - x_2. \end{aligned}$$

3.2.1(L) continued

Thus, in terms of x_1 and x_2, (2) becomes

$$(x_1,x_2,x_1 + x_2,6x_1 - x_2). \tag{5}$$

[In terms of c_1 and c_2, (5) says that $3c_1 + 7c_2 = (c_1 + 2c_2) + (2c_1 + 5c_2)$ and $4c_1 + 7c_2 = 6(c_1 + 2c_2) - (2c_1 + 5c_2)$, so that the last two components of (2) are then expressed in terms of the first two components.]

The advantage of (5) over (2) is that we can now tell by inspection whether $(x_1,x_2,x_3,x_4) \in S(\alpha_1,\alpha_2)$. Namely

$$(x_1,x_2,x_3,x_4) \in S(\alpha_1,\alpha_2) \leftrightarrow$$

$$\left.\begin{array}{l} x_3 = x_1 + x_2 \\ \\ \text{and} \\ \\ x_4 = 6x_1 - x_2 \end{array}\right\} \tag{6}$$

c. Letting $\alpha_3 = (3,7,8,9)$, we see from (6) that $\alpha_3 \notin S(\alpha_1,\alpha_2)$. Namely, in this case, $x_1 = 3$, $x_2 = 7$, and $x_3 = 8$; hence, $x_3 \neq x_1 + x_2$.

d. Knowing that $\beta = (3,7,y,z)$, we see from (6) that

$$\beta \in S(\alpha_1,\alpha_2) \leftrightarrow y = 3 + 7 \text{ and } z = 6(3) - 7.$$

Hence, for β to belong to $S(\alpha_1,\alpha_2)$, it must be that

$$\beta = (3,7,10,11).$$

We shall revisit this exercise as a note to Exercise 8.2.2.

3.2.2(L)

The main aim of this exercise is to illustrate some "tricks of the trade" in finding the space spanned by a set of vectors. We have chosen some special cases, each of which shall be expanded within

3.2.2(L) continued

the context of our solution, to show how we can replace vectors in the given set by other vectors which span the same space. This new technique will supply us with a "neater" way of obtaining the same results as those of the type obtained in the previous exercise.

a. $W = S(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \left\{\sum_{i=1}^{4} c_i\alpha_i : \; c_i \in R\right\}.$

Thus, $\beta \in W$ means there exist real numbers c_1, c_2, c_3, and c_4 such that

$$\beta = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4. \tag{1}$$

But since vector addition is associative and commutative, we have from (1) that

$$\begin{aligned}\beta &= c_1\alpha_1 + (c_2\alpha_2 + c_3\alpha_3) + c_4\alpha_4\\ &= c_1\alpha_1 + (c_3\alpha_3 + c_2\alpha_2) + c_4\alpha_4\\ &= (c_1\alpha_1 + c_3\alpha_3) + (c_2\alpha_2 + c_4\alpha_4)\\ &= (c_1\alpha_1 + c_3\alpha_3) + (c_4\alpha_4 + c_2\alpha_2)\\ &= c_1\alpha_1 + c_3\alpha_3 + c_4\alpha_4 + c_2\alpha_2. \qquad (2)\end{aligned}$$

<u>Note</u>

(i) The significance of this part of the exercise is to show that the space spanned by any set of vectors, $\{\alpha_1, \ldots, \alpha_n\}$ does not depend on the order in which the vectors are listed. True, we have taken the special case n = 4 and the particular reordering given by (2), <u>but</u> in the same way we derived (2), we could have shown how other rearrangements could be made, and the case n = 4 was chosen only to get away from the usual geometric association of ideas. Any other value of n would work just as well.

3.2.2(L) continued

(ii) In terms of linear structure, this exercise is analogous to the statement that the solution set of a system of algebraic equations does not depend on the order in which the equations are written.

b. Let

$$W_1 = S(\alpha_1,\alpha_2,\alpha_3) = \{c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 : c_1,c_2,c_3 \in R\} \tag{3}$$

and

$$W_2 = S(3\alpha_1,\alpha_2,\alpha_3) = \{k_1 3\alpha_1 + k_2\alpha_2 + k_3\alpha_3 : k_1,k_2,k_3 \in R\}. \tag{4}$$

Then, if $\beta \in W_1$,

$$\beta = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3$$

$$= \frac{c_1}{3}(3\alpha_1) + c_2\alpha_2 + c_3\alpha_3, \tag{5}$$

or letting $k_1 = \frac{c_1}{3}$, $k_2 = c_2$, and $k_3 = c_3$,

$$\beta = k_1(3\alpha_1) + k_2\alpha_2 + k_3\alpha_3$$

$$\in S(3\alpha_1,\alpha_2,\alpha_3) = W_2.$$

That is, $\beta \in W_1 \rightarrow \beta \in W_2$. Hence,

$$W_1 \subset W_2. \tag{6}$$

Similarly,

$\gamma \in W_2 \rightarrow$

$\gamma = 3k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3$,

so letting $c_1 = 3k_1$, $c_2 = k_2$, $c_3 = k_3$,

3.2.2(L) continued

$$\gamma = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 \in S(\alpha_1,\alpha_2,\alpha_3) = W_1.$$

Therefore,

$$W_2 \subset W_1. \tag{7}$$

Hence, (6) and (7), together, imply that $W_1 = W_2$.

Note

(i) Again what is important here is the fact that the space spanned by $\{\alpha_1, \ldots, \alpha_n\}$ is unchanged if one of the vectors α_i is replaced by a non-zero multiple $c_i\alpha_i$. In our example, we took $i = 1$, but by part (a) this was no loss of generality. Namely, whatever α_i was replaced by $c_i\alpha_i$, we could have rearranged $\{\alpha_1, \ldots, \alpha_n\}$ so that α_i appeared first in the listing. Moreover, we picked $c_i = 3$. All that was important was that $c_i \neq 0$. For if $c_i = 0$, we see from (5) that we would have to divide by 0, which is not permitted.

(ii) The analogy here to systems of equations is that if we replace one equation by a constant multiple of that equation, we do not change the solution set of the system.

c. $W_1 = S(\alpha_1,\alpha_2,\alpha_3)$

$$W_2 = S(\alpha_1 + \alpha_2, \alpha_2, \alpha_3).$$

Hence,

$$\beta \in W_1 \rightarrow$$

$$\begin{aligned}\beta &= c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3\\ &= c_1\alpha_1 + [c_1\alpha_2 + (c_2 - c_1)\alpha_2] + c_3\alpha_3\\ &= c_1(\alpha_1 + \alpha_2) + (c_2 - c_1)\alpha_2 + c_3\alpha_3 \in S(\alpha_1 + \alpha_2,\alpha_2,\alpha_3).\end{aligned}$$

Therefore,

$$W_1 \subset W_2. \tag{8}$$

3.2.2(L) continued

Similarly,

$\gamma \in W_2 \rightarrow$

$$\gamma = c_1(\alpha_1 + \alpha_2) + c_2\alpha_2 + c_3\alpha_3$$

$$= c_1\alpha_1 + c_1\alpha_2 + c_2\alpha_2 + c_3\alpha_3$$

$$= c_1\alpha_1 + (c_1 + c_2)\alpha_2 + c_3\alpha_3 \in S(\alpha_1,\alpha_2,\alpha_3).$$

Therefore,

$$W_2 \subset W_1. \tag{9}$$

So comparing (8) and (9), we have that $W_2 = W_1$.

Note

(i) What we have illustrated here is that if we are studying the space spanned by $\{\alpha_1, \ldots, \alpha_n\}$, we may replace any α_i by itself plus any multiple of another, say, α_i by $\alpha_i + k_j\alpha_j$ (where in our example $i = 1$, $j = 2$ and for computational simplicity, $k_j = 1$), without changing the space spanned by the vectors.

(ii) Since the three properties developed in parts (a), (b) and (c) are precisely the properties that one needs in order to use row-reduced matrix techniques, it should be clear that this matrix technique may be used to find the space spanned by a given set of vectors. To illustrate this idea, we shall revisit Exercise 3.2.1 in the form of a note.

NOTE ON MATRIX CODING SYSTEM

Suppose we want to determine the space spanned by $\alpha_1 = (1,2,3,4)$ and $\alpha_2 = (2,5,7,7)$. We may use a 2 by 4 matrix in which the first row represents the components of α_1 and the second row, the components of α_2. We then have

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 7 & 7 \end{bmatrix}. \tag{1}$$

3.2.2(L) continued

Now we have seen in this exercise that the space spanned by α_1 and α_2 is not altered if we replace α_1 by a non-zero multiple of α_1, say, $-2\alpha_1$ [which is precisely the multiple of the first row of (1) that must be added to the second row when we use the usual row-reduction technique].

Thus, the matrix

$$\begin{bmatrix} -2 & -4 & -6 & -8 \\ 2 & 5 & 7 & 7 \end{bmatrix}, \qquad (2)$$

which is row-equivalent to (1), tells us that the space spanned by α_1 and α_2 is the same as that spanned by $-2\alpha_1$ and α_2.

In fact, proceeding quite mechanically, we have

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 7 & 7 \end{bmatrix} \sim \qquad (1)$$

$$\begin{bmatrix} -2 & -4 & -6 & -8 \\ 2 & 5 & 7 & 7 \end{bmatrix} \sim \qquad (2)$$

$$\begin{bmatrix} -2 & -4 & -6 & -8 \\ 0 & 1 & 1 & -1 \end{bmatrix} \sim \qquad (3)$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & -1 \end{bmatrix}. \qquad (4)^*$$

In terms of our code, (1), (2), (3), and (4) say that

$$S(\alpha_1,\alpha_2) = S(-2\alpha_1,\alpha_2) = S(-2\alpha_1,\alpha_2 - 2\alpha_1) = S(\alpha_1,\alpha_2 - 2\alpha_1).$$

*We usually obtain (4) from (1) by the single step of replacing the second row of (1) by the second minus twice the first. We have included (2) and (3) to emphasize the validity our claim that (1) and (4) code the same space in terms of the basic properties described in this exercise.

3.2.2(L) continued

In still other words, if we let $\beta_2 = \alpha_2 - 2\alpha_1 = (0,1,1,-1)$, then $S(\alpha_1,\alpha_2) = S(\alpha_1,\beta_2)$.

If we now complete the row-reduction of (1), we see from (4) that

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 7 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 6 \\ 0 & 1 & 1 & -1 \end{bmatrix}. \qquad (5)$$

Using our code, (5) tells us that

$S(\alpha_1,\alpha_2) = S(\beta_1,\beta_2)$

where

$\beta_1 = (1,0,1,6)$

and

$\beta_2 = (0,1,1,-1)$.

Now, β_1 and β_2 have a very nice form which help us express $S(\alpha_1,\alpha_2)$ more conveniently. Namely,

$\gamma \in S(\alpha_1,\alpha_2) \rightarrow$

$\gamma \in S(\beta_1,\beta_2) \rightarrow$

$\gamma = x_1\beta_1 + x_2\beta_2 \rightarrow$

$\gamma = x_1(1,0,1,6) + x_2(0,1,1,-1) \rightarrow$

$\gamma = (x_1,0,x_1,6x_1) + (0,x_2,x_2,-x_2) \rightarrow$

$\gamma = (x_1,x_2,x_1 + x_2,6x_1 - x_2)$,

which checks with our result in Exercise 3.2.1.

3.2.2(L) continued

From another perspective, we are saying that

$$(x_1,x_2,x_3,x_4) \in S(\alpha_1,\alpha_2) \leftrightarrow$$

$$(x_1,x_2,x_3,x_4) = x_1\beta_1 + x_2\beta_2$$

so that $\{\beta_1,\beta_2\}$ seems to be a natural coordinate system (basis) for $S(\alpha_1,\alpha_2)$.

Again, with respect to Exercise 3.2.1, observe that

$$\begin{aligned} 3\beta_1 + 7\beta_2 &= 3(1,0,1,6) + 7(0,1,1,-1) \\ &= (3,0,3,18) + (0,7,7,-7) \\ &= (3,7,10,11) \end{aligned}$$

which checks with our result in Exercise 3.2.2(d).

Notice also that we may use the augmented matrix technique to convert from the α's to the β's; namely, we may write

$$\begin{array}{c} \qquad\qquad\qquad\qquad\quad \alpha_1 \quad \alpha_2 \\ \left[\begin{array}{cccc|cc} 1 & 2 & 3 & 4 & 1 & 0 \\ 2 & 5 & 7 & 7 & 0 & 1 \end{array}\right] \end{array} \sim$$

$$\left[\begin{array}{cccccc} 1 & 2 & 3 & 4 & 1 & 0 \\ 0 & 1 & 1 & -1 & -2 & 1 \end{array}\right] \sim$$

$$\begin{array}{c} \qquad\qquad\qquad\qquad\quad \alpha_1 \quad \alpha_2 \\ \left[\begin{array}{cccc|cc} 1 & 0 & 1 & 6 & 5 & -2 \\ 0 & 1 & 1 & -1 & -2 & 1 \end{array}\right] \end{array}$$

from which we see that

3.2.2(L) continued

$$\left.\begin{aligned}\beta_1 &= 5\alpha_1 - 2\alpha_2\\ \beta_2 &= -2\alpha_1 + \alpha_2\end{aligned}\right\}.$$

We shall continue to reinforce this idea in the remaining exercises.

3.2.3(L)

a. Using our matrix coding system, we have

$$\begin{bmatrix}1 & 2 & 3 & 4\\ 2 & 5 & 7 & 7\\ 3 & 7 & 8 & 9\end{bmatrix} \sim \begin{bmatrix}1 & 2 & 3 & 4\\ 0 & 1 & 1 & -1\\ 0 & 1 & -1 & -3\end{bmatrix}$$

$$\sim \begin{bmatrix}1 & 0 & 1 & 6\\ 0 & 1 & 1 & -1\\ 0 & 0 & -2 & -2\end{bmatrix}$$

$$\sim \begin{bmatrix}1 & 0 & 1 & 6\\ 0 & 1 & 1 & -1\\ 0 & 0 & -1 & -1\end{bmatrix}$$

$$\sim \begin{bmatrix}1 & 0 & 0 & 5\\ 0 & 1 & 0 & -2\\ 0 & 0 & 1 & 1\end{bmatrix} . \qquad (1)$$

From (1), we see that

$$S(\alpha_1,\alpha_2,\alpha_3) = S(\beta_1,\beta_2,\beta_3)$$

where

3.2.3(L) continued

$$\left.\begin{aligned} \beta_1 &= (1,0,0,5) \\ \beta_2 &= (0,1,0,-2) \\ \beta_3 &= (0,0,1,1) \end{aligned}\right\} \qquad (2)$$

Therefore, $\beta_1, \beta_2, \beta_3$ belong to $S(\alpha_1, \alpha_2, \alpha_3)$ and $(x_1, x_2, x_3, x_4) \in V$ belongs to

$S(\alpha_1, \alpha_2, \alpha_3) \leftrightarrow$

$$(x_1, x_2, x_3, x_4) = x_1\beta_1 + x_2\beta_2 + x_3\beta_3. \qquad (3)$$

b. From (2),

$$x_1\beta_1 + x_2\beta_2 + x_3\beta_3 = (x_1, 0, 0, 5x_1) + (0, x_2, 0, -2x_2) + (0, 0, x_3, x_3)$$

$$= (x_1, x_2, x_3, 5x_1 - 2x_2 + x_3). \qquad (4)$$

Combining the results of (3) and (4), we see that

$(x_1, x_2, x_3, x_4) \in S(\alpha_1, \alpha_2, \alpha_3)$

if and only if

$$x_4 = 5x_1 - 2x_2 + x_3. \qquad (5)$$

[As a check of (5), observe that (1,2,3,4), (2,5,7,7), and (3,7,8,9) each obey (5). Namely, with $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, we have from (5) that $x_4 = 5 - 4 + 3 = 4$; with $x_1 = 2$, $x_2 = 5$, $x_3 = 7$, we have $x_4 = 10 - 10 + 7 = 7$; and with $x_1 = 3$, $x_2 = 7$, $x_3 = 8$, we have $x_4 = 15 - 14 + 8 = 9$.]

At any rate, we have from (5) that

$(4,9,13,y) \in S(\alpha_1, \alpha_2, \alpha_3) \leftrightarrow$

$y = 5(4) - 2(9) + 13 = 15.$

3.2.3(L) continued

Hence, in particular,

$(4,9.13,14) \notin S(\alpha_1,\alpha_2,\alpha_3)$.

c. Here we emphasize the salient feature of the vectors obtained by our row-reduced matrix technique. Namely, the very form of β_1, β_2, and β_3 guarantees that $\{\beta_1,\beta_2,\beta_3\}$ is linearly independent. Namely, suppose

$$x_1\beta_1 + x_2\beta_2 + x_3\beta_3 = 0. \tag{6}$$

Then

$$x_1(1,0,0,5) + x_2(0,1,0,-2) + x_3(0,0,1,1) = 0.$$

Hence,

$$(x_1,0,0,5x_1) + (0,x_2,0,-2x_2) + (0,0,x_3,x_3) = 0 \ [= (0,0,0,0)].$$

Therefore,

$$(x_1,x_2,x_3,5x_1 - 2x_2 + x_3) = (0,0,0,0)$$

so that

$$x_1 = 0$$

$$x_2 = 0$$

$$x_3 = 0$$

[and $5x_1 - 2x_2 + x_3 = 0$].

Since $x_1 = x_2 = x_3 = 0$, we see from (6) that $\{\beta_1,\beta_2,\beta_3\}$ is linearly independent.

More generally, when we employ the row-reduced matrix idea to compute $S(u_1, \ldots, u_n)$, we look at our final reduced matrix and

3.2.3(L) continued

delete the rows which consist entirely of zeroes. The number of remaining rows will turn out to be the dimension of $S(u_1, \ldots, u_n)$. In particular, if we name the non-zero rows of the reduced matrix by $v_1, \ldots,$ and v_m (where $m \leqslant n$) then

$$S(u_1, \ldots, u_n) = S(v_1, \ldots, v_m)$$

and the set $\{v_1, \ldots, v_m\}$ is linearly independent.

3.2.4(L)

a. To find the space spanned by $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ where $\alpha_1 = (1,2,3,4)$, $\alpha_2 = (2,3,5,5)$, $\alpha_3 = (2,4,7,6)$ and $\alpha_4 = (-1,2,3,4)$, we row-reduce the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 5 \\ 2 & 4 & 7 & 6 \\ -1 & 2 & 3 & 4 \end{bmatrix}$$

Namely,

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 5 \\ 2 & 4 & 7 & 6 \\ -1 & 2 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 1 & -2 \\ 0 & 4 & 6 & 8 \end{bmatrix} \tag{1}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 2 & -4 \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3}$$

3.2.4(L) continued

From (3), we see that

$$S(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = S(\beta_1,\beta_2,\beta_3,\beta_4)$$

where

$$\left.\begin{aligned} \beta_1 &= (1,0,0,0) \\ \beta_2 &= (0,1,0,5) \\ \beta_3 &= (0,0,1,-2) \\ \beta_4 &= (0,0,0,0) \end{aligned}\right\} \qquad (4)$$

But the 0-vector is redundant in any spanning set. Namely, $c_1\alpha_1 + c_2\alpha_2 + c_3\vec{0} = c_1\alpha_1 + c_2\alpha_2$, etc.

Hence,

$$S(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = S(\beta_1,\beta_2,\beta_3)$$

where β_1, β_2, and β_3 are as in (4).

Consequently,

$$(x_1,x_2,x_3,x_4) \in S(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leftrightarrow$$

$$\begin{aligned} (x_1,x_2,x_3,x_4) &= x_1\beta_1 + x_2\beta_2 + x_3\beta_3 \\ &= (x_1,0,0,0) + (0,x_2,0,5x_2) + (0,0,x_3,-2x_3) \\ &= (x_1,x_2,x_3,5x_2 - 2x_3). \end{aligned} \qquad (5)$$

That is, for (x_1,x_2,x_3,x_4) to belong to $S(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ it is necessary and sufficient that $x_4 = 5x_2 - 2x_3$. Notice that $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ all have this property.

For example, with (1,2,3,4), $x_2 = 2$, $x_3 = 3$ whence $5x_2 - 2x_3 = 10 - 6 = 4 = x_4$.

3.2.4(L) continued

b. We now use the augmented matrix technique to see why $\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$ spanned less than 4-dimensions. We have

$$\begin{array}{c} \begin{array}{cccc|cccc} & & & & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{array} \\ \left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 2 & 3 & 5 & 5 & 0 & 1 & 0 & 0 \\ 2 & 4 & 7 & 6 & 0 & 0 & 1 & 0 \\ -1 & 2 & 3 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \sim \end{array}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -3 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & -2 & 0 & 1 & 0 \\ 0 & 4 & 6 & 8 & 1 & 0 & 0 & 1 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 1 & -2 & -3 & 2 & 0 & 0 \\ 0 & -1 & -1 & -3 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & -2 & 0 & 1 & 0 \\ 0 & 0 & 2 & -4 & -7 & 4 & 0 & 1 \end{bmatrix} \sim$$

$$\begin{array}{c} \begin{array}{cccc|cccc} & & & & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{array} \\ \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & -1 & 0 & -5 & -4 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -3 & 4 & -2 & 1 \end{array}\right] \end{array} \tag{6}$$

From (6), we may conclude at once that

$$\left.\begin{aligned} (1,0,0,0) &= -\alpha_1 + 2\alpha_2 - \alpha_3 \\ (0,1,0,5) &= 4\alpha_1 - \alpha_2 - \alpha_3 \\ (0,0,1,-2) &= -2\alpha_1 + \alpha_3 \end{aligned}\right\} \tag{7}$$

<u>and</u>

$$(0,0,0,0) = -3\alpha_1 + 4\alpha_2 - 2\alpha_3 + \alpha_4.$$

3.2.4(L) continued

That is,

$$\alpha_4 = 3\alpha_1 - 4\alpha_2 + 2\alpha_3. \tag{8}$$

In other words, we see from (7) how to express β_1, β_2, and β_3 as linear combinations of α_1, α_2, and α_3. From equation (8), we see that α_4 is "redundant" in the sense that it is a linear combination of α_1, α_2, and α_3.

c. Since $5(7) - 2(12) = 11$, $(4,7,12,11) \in S(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$. In fact

$(4,7,12,11) = 4\beta_1 + 7\beta_2 + 12\beta_3$,*

so by (7),

$$(4,7,12,11) = 4(-\alpha_1 + 2\alpha_2 - \alpha_3) + 7(4\alpha_1 - \alpha_2 - \alpha_3) + 12(-2\alpha_1 + \alpha_3)$$

$$= \alpha_2 + \alpha_3. \tag{9}$$

<u>Check</u>

$$\alpha_2 + \alpha_3 = (2,3,5,5) + (2,4,7,6)$$

$$= (4,7,12,11).$$

Now, from (8),

$$3\alpha_1 - 4\alpha_2 + 2\alpha_3 - \alpha_4 = 0. \tag{10}$$

Hence, for any real number c, (10) implies that

$$3c\alpha_1 - 4c\alpha_2 + 2c\alpha_3 - c\alpha_4 = 0. \tag{11}$$

Combining (9) and (11), we have

*Notice how we "pick off" this information simply by inspection.

3.2.4(L) continued

$$\begin{aligned}(4,7,12,11) &= \alpha_2 + \alpha_3 \\ &= \alpha_2 + \alpha_3 + 0 \\ &= \alpha_2 + \alpha_3 + (3c\alpha_1 - 4c\alpha_2 + 2c\alpha_3 - c\alpha_4) \\ &= 3c\alpha_1 + (1 - 4c)\alpha_2 + (1 + 2c)\alpha_3 - c\alpha_4. \qquad (12)\end{aligned}$$

Thus, from (12), we see that there are as many ways of expressing (4,7,12,11) as a linear combination of α_1, α_2, α_3, and α_4 as there are ways of choosing a value of c.

In other words, since $\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$ is a linearly dependent set, every member of $S(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ may be expressed as a linear combination of α_1, α_2, α_3, and α_4 in infinitely many ways.

Note

This exercise is a concrete illustration of a more general result; namely, any finite set of vectors contains a linearly independent subset which generates (spans) the same subspace. This linearly independent subset can always be obtained by our row-reduced matrix technique.

3.2.5(L)

a. Using our matrix code for α_1, α_2, α_3, α_4, and α_5, we have

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 7 & 8 \\ 1 & 3 & 2 \\ 1 & -2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & -4 & 4 \end{bmatrix} \qquad (1)$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix} \qquad (2)$$

3.2.5(L) continued

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3}$$

From (3), we see that $S(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)$ is the same space as that spanned by β_1 and β_2 where $\beta_1 = (1,0,5)$ and $\beta_2 = (0,1,-1)$. [The last three rows of (3) all represent the 0-vector (0,0,0) and this adds nothing to the space spanned by β_1 and β_2.]

We next observe that

$$\begin{aligned} S(\beta_1,\beta_2) &= \{x_1\beta_1 + x_2\beta_2 : x_1,x_2 \in R\} \\ &= \{x_1(1,0,5) + x_2(0,1,-1) : x_1,x_2 \in R\} \\ &= \{(x_1,0,5x_1) + (0,x_2,-x_2) : x_1,x_2 \in R\} \\ &= \{(x_1,x_2,5x_1 - x_2) : x_1,x_2 \in R\}. \end{aligned} \tag{4}$$

From (4), we see that each element $(x_1,x_2,x_3) \in S(\beta_1,\beta_2)$ can be expressed in one and only one way as a linear combination of β_1 and β_2. In particular,

$$(x_1,x_2,x_3) \in S(\beta_1,\beta_2) = S(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) \leftrightarrow$$

$$(x_1,x_2,x_3) = x_1\beta_1 + x_2\beta_2.$$

Hence,

$$\dim S(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) = \dim S(\beta_1,\beta_2) = 2.$$

b. Using the augmented matrix idea in which our last five columns denote α_1, α_2, α_3, α_4, and α_5, respectively, we have

3.2.5(L) continued

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 & 0 & 0 \\ 2 & 4 & 6 & 0 & 1 & 0 & 0 & 0 \\ 3 & 7 & 8 & 0 & 0 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 & 0 & 0 & 1 & 0 \\ 1 & -2 & 7 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -3 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & -4 & 4 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 5 & 7 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -13 & 0 & 4 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{c} \\ \sim \end{array} \begin{array}{c} \begin{array}{ccc|ccccc} & & & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \end{array} \\ \left[\begin{array}{ccc|ccccc} 1 & 0 & 5 & 7 & 0 & -2 & 0 & 0 \\ 0 & 1 & -1 & -3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -13 & 0 & 4 & 0 & 1 \end{array}\right] \end{array}$$

Hence,

$\beta_1 = (1,0,5) = 7\alpha_1 - 2\alpha_3$

$\beta_2 = (0,1,-1) = -3\alpha_1 + \alpha_3$

and

$$\left.\begin{array}{l} -2\alpha_1 + \alpha_2 = 0 \\ 2\alpha_1 - \alpha_3 + \alpha_4 = 0 \\ -13\alpha_1 + 4\alpha_3 + \alpha_5 = 0 \end{array}\right\} \text{ i.e. } \left\{\begin{array}{l} \alpha_2 = 2\alpha_1 \\ \alpha_4 = -2\alpha_1 + \alpha_3 \\ \alpha_5 = 13\alpha_1 - 4\alpha_3 \end{array}\right.$$

Note

Geometrically speaking, if we view each 3-tuple as a vector in $\vec{i}$, $\vec{j}$, and $\vec{k}$ components originating at the origin, then $\vec{i} + 2\vec{j} + 3\vec{k}$, $2\vec{i} + 4\vec{j} + 6\vec{k}$, $3\vec{i} + 7\vec{j} + 8\vec{k}$, $\vec{i} + 3\vec{j} + 2\vec{k}$, and $\vec{i} - 2\vec{j} + 7\vec{k}$ all lie

3.2.5(L) continued

in the same plane. This plane is determined by the pair of vectors $\vec{\alpha}_1 = \vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{\alpha}_3 = 3\vec{i} + 7\vec{j} + 8\vec{k}$. It is also determined by the pair of vectors $\vec{\beta}_1 = \vec{i} + 5\vec{k}$ and $\vec{\beta}_2 = \vec{j} - \vec{k}$.

In terms of $\vec{\beta}_1$- and $\vec{\beta}_2$-components, $x_1\vec{i} + x_2\vec{j} + x_3\vec{k}$ lies in the plane $\leftrightarrow$

$$x_1\vec{i} + x_2\vec{j} + x_3\vec{k} = x_1\vec{\beta}_1 + x_2\vec{\beta}_2,$$

so just as in equation (4), we see that the equation of the plane is

$$x_3 = 5x_1 - x_2$$

(or in more common notation, $z = 5x - y$).

3.2.6(L)

a. In a manner of speaking, this is the most important result in the first three units. What it says is that if we have a set of m linearly independent vectors in V, then no fewer than m vectors can span V. To make this a bit more concrete, we are saying, for example, that if v_1, v_2, v_3, and v_4 all belong to a vector space V, and if $\{v_1,v_2,v_3,v_4\}$ is linearly independent, then no set of three, or less, elements in V can span V. As a still more concrete illustration, since $\vec{i}$, $\vec{j}$, and $\vec{k}$ are linearly independent, no fewer than three vectors can span xyz-space.

Before we prove the theorem, let us explain why the result (assuming its true) is so important in the discussion of the dimension of a vector space. To begin with, let us notice that the constructive techniques used in the lecture seem to indicate that the answer might depend on how we choose our vectors. In other words, suppose we carry out the construction described in the lecture and find two sets of linearly independent vectors that span V. How do we know that these two sets have the same number of elements? And if they don't have the same number, then it is ambiguous to define <u>the</u> dimension of a vector space to be the number of elements in a linearly independent set which spans V.

3.2.6(L) continued

If the theorem is true, let us suppose that $\{u_1, \ldots, u_r\}$ is one set of linearly independent vectors which span V and that $\{v_1, \ldots, v_s\}$ is another such set. Then by the theorem, since $\{u_1, \ldots, u_r\}$ span V and $\{v_1, \ldots, v_s\}$ is linearly independent, $r \geqslant s$. Reversing the roles of the two sets we have that $\{v_1, \ldots, v_s\}$ span V and that $\{u_1, \ldots, u_r\}$ is linearly independent. Hence, $s \geqslant r$.

But since $r \geqslant s$ and $s \geqslant r$, it can only be that $r = s$. In particular, this proves that if $\{u_1, \ldots, u_r\}$ is one set of linearly independent vectors which span V, then any other set of linearly independent vectors which span V must also have r elements. It can also be shown in this case that any set of r linearly independent vectors span V. (Namely, we can express these r vectors as linear combinations of $u_1, \ldots, u_r$, etc.)

We then define a basis for V to be any set of linearly independent vectors which span V, whereupon we may then unambiguously define the dimension of V to be the number of elements in any basis of V, since all bases have the same number of elements.

In other words, when we say that the dimension of V is r (written $\dim V = r$), we mean that there exists a set of r linearly independent vectors which span V. If $\{u_1, \ldots, u_r\}$ is such a set, then we write

$$V = V[u_1, \ldots, u_r],$$

or more simply,

$$V = [u_1, \ldots, u_r].$$

Returning to the proof of the theorem, we let

$$S_1 = \{\beta_1, \alpha_1, \ldots, \alpha_n\}. \tag{1}$$

Certainly, S_1 spans V since $\{\alpha_1, \ldots, \alpha_n\}$ already spans V and S_1 is linearly dependent since β_1 is a linear combination of $\alpha_1, \ldots, \alpha_n$. Hence, <u>in the order given by (1)</u>, at least one of the members of S is a linear combination of the preceding ones. It can't be

3.2.6(L) continued

β_1, so it must be one of the α's. This α can be deleted from S_1 without changing the space spanned by S_1. Let us delete such an α and renumber the remaining α's to be $\alpha_1, \ldots, \alpha_{n-1}$. Thus,

$$\{\beta_1, \alpha_1, \ldots, \alpha_{n-1}\}$$

spans V also.

Now let

$$S_2 = \{\beta_1, \beta_2, \alpha_1, \ldots, \alpha_{n-1}\}. \tag{2}$$

Since $\{\beta_1, \alpha_1, \ldots, \alpha_{n-1}\}$ spans V, so also does S_2 since

$$\{\beta_1, \alpha_1, \ldots, \alpha_{n-1}\} \subset \{\beta_1, \beta_2, \alpha_1, \ldots, \alpha_{n-1}\}$$

and since β_2 is a linear combination of $\beta_1, \alpha_1, \ldots \alpha_{n-1}$, we have that S_2 is a linearly dependent set.

Hence, in the order given in (2), one of the elements of S_2 can be expressed as a linear combination of the preceding ones. But, since the β's are linearly independent, β_2 cannot be a scalar multiple of β_1; hence, it must again be one of the α's which is expendable. Deleting this α, let's again renumber the α's and conclude that

$$\{\beta_1, \beta_2, \alpha_1, \ldots, \alpha_{n-2}\}$$

also spans V. We now let

$$S_3 = \{\beta_1, \beta_2, \beta_3, \alpha_1, \ldots, \alpha_{n-2}\}$$

and apply the same argument as before. Since S_3 is linearly dependent, the β's are linearly independent it must be one of the α's which is redundant. Continuing inductively in this manner, each time we tack on a β, we must be able to delete an α. In particular, then, there are at least as many α's as there are β's.

3.2.6(L) continued

As a final note to this part of the exercise, let us observe that the procedure outlined here tells us how to augment any set of linearly independent vectors into a basis. Namely, we use the procedure outlined in our proof whereby we list one of our independent vectors followed by a set of spanning vectors and then using some computational technique such as row-reducing matrices, we find that one of the spanning vectors can be deleted. We then augment our set of vectors by listing the next member of the linearly independent set and finding another of the spanning vectors to delete. We continue in this way until the last member of the linearly independent set is added and we may then use the row-reduced technique to prune out the remaining redundancies. What we are then left with is a basis which includes the originally given set of linearly independent vectors.

This technique will be employed in more concrete form in the following units of this block, but for now, we close with the example given in (6).

b. The main aim here is to add concreteness to part (a) and at the same time, to show one way of augmenting a set of linearly independent vectors to form a basis. What we do is augment $\{\alpha_1,\alpha_2,\alpha_3\}$, one at a time by u_1,u_2,u_3, etc., until we have a basis for E^5. Thus, we begin by row-reducing

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 3 & 3 \\ 2 & 3 & 4 & 3 & 6 \end{bmatrix}$$

to make sure that $\{\alpha_1,\alpha_2,\alpha_3\}$ is linearly independent. This leads to

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 1 & 2 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 2 \end{bmatrix} \qquad (1)$$

so that $\{\alpha_1,\alpha_2,\alpha_3\}$ is linearly independent.

From (1), we see that $S(\alpha_1,\alpha_2,\alpha_3) = S(\beta_1,\beta_2,\beta_3)$ where $\beta_1 = (1,0,0,-1,-1)$, $\beta_2 = (0,1,0,3,0)$, $\beta_3 = (0,0,1,-1,2)$. We

3.2.6(L) continued

augment $S(\alpha_1,\alpha_2,\alpha_3)$ [$= S(\beta_1,\beta_2,\beta_3)$] by u_1 to obtain

$$\begin{bmatrix}1&0&0&-1&-1\\0&1&0&3&0\\0&0&1&-1&2\\1&0&0&0&0\end{bmatrix}\sim\begin{bmatrix}1&0&0&-1&-1\\0&1&0&3&0\\0&0&1&-1&2\\0&0&0&1&1\end{bmatrix}\sim\begin{bmatrix}1&0&0&0&0\\0&1&0&0&-3\\0&0&1&0&3\\0&0&0&1&1\end{bmatrix}\quad(2)$$

From (2), we see that $S(\alpha_1,\alpha_2,\alpha_3,u_1)$ is a 4-dimensional, so we next look at $S(\alpha_1,\alpha_2,\alpha_3,u_1,u_2)$ by row-reducing

$$\begin{bmatrix}1&0&0&0&0\\0&1&0&0&-3\\0&0&1&0&3\\0&0&0&1&1\\0&1&0&0&0\end{bmatrix}\sim\begin{bmatrix}1&0&0&0&0\\0&1&0&0&-3\\0&0&1&0&3\\0&0&0&1&1\\0&0&0&0&3\end{bmatrix}\sim\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}\quad(3)$$

We see from (3) that $E^5 = S(\alpha_1,\alpha_2,\alpha_3,u_1,u_2)$. Had u_2 been a linear combination of $\alpha_1,\alpha_2,\alpha_3$, and u_1, we would have deleted it and used row-reduction on $S(\alpha_1,\alpha_2,\alpha_3,u_1,u_3)$ etc., until we wound up with the five elements which span V.

3.2.7

a. $$\begin{bmatrix}1&3&-1&2\\2&0&1&3\\-1&1&0&0\end{bmatrix}\sim\begin{bmatrix}1&3&-1&2\\0&-6&3&-1\\0&4&-1&2\end{bmatrix}$$

$$\sim\begin{bmatrix}4&12&-4&8\\0&-12&6&-2\\0&12&-3&6\end{bmatrix}$$

$$\sim\begin{bmatrix}4&0&2&6\\0&-12&6&-2\\0&0&3&4\end{bmatrix}$$

3.2.7 continued

$$\sim\begin{bmatrix} 12 & 0 & 6 & 18 \\ 0 & -12 & 6 & -2 \\ 0 & 0 & -6 & -8 \end{bmatrix}$$

$$\sim\begin{bmatrix} 12 & 0 & 0 & 10 \\ 0 & -12 & 0 & -10 \\ 0 & 0 & -6 & -8 \end{bmatrix}$$

$$\sim\begin{bmatrix} 1 & 0 & 0 & \frac{5}{6} \\ 0 & 1 & 0 & \frac{5}{6} \\ 0 & 0 & 1 & \frac{4}{3} \end{bmatrix} \qquad (1)$$

From (1), we see that $S(\alpha_1,\alpha_2,\alpha_3) = S(\beta_1,\beta_2,\beta_3)$ where $\beta_1 = (1,0,0,\frac{5}{6})$, $\beta_2 = (0,1,0,\frac{5}{6})$, and $\beta_3 = (0,0,1,\frac{4}{3})$.

Moreover,

$$(x_1,x_2,x_3,x_4) \in S(\alpha_1,\alpha_2,\alpha_3) = S(\beta_1,\beta_2,\beta_3) \leftrightarrow$$

$$\begin{aligned}(x_1,x_2,x_3,x_4) &= x_1\beta_1 + x_2\beta_2 + x_3\beta_3 \\ &= \left(x_1,0,0,\frac{5x_1}{6}\right) + \left(0,x_2,0,\frac{5x_2}{6}\right) + \left(0,0,x_3,\frac{4x_3}{3}\right) \\ &= \left(x_1,x_2,x_3,\frac{5x_1}{6} + \frac{5x_2}{6} + \frac{4x_3}{3}\right) \\ &= \left(x_1,x_2,x_3,\frac{5x_1 + 5x_2 + 8x_3}{6}\right).\end{aligned}$$

Hence,

$$\begin{aligned}S(\alpha_1,\alpha_2,\alpha_3) &= \left\{(x_1,x_2,x_3,x_4):\ x_4 = \frac{5x_1 + 5x_2 + 8x_3}{6}\right\} \\ &= \left\{(x_1,x_2,x_3,x_4):\ 5x_1 + 5x_2 + 8x_3 - 6x_4 = 0\right\}.\end{aligned}$$

3.2.7 continued

b. The dimension of $W = S(\alpha_1,\alpha_2,\alpha_3) = 3$. Any three linearly independent members of W span W and conversely any set of three elements of W which span W are linearly independent. The "natural" basis for W is $\{\beta_1,\beta_2,\beta_3\}$ since $(x_1,x_2,x_3,x_4) \in W \leftrightarrow$

$$(x_1,x_2,x_3,x_4) = x_1\beta_1 + x_2\beta_2 + x_3\beta_3.$$

3.2.8

a.

$$\begin{array}{c} \begin{array}{ccccccc} & & & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{array} \\ \left[\begin{array}{ccc:cccc} 1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 2 & 5 & 4 & 0 & 1 & 0 & 0 \\ 3 & 8 & 9 & 0 & 0 & 1 & 0 \\ 4 & 9 & 9 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \sim \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 & 0 \\ 0 & 2 & 0 & -3 & 0 & 1 & 0 \\ 0 & 1 & -3 & -4 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 7 & 5 & -2 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -2 & -1 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -9 & -9 & 0 & 7 \\ 0 & 1 & 0 & 2 & 3 & 0 & -2 \\ 0 & 0 & 0 & -7 & -6 & 1 & 4 \\ 0 & 0 & 1 & 2 & 1 & 0 & -1 \end{bmatrix} \qquad (1)$$

From (1), we conclude that

$$\left.\begin{aligned} (1,0,0) &= -9\alpha_1 - 9\alpha_2 + 7\alpha_4 \\ (0,1,0) &= 2\alpha_1 + 3\alpha_2 - 2\alpha_4 \\ (0,0,1) &= 2\alpha_1 + \alpha_2 - \alpha_4 \end{aligned}\right\} \qquad (2)$$

and

b. $$(0,0,0) = -7\alpha_1 - 6\alpha_2 + \alpha_3 + 4\alpha_4. \qquad (3)$$

3.2.8 continued

Hence,

$$\alpha_3 = 7\alpha_1 + 6\alpha_2 - 4\alpha_4. \tag{4}$$

[Of course, if we so desire, we may use (3) to obtain

$$\alpha_4 = \frac{7}{4}\alpha_1 + \frac{3}{2}\alpha_2 - \frac{1}{4}\alpha_3. \tag{4'}$$

From (4') and (2), we could express (1,0,0), (0,1,0), and (0,0,1) as linear combinations of α_1, α_2, and α_3. Equation (4') is more conventional in the sense that for linear dependence, we like to express a vector as a linear combination of its predecessors.]

c. $(2,1,4) = 2(1,0,0) + 1(0,1,0) + 4(0,0,1)$

$$\begin{aligned} &= 2(-9\alpha_1 - 9\alpha_2 + 7\alpha_4) + (2\alpha_1 + 3\alpha_2 - 2\alpha_4) + \\ &\quad + 4(2\alpha_1 + \alpha_2 - \alpha_4) \\ &= -8\alpha_1 - 11\alpha_2 + 8\alpha_4. \end{aligned}$$

3.2.9(L)

The main aim of this exercise is to illustrate the existence of an infinite-dimensional vector space. First of all, since every polynomial is continuous and since the set of all continuous functions is a vector space (with respect to the usual meanings of the sum of two functions and the product of a scalar and a function), we know that the set of all polynomials is at least a sub<u>set</u> of the space of continuous functions. To prove that this subset is a subspace, we need only know that the sum of two members of this subset belongs to the subset as does any scalar multiple of a member of the subset.

Clearly, the sum of two polynomials is a polynomial and a scalar multiple of a polynomial is also a polynomial. Therefore, the set of all polynomials is itself a vector space. But we know from our treatment of elementary calculus that the powers of x are linearly

3.2.9(L) continued

independent. That is, x^n cannot be expressed as a linear combination of 1, x, ..., and x^{n-1}.

In other words, for any value of n no matter how large $\{1, x, \ldots, x^n\}$ is a linearly independent set. Yet, the space spanned by $\{1, x, \ldots, x^n\}$ can never yield the entire space P since $x^{n+1} \in P$ but $x^{n+1} \notin S(1, x, \ldots, x^n)$.

Hence, in the language of today's lecture with $\alpha_n = x^{n-1}$, we see that for each n

$$S(\alpha_1,\ldots,\alpha_{n+1}) \supsetneqq S(\alpha_1,\ldots,\alpha_n)$$

but for no n will $S(\alpha_1, \ldots, \alpha_n)$ yield all of our vector space. That is, the constructive device described at the end of the lecture never terminates and accordingly, we refer to the space of all polynomials in x as an infinite-dimensional vector space.

As a final note to this exercise, recall that since every analytic function may be represented as a power series, the space spanned by the finite set $\{1, x, x^2, \ldots, x^n, \ldots\}$ is the space of all analytic functions.