Unit 4: Volumes and Masses of More General Solids

5.4.1(L)

Let us first observe that if S were homogeneous (i.e., of constant density), we would not think in terms of triple integrals. Namely, we would find the volume of S simply by computing

$$\int_0^1 \int_0^1 (x + y + 1)\,dy\,dx \tag{1}$$

and we would then find the mass by multiplying this result by the density.

Now, while it would be quite artificial, we could certainly observe that x + y + 1 is equivalent to

$$\int_0^{x+y+1} dz. \tag{2}$$

The result of substituting (2) into (1) is that we obtain the volume of S as the triple integral

$$\int_0^1 \int_0^1 \left[\int_0^{x+y+1} dz\right]dy\,dx = \int_0^1 \int_0^1 \int_0^{x+y+1} dz\,dy\,dx. \tag{3}$$

This discussion is simply meant to reinforce the notion that one does not need triple integrals to compute volumes. Rather, the use of triple integrals enters the picture when we must limit our changes in x, y, and z to be small. In particular, in the present exercise, we want the mass of S, and since the density of the solid depends on x, y, and z, the approximation that the density may be viewed as being constant requires that S be viewed as being partitioned into small parallelepipeds.

The volume of one such parallelepiped, ΔS_{ijk}, is $\Delta x_i \Delta y_j \Delta z_k = \Delta V_{ijk}$, and the approximate density is $\rho(a_i, b_j, c_k) = a_i b_j c_k$ where (a_i, b_j, c_k) is any point in ΔS_{ijk}. Then the approximate mass of S would be obtained by summing the $\rho(a_i, b_j, c_k)\Delta V_{ijk}$'s.

5.4.1(L) continued

In fact, if we assume that $\underline{\rho}_{ijk}$ is the minimum density in ΔS_{ijk} and that $\overline{\rho}_{ijk}$ is the maximum density in ΔS_{ijk}, then the mass M of the solid S is bounded by

$$\sum_{k=1}^{p} \sum_{j=1}^{m} \sum_{i=1}^{n} \underline{\rho}_{ijk} \Delta V_{ijk} < M < \sum_{k=1}^{p} \sum_{j=1}^{m} \sum_{i=1}^{n} \overline{\rho}_{ijk} \Delta V_{ijk}.$$

If we let the size of the partitions approach 0 and if $\rho(x,y,z)$ is continuous, we see that

$$M = \lim_{\begin{subarray}{l}\Delta x_i \\ \Delta y_j \\ \Delta z_k\end{subarray} \to 0} \left\{ \sum_{k=1}^{p} \sum_{j=1}^{m} \sum_{i=1}^{n} \overline{\rho}_{ijk} \Delta V_{ijk} \right\}, \text{ etc.} \tag{4}$$

Notice that our discussion here is equivalent to our discussion in Unit 1, except that we are now involved with triple sums rather than double sums.

If the limit in (4) exists, we write it as

$$\iiint_{S} \rho(x,y,z) \, dV_S \tag{5}$$

and this, in turn, may be viewed as an iterated integral. In terms of the present exercise, (5) becomes

$$\int_0^1 \int_0^1 \int_0^{x+y+1} xyz \; dz \; dy \; dx. \tag{6}$$

[Notice that we obtain the limits of integration just as we did in double integration. Namely, for a fixed (x,y), z varies from the xy-plane $(z = 0)$ to the plane $z = x + y + 1$, etc.]

The key point is that while the triple integral in (3) is artificial, the triple integral in (6) is essential since the integrand is affected by a change in z.

5.4.1(L) continued

At any rate, the actual mechanics work exactly as they did with double integrals and we obtain

$$M = \int_0^1 \int_0^1 \int_0^{x+y+1} xyz \ dz \ dy \ dx$$

$$= \int_0^1 \int_0^1 \left[\int_0^{x+y+1} xyz \ dz \right] dy \ dx$$

$$= \int_0^1 \int_0^1 \left[\frac{1}{2} xyz^2 \ \Big|_{z=0}^{x+y+1} \right] dy \ dx$$

$$= \int_0^1 \int_0^1 \frac{1}{2} xy(x + y + 1)^2 dy \ dx$$

$$= \int_0^1 \int_0^1 \frac{1}{2} xy(x^2 + y^2 + 1 + 2xy + 2x + 2y) dy \ dx$$

$$= \int_0^1 \left[\int_0^1 \left(\frac{1}{2} x^3 y + \frac{1}{2} xy^3 + \frac{1}{2} xy + x^2 y^2 + x^2 y + xy^2 \right) dy \right] dx$$

$$= \int_0^1 \left[\frac{1}{4} x^3 y^2 + \frac{1}{8} xy^4 + \frac{1}{4} xy^2 + \frac{1}{3} x^2 y^3 + \frac{1}{2} x^2 y^2 + \frac{1}{3} xy^3 \ \Big|_{y=0}^{1} \right] dx$$

$$= \int_0^1 \left(\frac{1}{4} x^3 + \frac{1}{8} x + \frac{1}{4} x + \frac{1}{3} x^2 + \frac{1}{2} x^2 + \frac{1}{3} x \right) dx$$

$$= \int_0^1 \left(\frac{1}{4} x^3 + \frac{5}{6} x^2 + \frac{17}{24} x \right) dx$$

$$= \frac{1}{16} x^4 + \frac{5}{18} x^3 + \frac{17}{48} x^2 \ \Big|_{x=0}^{1}$$

$$= \frac{1}{16} + \frac{5}{18} + \frac{17}{48} = \frac{25}{36}.$$

5.4.2(L)

The key point here is that we use the axiom that the whole equals the sum of its parts. Specifically, we first compute

$$\iint\limits_{R} (x^2 + y^2 + 3)\, dA_R \tag{1}$$

which denotes the volume of the portion of the cylinder between the xy-plane and $z = x^2 + y^2 + 3$.

Then we compute

$$\iint\limits_{R} (x + y + 1)\, dA_R \tag{2}$$

which denotes the volume of the portion of the cylinder between the xy-plane and $z = x + y + 1$.

Consequently,

$$\iint\limits_{R} (x^2 + y^2 + 3)\, dA_R - \iint\limits_{R} (x + y + 1)\, dA_R \tag{3}$$

must represent the required volume since the required volume is of the solid consisting of the portion of the cylinder between the xy-plane and $z = x^2 + y^2 + 3$, <u>with the portion between the xy-plane and $z = x + y + 1$ deleted</u>.

At any rate, since R is given by
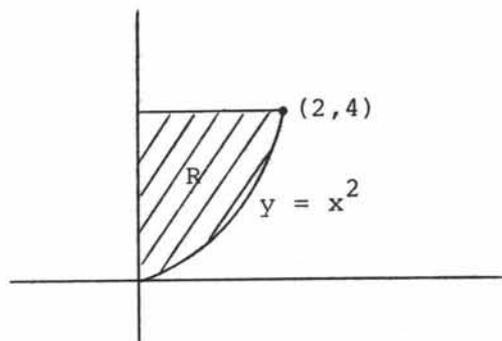
5.4.2(L) continued

we have,

$$\iint_R (x^2 + y^2 + 3)\,dA_R - \iint_R (x + y + 1)\,dA_R =$$

$$\iint_R [(x^2 + y^2 + 3) - (x + y + 1)]\,dA_R =$$

$$\int_0^4 \int_0^{\sqrt{y}} (x^2 + y^2 - x - y + 2)\,dx\,dy.* \tag{4}$$

Thus, the required volume is

$$\int_0^4 \left[\int_0^{\sqrt{y}} (x^2 + y^2 - x - y + 2)\,dx \right] dy =$$

$$\int_0^4 \left[\frac{1}{3}x^3 + xy^2 - \frac{1}{2}x^2 - xy + 2x \; \Big|_{x=0}^{\sqrt{y}} \right] dy =$$

$$\int_0^4 \left(\frac{1}{3}\,y^{\frac{3}{2}} + y^{\frac{5}{2}} - \frac{1}{2}y - y^{\frac{3}{2}} + 2y^{\frac{1}{2}} \right) dy =$$

$$\int_0^4 \left(y^{\frac{5}{2}} - \frac{2}{3}\,y^{\frac{3}{2}} - \frac{1}{2}y + 2y^{\frac{1}{2}} \right) dy$$

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*Recall, once again, that our limits of integration are determined
by R not by the integrand. We elected to write the integral in
the order dx dy rather than dy dx to simplify the limits of inte-
gration. That is, (4) gives us 0 as the lower limit on each inte-
gral. Had we used the order dy dx, (4) would have been replaced by

$$\int_0^2 \int_{x^2}^4 (x^2 + y^2 - x - y + 2)\,dy\,dx.$$

Obviously, the correct answer should be obtained from either
integral.

5.4.2(L) continued

Therefore,

$$\int_0^4 \left[\int_0^{\sqrt{y}} (x^2 + y^2 - x - y + 2)\,dx \right] dy =$$

$$\frac{2}{7} y^{\frac{7}{2}} - \frac{4}{15} y^{\frac{5}{2}} - \frac{1}{4} y^2 + \frac{4}{3} y^{\frac{3}{2}} \Big|_{y=0}^{4} =$$

$$\frac{2}{7}(2)^7 - \frac{4}{15}(2)^5 - \frac{1}{4}(16) + \frac{4}{3}(2)^3 =$$

$$\frac{256}{7} - \frac{128}{15} - 4 + \frac{32}{3} =$$

$$36\frac{4}{7} - 8\frac{8}{15} - 4 + 10\frac{2}{3} =$$

$$34\frac{74}{105}.$$

Notice again that there was no need to use triple integrals even
though (4) could have been written as

$$\int_0^4 \int_0^{\sqrt{y}} \left[\int_0^{x^2+y^2-x-y+2} dz \right] dx\, dy.$$

As a final note, we need only check that $z = x^2 + y^2 + 3$ always
lies above $z = x + y + 1$. Algebraically, this means that for each
(x,y), $(x^2 + y^2 + 3) > (x + y + 1)$.
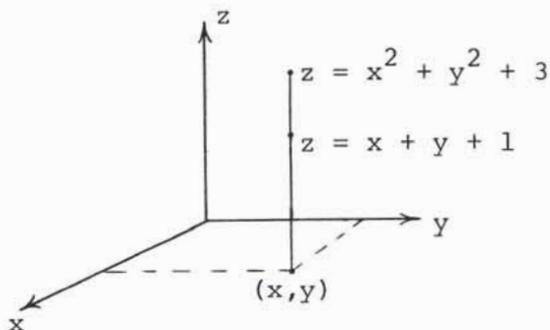
5.4.2(L) continued

Now

$$x^2 + y^2 + 3 > x + y + 1 \leftrightarrow$$

$$x^2 + y^2 - x - y + 2 > 0.$$

But

$$x^2 + y^2 - x - y + 2 =$$

$$(x^2 - x + \tfrac{1}{4}) + (y^2 - y + \tfrac{1}{4}) + \tfrac{3}{2} =$$

$$\underbrace{\left(x - \tfrac{1}{2}\right)^2}_{\geqslant 0} + \underbrace{\left(y - \tfrac{1}{2}\right)^2}_{\geqslant 0} + \tfrac{3}{2} \geqslant \tfrac{3}{2} > 0.$$

Hence, the surface $z = x^2 + y^2 + 3$ always lies above the surface $z = x + y + 1$.

5.4.3

For a fixed (x,y), z ranges from 0 to $x^2 + y^2 + 3$. Hence, the mass of S is given by

$$\int_0^4 \int_0^{\sqrt{y}} \int_0^{x^2+y^2+3} xyz \; dz \; dx \; dy =$$

$$\int_0^4 \int_0^{\sqrt{y}} \tfrac{1}{2} xyz^2 \Big|_{z=0}^{x^2+y^2+3} dx \; dy =$$

$$\int_0^4 \left[\int_0^{\sqrt{y}} \tfrac{1}{2}xy(x^2 + y^2 + 3)^2 dx \right] dy =$$

$$\int_0^4 \left[\int_0^{\sqrt{y}} \tfrac{1}{2}xy(x^4 + y^4 + 9 + 2x^2y^2 + 6x^2 + 6y^2) dx \right] dy =$$

5.4.3 continued

Therefore,

$$\int_0^4 \int_0^{\sqrt{y}} \int_0^{x^2+y^2+3} xyz \, dz \, dx \, dy =$$

$$\int_0^4 \left[\int_0^{\sqrt{y}} \left(\frac{1}{2}x^5 y + \frac{1}{2}xy^5 + \frac{9}{2}xy + x^3 y^3 + 3x^3 y + 3xy^2 \right) dx \right] dy =$$

$$\int_0^4 \left[\frac{1}{12}x^6 y + \frac{1}{4}x^2 y^5 + \frac{9}{4}x^2 y + \frac{1}{4}x^4 y^3 + \frac{3}{4}x^4 y + \frac{3}{2}x^2 y^2 \, \Big|_{x=0}^{\sqrt{y}} \right] dy$$

$$\int_0^4 \left(\frac{1}{12}y^4 + \frac{1}{4}y^6 + \frac{9}{4}y^2 + \frac{1}{4}y^5 + \frac{3}{4}y^3 + \frac{3}{2}y^3 \right) dy =$$

$$\frac{1}{60}y^5 + \frac{1}{28}y^7 + \frac{3}{4}y^3 + \frac{1}{24}y^6 + \frac{3}{16}y^4 + \frac{3}{8}y^4 \, \Big|_{y=0}^4 =$$

$$\frac{1}{60}(4)^5 + \frac{1}{28}(4)^7 + \frac{3}{4}(4)^3 + \frac{1}{24}(4)^6 + \frac{9}{16}(4)^4 =$$

$$\frac{4^3}{15} + \frac{4^6}{7} + 3(4)^2 + \frac{2}{3}(4)^4 + 9(4)^2 =$$

$$16\left(\frac{16}{15} + \frac{256}{7} + 3 + \frac{32}{3} + 9\right) = 16\left(1\frac{1}{15} + 36\frac{4}{7} + 3 + 10\frac{2}{3} + 9\right) = 964\frac{92}{105}.$$

5.4.4(L)

The main twist to this exercise is that we are not explicitly
given the cylinder which is sliced by the two surfaces. In
problems like this, the technique is to eliminate z from the two
equations of the surfaces. Quite in general, if $z = f_t(x,y)$ de-
notes the top surface and $z = f_b(x,y)$ denotes the bottom surface,
then we may equate the two expressions for z to obtain

$$f_t(x,y) = f_b(x,y). \tag{1}$$

5.4.4(L) continued

At first glance, it might appear that equation (1) gives the curve
of intersection between the two surfaces. A second glance at
equation (1), however, should soon convince us that this, in
general, is not the case. In particular, notice that equation (1)
can be put into the form

$$g(x,y) = 0 \tag{2}$$

where $g = f_t - f_b$, and the equation $g(x,y) = 0$ is the equation of
a curve <u>in the xy-plane</u> (or at least in a plane parallel to the
xy-plane), while in general the curve of intersection of two sur-
faces will not lie in this plane.

In fact, if we recall that the equation of a cylinder looks like
the equation of a curve in the xy-plane,* we soon suspect that
equation (2) yields the cylinder which contains the intersection
of the surfaces $z = f_t(x,y)$ and $z = f_b(x,y)$.

More specifically,

$$\{(x,y): f_t(x,y) = f_b(x,y)\}$$

denotes the set of points (x,y) which have the same z-value so
that this set may be viewed as the <u>projection</u> of the curve of
intersection onto the xy-plane. [Figure 16.13 and the subsequent
discussion in Thomas, Section 16.5, illustrates this point very
nicely.]

At any rate, with respect to the given surfaces in these exercises,
we obtain

$$\frac{1}{2}(x^2 + y^2 + 1) = x^2 + y^2$$

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*For example, $x^2 + y^2 - 1 = 0$ is the circle centered at $(0,0)$ with
radius 1 when we view the equation in the form $\{(x,y): x^2 + y^2 - 1
= 0\}$, but it is the right circular cylinder when viewed in the
form $\{(x,y,z): x^2 + y^2 - 1 = 0\}$.

5.4.4(L) continued

or

$$\frac{1}{2}x^2 + \frac{1}{2}y^2 = \frac{1}{2}.$$

Hence, the curve of intersection projects onto the circle $x^2 + y^2 = 1$ in the xy-plane.

In still other words, the given solid is the portion of the cylinder $x^2 + y^2 = 1$ between $z = x^2 + y^2$ and $z = \frac{1}{2}(x^2 + y^2 + 1)$.

To find which is the top surface and which is the bottom, we pick a point in the projected region of the xy-plane and see which curve is the upper curve above that point. For example, $(0,0)$ is in the region $x^2 + y^2 = 1$. Then the corresponding point on $z = x^2 + y^2$ is $(0,0,0)$, while on $\frac{1}{2}(x^2 + y^2 + 1)$ it is $(0,0,\frac{1}{2})$. Thus, at least above the point $(0,0)$ $z = \frac{1}{2}(x^2 + y^2 + 1)$ is the upper surface.

We then observe that wherever the curves interchange positions, this will be reflected by the fact that the projection of the curve of intersection will contain a curve corresponding to where the surfaces crossed. In this example, the fact that the projection of the curve of intersection in the xy-plane is the circle $x^2 + y^2 = 1$ insures that whichever surface was the top surface at one point inside the circle is the top surface at all points inside the circle. The circle, itself, represents the x and y coordinates of those points at which the two surfaces intersect (i.e., at those points neither surface is higher than the other). Finally, at those points outside the circle, the surface $z = x^2 + y^2$ lies above the surface $z = \frac{1}{2}(x^2 + y^2 + 1)$. Again, as a check, pick, for example, the point $(1,1)$ which lies outside the circle. The corresponding point on $z = x^2 + y^2$ is $(1,1,2)$ while the corresponding point of $z = \frac{1}{2}(x^2 + y^2 + 1)$ is $(1,1,\frac{3}{2})$. The fact that the surfaces do not again interchange positions is reflected by the fact that the projection of the curve of intersection includes nothing besides the circle $x^2 + y^2 = 1$.

Returning to the given problem, we seek the volume of that portion of the cylinder $x^2 + y^2 = 1$ bounded below by $z = x^2 + y^2$ and above by $z = \frac{1}{2}(x^2 + y^2 + 1)$.

5.4.4(L) continued

Thus, the required volume is

$$\iint_R \frac{1}{2}(x^2 + y^2 + 1)\,dA_R - \iint_R (x^2 + y^2)\,dA_R$$

$$= \iint_R (\frac{1}{2} - \frac{1}{2}x^2 - \frac{1}{2}y^2)\,dA_R$$

where R is:
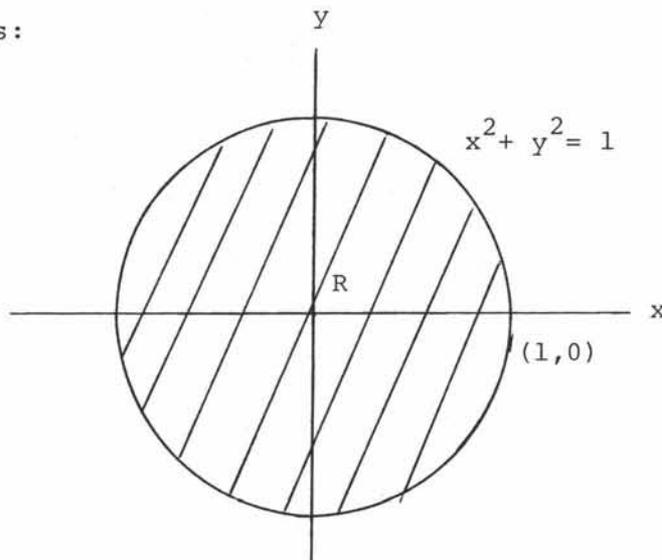
Therefore the volume is

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (\frac{1}{2} - \frac{1}{2}x^2 - \frac{1}{2}y^2)\,dy\,dx. \tag{1}$$

To evaluate (1) we observe that it is in our best interests to use polar coordinates, noting that R is simply $\{(r,\theta):$ $0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$. Moreover, our integrand becomes $\frac{1}{2} - \frac{1}{2}(x^2 + y^2) = \frac{1}{2} - \frac{1}{2}r^2$ and our element of area is $\frac{\partial(x,y)}{\partial(r,\theta)} = rdrd\theta$.

Thus

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (\frac{1}{2} - \frac{1}{2}x^2 - \frac{1}{2}y^2)\,dy\,dx$$

5.4.4(L) continued

$$= \int_0^{2\pi} \int_0^1 (\tfrac{1}{2} - \tfrac{1}{2} r^2) r \ dr \ d\theta$$

$$= \int_0^{2\pi} \left[\int_0^1 (\tfrac{1}{2} r - \tfrac{1}{2} r^3) dr \right] d\theta$$

$$= \int_0^{2\pi} \left. \tfrac{1}{4} r^2 - \tfrac{1}{8} r^4 \right|_0^1 d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{8} d\theta$$

$$= \tfrac{\pi}{4} \ .$$

5.4.5

Eliminating z from $x^2 + y^2 + z^2 = 4$ and $\sqrt{2}\, z = x^2 + y^2$ we obtain

$$x^2 + y^2 + (\frac{x^2 + y^2}{\sqrt{2}})^2 = 4.$$

Hence,

$$2x^2 + 2y^2 + (x^2 + y^2)^2 = 8 \tag{1}$$

represents the projection of the curve of intersection into the xy-plane.

Equation (1) may be simplified by a switch to polar coordinates since we then obtain

$$2x^2 + 2y^2 + (x^2 + y^2)^2 = 8$$
or
$$2(x^2 + y^2) + (x^2 + y^2)^2 = 8$$
or

5.4.4(L) continued

$$2r^2 + r^4 = 8$$

or

$$r^4 + 2r^2 - 8 = 0$$

or

$$(r^2 + 4)(r^2 - 2) = 0. \tag{2}$$

Since $r^2 + 4 \neq 0$, equation (2) tells us that the cylinder which contains the curve of intersection is $r^2 - 2 = 0$ or $r = \sqrt{2}$. In other words, the curve of intersection projects onto the circle in the xy-plane centered at (0,0) with radius 2.

So the required volume is given by

$$\iint\limits_{R} \left[\sqrt{4 - x^2 - y^2} - \frac{(x^2 + y^2)}{\sqrt{2}} \right] dA_R$$

where R is the region $x^2 + y^2 \leq 2$; or

$$\int_0^{2\pi} \int_0^{\sqrt{2}} \left(\sqrt{4 - r^2} - \frac{r^2}{\sqrt{2}} \right) r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[\int_0^{\sqrt{2}} \left(r\sqrt{4 - r^2} - \frac{r^3}{\sqrt{2}} \right) dr \right] d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{3}(4 - r^2)^{\frac{3}{2}} - \frac{r^4}{4\sqrt{2}} \right]_{r=0}^{\sqrt{2}} d\theta$$

$$= \int_0^{2\pi} \left\{ \left[-\frac{1}{3}(4 - 2)^{\frac{3}{2}} - \frac{4}{4\sqrt{2}} \right] - \left[-\frac{1}{3}(4 - 0)^{\frac{3}{2}} \right] \right\} d\theta$$

$$= \int_0^{2} \left[-\frac{1}{3} 2^{\frac{3}{2}} - \frac{1}{\sqrt{2}} + \frac{1}{3}(4)^{\frac{3}{2}} \right] d\theta$$

5.4.4(L) continued

$$= 2\pi \left[-\frac{2}{3} \sqrt{2} - \frac{\sqrt{2}}{2} + \frac{8}{3} \right]$$

$$= 2\pi \frac{-4\sqrt{2} - 3\sqrt{2} + 16}{6}$$

$$= \frac{\pi}{3} [16 - 7\sqrt{2}].$$

5.4.6

The cylinder which contains the curve of intersection is obtained from the equation

$$x^2 + 9y^2 = 18 - x^2 - 9y^2$$

or

$$2x^2 + 18y^2 = 18$$

or

$$\frac{x^2}{9} + \frac{y^2}{1} = 1,$$

which is the ellipse whose semi-axes are 3 and 1. That is
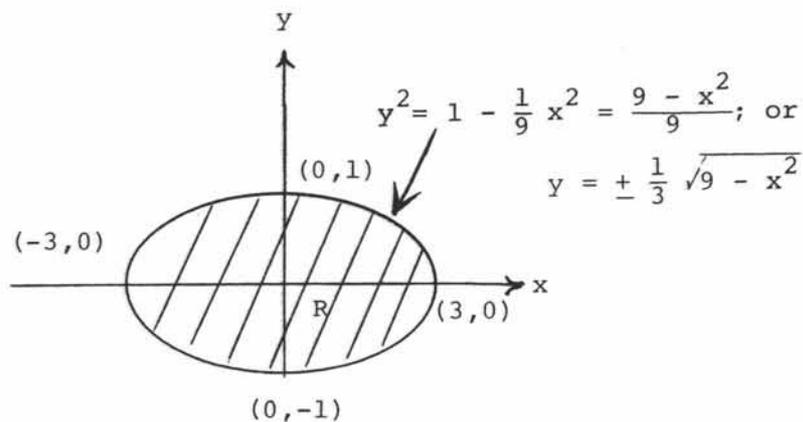
5.4.6 continued

Looking at $(0,0)$ we see that it corresponds to the point $(0,0,0)$ on the surface $z = x^2 + 9y^2$ and to the point $(0,0,18)$ on the surface $z = 18 - x^2 - 9y^2$. Hence, $z = 18 - x^2 - 9y^2$ is the top surface throughout R. Consequently, the required volume is given by

$$\iint\limits_{R} \left[(18 - x^2 - 9y^2) - (x^2 + 9y^2) \right] \, dA_R$$

$$= \int_{-3}^{3} \int_{-\frac{1}{3}\sqrt{9-x^2}}^{\frac{1}{3}\sqrt{9-x^2}} (18 - 2x^2 - 18y^2) \, dy \, dx$$

$$\int_{-3}^{3} \left. 18y - 2x^2 y - 6y^3 \right|_{y = -\frac{1}{3}\sqrt{9-x^2}}^{\frac{1}{3}\sqrt{9-x^2}} dx$$

$$= 2 \int_{-3}^{3} \left[6\sqrt{9-x^2} - \frac{2}{3} x^2 \sqrt{9-x^2} - \frac{6}{27}(9-x^2)^{\frac{3}{2}} \right] dx. \qquad (1)$$

The integral in (1) suggests trigonometric substitution. Namely,

$$3 \sin \theta = x$$
$$3 \cos \theta \, d\theta = dx$$
$$\sqrt{9 - x^2} = 3 \cos \theta$$
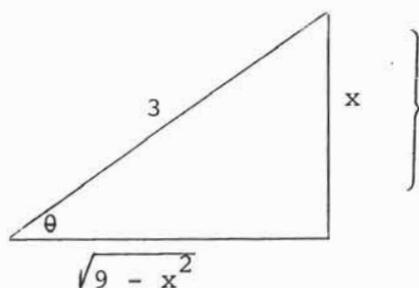
5.4.6 continued

Therefore,

$$2 \int_{-3}^{3} [6 \sqrt{9 - x^2} - \frac{2}{3} x^2 \sqrt{9 - x^2} - \frac{6}{27} (9 - x^2)^{\frac{3}{2}}] \, dx$$

$$= 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [18 \cos \theta - \frac{2}{3}(9 \sin^2\theta) 3 \cos \theta - \frac{6}{27}(3 \cos \theta)^3] 3 \cos \theta \, d\theta$$

$$= 6 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (18 \cos \theta - 18 \sin^2\theta \cos \theta - 6 \cos^3\theta) \cos \theta \, d\theta$$

$$= 36 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (3 \cos^2\theta - 3 \sin^2\theta \cos^2\theta - \cos^4\theta) d\theta$$

$$= 72 \int_{0}^{\frac{\pi}{2}} [3 \cos^2\theta (1 - \sin^2\theta) - \cos^4\theta] d\theta$$

$$= 72 \int_{0}^{\frac{\pi}{2}} (3 \cos^4\theta - \cos^4\theta) d\theta$$

$$= 144 \int_{0}^{\frac{\pi}{2}} \cos^4\theta \, d\theta \qquad (2)$$

$$= 144 \int_{0}^{\frac{\pi}{2}} [\frac{1 + \cos 2\theta}{2}]^2 \, d\theta$$

5.4.6 continued

$$= 36 \int_0^{\frac{\pi}{2}} (1 + 2 \cos 2\theta + \cos^2 2\theta) d\theta$$

$$= 36 \int_0^{\frac{\pi}{2}} [1 + 2 \cos 2\theta + \frac{1 + \cos 4\theta}{2}] d\theta$$

$$= 18 \int_0^{\frac{\pi}{2}} (3 + 4 \cos 2\theta + \cos 4\theta) d\theta$$

$$= 18 [3\theta + 2 \sin 2\theta + \frac{1}{4} \sin 4\theta \Big|_{\theta=0}^{\frac{\pi}{2}}]$$

$$= 18 \ (\frac{3\pi}{2}).$$

$$= 27 \pi \hspace{8cm} (3)$$

Note:

Given

$$\iint_R (18 - 2x^2 - 18y^2) dA_R$$

where R was the region $\{(x,y): \frac{x^2}{9} + y^2 \le 1\}$, it might have been a bit simpler computationally to make the change of variables

$$u = \frac{x}{3} \quad \text{or} \quad x = 3u$$
$$v = y \hspace{2cm} y = v .$$

In this case

$$\frac{\partial (x,y)}{\partial (u,v)} = \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} = 3,$$

and our transformed integral would be

5.4.6 continued

$$\iint\limits_{u^2+ v^2 \leq 1} (18 - 18u^2 - 18v^2)3 \, dv \, du$$

$$= 54 \int_{-1}^{1} \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} (1 - u^2 - v^2) \, dv \, du$$

$$= 54 \int_{-1}^{1} v - u^2v - \frac{1}{3} v^3 \, \Bigg|_{v=- \sqrt{1-u^2}}^{\sqrt{1-u^2}}$$

$$= 108 \int_{-1}^{1} v - u^2v - \frac{1}{3} v^3 \, \Bigg|_{v=0}^{\sqrt{1-u^2}} du$$

$$= 108 \int_{-1}^{1} \sqrt{1 - u^2} - u^2 \sqrt{1 - u^2} - \frac{1}{3}(1 - u^2)^{\frac{3}{2}}$$

$$= 216 \int_{0}^{1} \sqrt{1 - u^2} \left(1 - u^2 - \frac{1}{3} [1 - u^2]\right) du$$

$$= 72 \int_{0}^{1} 1 - u^2 (2 - 2u^2) \, du$$

$$= 144 \int_{0}^{1} (1 - u^2)^{\frac{3}{2}} \, du$$

$$\sin \theta = u$$
$$\cos \theta = \sqrt{1-u^2}$$
$$\cos \theta d\theta = du$$
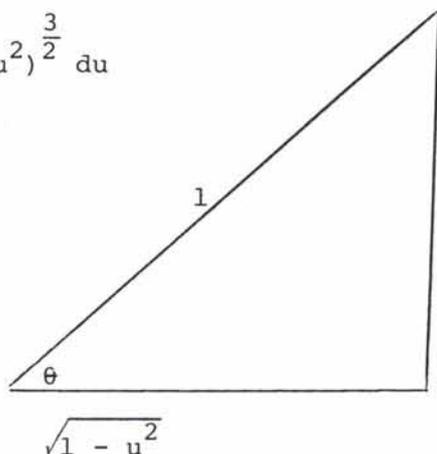
5.4.6 continued

Hence,

$$144 \int_0^1 (1 - u^2)^{\frac{3}{2}} du$$

$$= 144 \int_0^{\frac{\pi}{2}} [\cos^3 \theta] \cos \theta d\theta$$

$$= 144 \int_0^{\frac{\pi}{2}} \cos^4 \theta d\theta$$

which agrees with equation (2).

While this may not have been a big improvement over our original approach it does indicate how a change of variables can simplify the arithmetic involved in computing a double integral.

5.4.7

We may view the solid as being the portion of the cylinder $x^2 + y^2 = a^2$ bounded above by $z = \pm \sqrt{a^2 - x^2}$ (i.e., the upper portion of $x^2 + z^2 = a^2$) and below by $z = - \sqrt{a^2 - x^2}$.

Hence, letting $R = \{(x,y) : x^2 + y^2\} \le a^2$ we have that the given volume is

$$\iint_R [\sqrt{a^2 - x^2} - (-\sqrt{a^2 - x^2})] \, dA_R$$

$$= 2 \iint_R \sqrt{a^2 - x^2} \, dA_R$$

$$= 2 \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2} \, dy \, dx$$

5.4.7 continued

$$= 2 \int_{-a}^{a} y \sqrt{a^2 - x^2} \, \Big|_{y = -\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} \, dx$$

$$= 4 \int_{-a}^{a} (a^2 - x^2) \, dx$$

$$= 4 [a^2 x - \frac{1}{3} x^3 \, \Big|_{x = -a}^{a}]$$

$$= 8 [a^3 - \frac{1}{3} a^3]$$

$$= \frac{16a^3}{3} \, .$$

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Prof. Herbert Gross

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