Solutions

BLOCK 5:
MULTIPLE INTEGRATION

Pretest

1. 54

2. $1093 \frac{1}{2}$

3. 2

4. $\frac{\pi}{4} (1 - e^{-a^2})$

5. π

6. $\frac{\pi}{6} (5\sqrt{5} - 1)$

7. $\frac{54\pi}{\sqrt{5}}$

8. -4

Unit 1: Double Sums

5.1.1(L)

In the expression $\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{m} a_{ij} \right)$, it is assumed that in the paren-thesized-sum, i is treated as being constant (notice the "flavor" of the notion of independent variables). That is,

$$\sum_{j=1}^{m} a_{ij} = a_{i1} + \ldots + a_{im}.$$

Therefore,

$$\sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ij} \right) = \sum_{i=1}^{n} (a_{i1} + \ldots + a_{im})$$

$$= (a_{11} + \ldots + a_{1m}) + \ldots + (a_{n1} + \ldots + a_{nm}). \qquad (1)$$

From (1) we see that $\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{m} a_{ij}$ is the sum of mn terms, each of the form a_{ij} where i=1,...,n and j=1,...,m.

While this sum is independent of the order in which we add the terms, we still agree to adhere to the given definition for reasons which will become clearer in Exercise 5.1.4(L).

Similarly,

$$\sum_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij} \right) = \sum_{j=1}^{m} (a_{1j} + \ldots + a_{nj})$$

$$= (a_{11} + \ldots + a_{n1}) + \ldots + (a_{1m} + \ldots + a_{nm}). \qquad (2)$$

Except for the order, the mn terms in (2) are the same as those in (1), and the desired result is established.

5.1.1(L) continued

We would like to conclude our commentary on this exercise with an observation that may make it easier for you to visualize what we mean by a double sum. Notice that the mn numbers a_{ij} where $i = 1,\ldots,n$ and $j = 1,\ldots,m$ may be viewed, in matrix fashion, as an array of n rows and m columns. That is,

$$
\left.
\begin{array}{llll}
a_{11} & a_{12} & \cdots\cdots & a_{1m} \\
a_{21} & a_{22} & \cdots\cdots & a_{2m} \\
a_{n1} & a_{n2} & \cdots\cdots & a_{nm}
\end{array}
\right\}
\tag{3}
$$

The sum $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} = \sum_{i=1}^{n}(a_{i1} + \ldots + a_{im})$ may be viewed as the sum of the sum of each of the n rows in (3). In other words, $a_{i1} + \ldots + a_{im}$ is the sum of the terms in the ith row of (3) and we then sum over the n rows.

Schematically, to find $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}$, we have from (3),

$$
\begin{array}{lllll}
 & & & \text{sum} & \\
a_{11} & a_{12} & \cdots & a_{1m} & \rightarrow & a_{11} + \ldots + a_{1m} \\
a_{21} & a_{22} & \cdots & a_{2m} & \rightarrow & a_{21} + \ldots + a_{2m} \\
\cdot & & & & \\
\cdot & & & & \\
\cdot & & & & \\
a_{n1} & a_{n2} & \cdots & a_{nm} & \rightarrow & a_{n1} + \ldots + a_{nm}
\end{array}
\quad \text{sum} \quad \text{or} \quad
\left\{
\begin{array}{l}
(a_{11} + a_{12} + \ldots + a_{1m}) \\
(a_{21} + a_{22} + \ldots + a_{2m}) \\
\cdot \\
\cdot \\
\cdot \\
(a_{n1} + a_{n2} + \ldots + a_{nm})
\end{array}
\right.
$$

On the other hand, to form $\displaystyle\sum_{j=1}^{m}\sum_{i=1}^{n} a_{ij}$, we first form $\displaystyle\sum_{i=1}^{n} a_{ij} = a_{1j} + \ldots + a_{nj}$ which is equivalent to the sum of the terms in the jth column of (3), and we then sum over the m columns. That is, to form $\displaystyle\sum_{j=1}^{m}\sum_{i=1}^{n} a_{ij}$, we have

5.1.1(L) continued

$$
\begin{array}{llll}
a_{11} & a_{12} & \cdots\cdots & a_{1m} \\
a_{21} & a_{22} & \cdots\cdots & a_{2m} \\
\vdots & & & \vdots \\
a_{n1} & a_{n2} & \cdots\cdots & a_{nm} \\
\downarrow & \downarrow & & \downarrow
\end{array}
$$

sum each column → then add the sums of the columns. That is,

$$
\begin{bmatrix} a_{11} \\ + \\ a_{21} \\ + \\ \cdot \\ \cdot \\ \cdot \\ + \\ a_{n1} \end{bmatrix}
+
\begin{bmatrix} a_{12} \\ + \\ a_{22} \\ + \\ \cdot \\ \cdot \\ \cdot \\ + \\ a_{n2} \end{bmatrix}
+ \ldots +
\begin{bmatrix} a_{1m} \\ + \\ a_{2m} \\ + \\ \cdot \\ \cdot \\ \cdot \\ + \\ a_{nm} \end{bmatrix}
$$

In summary, both $\displaystyle\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}$ and $\displaystyle\sum_{j=1}^{m} \sum_{i=1}^{n} a_{ij}$ are ways of adding

the terms in (3). In the former case, we first sum the rows and then add these results, while in the latter case, we first sum the columns and then add these results.

5.1.2

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a. Using the matrix notation, we have

$$
\begin{array}{lll}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23}
\end{array}
$$

Then

$$
\sum_{i=1}^{2} \sum_{j=1}^{3} a_{ij} = \begin{array}{c} (a_{11} + a_{12} + a_{13}) \\ + \\ (a_{21} + a_{22} + a_{23}) \end{array} \tag{1}
$$

5.1.2 continued

while

$$\sum_{j=1}^{3} \sum_{i=1}^{2} a_{ij} = \begin{pmatrix} a_{11} \\ + \\ a_{21} \end{pmatrix} + \begin{pmatrix} a_{12} \\ + \\ a_{22} \end{pmatrix} + \begin{pmatrix} a_{13} \\ + \\ a_{23} \end{pmatrix} \tag{2}$$

Hence, from (1) and (2)

$$\sum_{i=1}^{2} \sum_{j=1}^{3} a_{ij} = \sum_{j=1}^{3} \sum_{i=1}^{2} a_{ij} = a_{11} + a_{12} + a_{13} + a_{21} + a_{22} + a_{23} \tag{3}$$

$$= a_{11} + a_{21} + a_{12} + a_{22} + a_{13} + a_{23} \tag{4}$$

Without reference to the matrix notation,

$$\sum_{i=1}^{2} \sum_{j=1}^{3} a_{ij} = \sum_{i=1}^{2} \left(\sum_{j=1}^{3} a_{ij} \right)$$

$$= \sum_{i=1}^{2} (a_{i1} + a_{i2} + a_{i3})$$

$$= (a_{11} + a_{12} + a_{13}) + (a_{21} + a_{22} + a_{23}),$$

which agrees with (3).

b. In this case, $a_{ij} = ij$, $i = 1,2,3,4$ and $j = 1,2,3$. Hence, our
    rectangular array is

$$\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{array} = \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \\ 4 & 8 & 12 \end{array}$$

Therefore,

5.1.2 continued

$$\sum_{i=1}^{4} \sum_{j=1}^{3} ij = (1+2+3) + (2+4+6) + (3+6+9) + (4+8+12) \qquad (5)$$

$$= 6 + 12 + 18 + 24$$

$$= 60$$

$$\sum_{j=1}^{3} \sum_{i=1}^{4} ij = \left| \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \right. + \left. \begin{matrix} 2 \\ 4 \\ 6 \\ 8 \end{matrix} \right| + \left. \begin{matrix} 3 \\ 6 \\ 9 \\ 12 \end{matrix} \right.$$

$$10 \quad + \quad 20 \quad + \quad 30 \ = \ 60$$

c. In this case, $i = 4$, $j = 3$, and $a_{ij} = i + j$. Hence, our rectangular array is given by

```
2  3  4
3  4  5
4  5  6
5  6  7
```

Therefore,

$$\sum_{j=1}^{3} \sum_{i=1}^{4} (i + j) = \begin{matrix} 2 \\ 3 \\ 4 \\ 5 \end{matrix} + \begin{matrix} 3 \\ 4 \\ 5 \\ 6 \end{matrix} + \begin{matrix} 4 \\ 5 \\ 6 \\ 7 \end{matrix}$$

$$14 + 18 + 22 = 54$$

$$\left[= \sum_{i=1}^{4} \sum_{j=1}^{3} (i + j) \right].$$

5.1.3(L)

Our main aim here is to establish a few formulas for dealing with double sums.

5.1.3(L) continued

a. $\displaystyle\sum_{i=1}^{n} \sum_{j=1}^{m} ca_{ij} = \sum_{i=1}^{n}\left(\sum_{j=1}^{m} ca_{ij}\right)$

$$= \sum_{i=1}^{n} ca_{i1} + \ldots + ca_{im}$$

$$= \sum_{i=1}^{n} c(a_{i1} + \ldots + a_{im})$$

$$= c\sum_{i=1}^{n} (a_{i1} + \ldots + a_{im})$$

$$= c[(a_{11} + \ldots + a_{1m}) + \ldots + (a_{n1} + \ldots + a_{nm})]$$

$$= c\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}. \tag{1}$$

Equation (1) shows us that a constant factor may be removed from within the double sum.

Matrix-wise

$$\sum_{i=1}^{n} \sum_{j=1}^{m} ca_{ij} = \begin{pmatrix} (ca_{11} + \ldots + ca_{1m} \\ + \\ \vdots \\ + \\ (ca_{n1} + \ldots + ca_{nm}) \end{pmatrix}$$

$$= c\begin{bmatrix} a_{11} + \ldots + a_{1m} \\ + \\ \vdots \\ + \\ a_{n1} + \ldots + a_{nm} \end{bmatrix}$$

$$= c\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}.$$

5.1.3(L) continued

b. $\displaystyle\sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j = \sum_{i=1}^{n}\left(\sum_{j=1}^{m} a_i b_j\right)$

$$= \sum_{i=1}^{n}(a_i b_1 + \dots + a_i b_m)$$

$$= (a_1 b_1 + \dots + a_1 b_m) + (a_2 b_1 + \dots + a_2 b_m) + \dots$$

$$+ (a_n b_1 + \dots + a_n b_m)$$

$$= a_1(b_1 + \dots + b_m) + a_2(b_1 + \dots + b_m) + \dots$$

$$+ a_n(b_1 + \dots + b_m)$$

$$= (a_1 + a_2 + \dots + a_n)(b_1 + \dots + b_m)$$

$$= \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{j=1}^{m} b_j\right).$$

c. As a check on Exercise 5.1.2, part (b), we have

$$\sum_{i=1}^{4} \sum_{j=1}^{3} ij = \left(\sum_{i=1}^{4} i\right)\left(\sum_{j=1}^{3} j\right)$$

$$= (1 + 2 + 3 + 4)(1 + 2 + 3)$$

$$= (10)(6)$$

$$= 60.$$

5.1.3(L) continued

d. $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{m}(a_i + b_j) = \sum_{i=1}^{n}\underbrace{(a_i + b_1) + (a_i + b_2) + \ldots + a_i + b_m)}_{m \text{ terms}}$

$$= \sum_{i=1}^{n}[ma_i + (b_1 + \ldots + b_m)]$$

$$= \underbrace{[ma_1 + (b_1 + \ldots + b_m)] + \ldots + [ma_n + (b_1 + \ldots + b_m)]}_{n \text{ terms}}$$

Therefore,

$$\sum_{i=1}^{n}\sum_{j=1}^{m}(a_i + b_j) = (ma_1 + \ldots + ma_n) + n(b_1 + \ldots + b_m)$$

$$= m(a_1 + \ldots + a_n) + n(b_1 + \ldots + b_m)$$

$$= m\sum_{i=1}^{n}a_i + n\sum_{j=1}^{m}b_j \qquad (2)$$

Notice that (2) tells us that, obvious or not,

$$\sum_{i=1}^{n}\sum_{j=1}^{m}(a_i + b_j) \neq \sum_{i=1}^{n}a_i + \sum_{j=1}^{m}b_j.$$

e. As a check of Exercise 5.1.2, part (c),

$$\sum_{i=1}^{4}\sum_{j=1}^{3}(i + j) = 3\sum_{i=1}^{4}i + 4\sum_{j=1}^{3}j$$

$$= 3(1 + 2 + 3 + 4) + 4(1 + 2 + 3)$$

$$= 30 + 24$$

$$= 54.$$

5.1.4(L)

In computing limits of double sums, we may have to contend with either of the forms,

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \quad \text{or} \quad \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$

While the order of summation is irrelevant for finite choices of m and n, the order may well make a difference for the <u>infinite</u> double sum. This should not be too surprising, since this fact was true even for single infinite sums (absolute convergence versus "plain old" convergence). In any event, when limits are involved we must, in general, make sure we add in the indicated order. In this particular exercise, notice that our rectangular array is given by

$$\left.\begin{array}{ccccccc} 1 - 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 - 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 - 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 1 - 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 1 - 1 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 1 - 1 & \ldots \end{array}\right\} \tag{1}$$

Notice that each row has 0 as its sum. That is,

$$\sum_{j=1}^{\infty} a_{ij} = 0$$

for each i. Hence,

$$\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{ij}\right) = \sum_{i=1}^{\infty} 0 = 0. \tag{2}$$

Schematically,

5.1.4(L) continued

$$\left.\begin{array}{l}(1 - 1 + 0 + 0 \ldots) \\ \qquad + \\ (0 + 1 - 1 + 0 + 0 \ldots) \\ \qquad + \\ (0 + 0 + 1 - 1 + 0 + 0 \ldots)\end{array}\right\} = \left.\begin{array}{l}0 \\ + \\ 0 \\ + \\ 0 \\ \vdots\end{array}\right\} = 0$$

On the other hand, the first column has 1 as a sum, while each of the other columns has 0 as a sum. That is,

$$\sum_{i=1}^{\infty} a_{i1} = 1 \quad \text{while} \quad \sum_{i=1}^{\infty} a_{ij} = 0, \text{ for } j = 2,3,4,\ldots$$

Thus,

$$\sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} a_{ij}\right) = \left(\sum_{i=1}^{\infty} a_{i1}\right) + \sum_{j=2}^{\infty} \left(\sum_{i=1}^{\infty} a_{ij}\right)$$

$$= 1 + \sum_{j=2}^{\infty} 0$$

$$= 1. \tag{3}$$

Comparing equations (2) and (3), we see that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = 0 \quad \text{while} \quad \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij} = 1.$$

As an aside, notice that if we sum diagonally as shown below, the sum diverges by oscillation. Namely,

5.1.4(L) continued

$$
\begin{array}{rrrrrrrl}
1 & | & 1 - 1 & 0 & 0 & 0 & 0 & \dots \\
-1 & | & 0 & 1 - 1 & 0 & 0 & 0 & \dots \\
1 & | & 0 & 0 & 1 - 1 & 0 & 0 & \dots \\
-1 & | & 0 & 0 & 0 & 1 - 1 & 0 & \dots \\
1 & | & 0 & 0 & 0 & 0 & 1 - 1 & \dots \quad \text{etc.}
\end{array}
$$

In other words, the sum is drastically affected by rearrangements of the terms.

The following theorem, stated without proof, gives us a situation

for which $\displaystyle\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$. Namely, if we let b_i denote

$\displaystyle\sum_{j=1}^{\infty} |a_{ij}|$, then if $\displaystyle\sum_{i=1}^{\infty} b_i$ converges (i.e. if $\displaystyle\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |a_{ij}| \right)$ converges)

then $\displaystyle\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$. This is the analog of absolute

convergence for single infinite sums. From our point of view, a

major point is that for most of the double series $\displaystyle\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$ we

encounter in our applications, it is true that $\displaystyle\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |a_{ij}| \right)$ does

converge. Consequently, in most cases, we can change the order of summation without changing the sum (but we must check

$\displaystyle\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |a_{ij}| \right)$ in each case).

In our present example, for a fixed i (i.e., a fixed row in our

array) $\displaystyle\sum_{j=1}^{\infty} |a_{ij}| = 1 + |-1| = 2$. That is, if $\displaystyle\sum_{j=1}^{\infty} |a_{ij}| = b_i$, then

$b_i = 2$. Therefore, $\displaystyle\sum_{i=1}^{\infty} b_i = \sum_{i=1}^{\infty} 2 = \infty$, so that the conditions

stated in the theorem do not apply.

5.1.4(L) continued

An important corollary of the theorem is that if $a_{ij} \geqslant 0$ for each

i and j [so that $\sum\limits_{i=1}^{\infty} \sum\limits_{j=1}^{\infty} a_{ij}$ is the same as $\sum\limits_{i=1}^{\infty} \sum\limits_{j=1}^{\infty} |a_{ij}|$], then

if $\sum\limits_{i=1}^{\infty} \sum\limits_{j=1}^{\infty} a_{ij}$ converges, $\sum\limits_{j=1}^{\infty} \sum\limits_{i=1}^{\infty} a_{ij}$ converges also, and to the
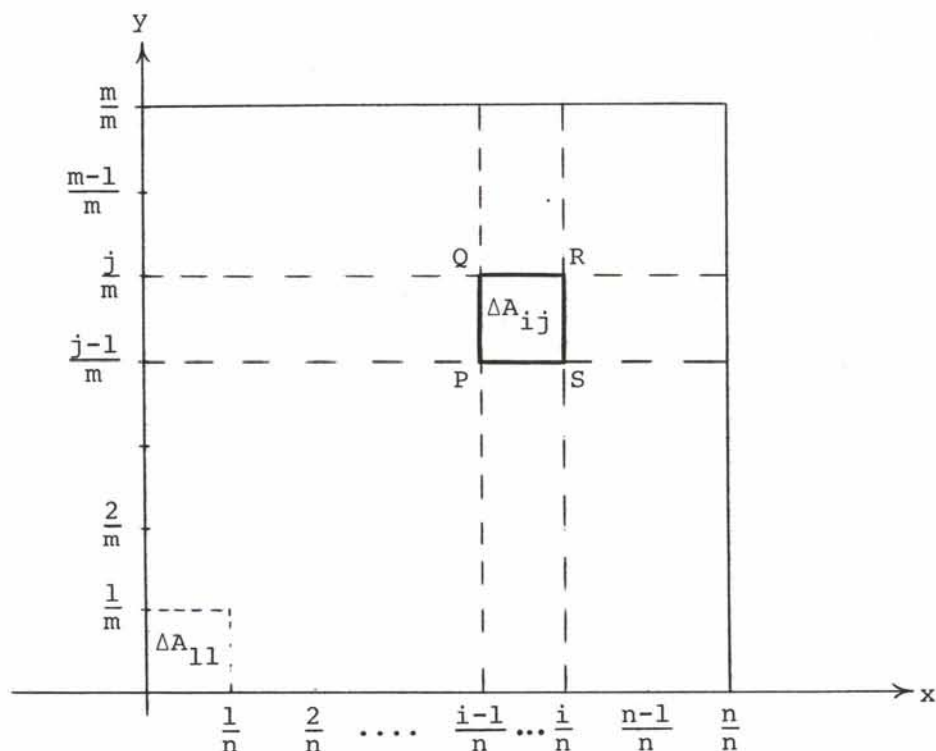
same sum.

5.1.5(L)

a.

Figure 1

Least density of PQRS occurs at $P(\frac{i-1}{n}, \frac{j-1}{m})$ since P is the point in
PQRS nearest the origin. Maximum density occurs at $R(\frac{i}{n}, \frac{j}{m})$ since
it is furthest from the origin.*

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*Notice that nearest and furthest are important only because
$\rho = x^2 + y^2$ which is the square of the distance from the origin to
(x,y).

5.1.5(L) continued

$$\underline{\Delta M_{ij}} = \left[\left(\frac{i-1}{n}\right)^2 + \left(\frac{j-1}{m}\right)^2\right]\Delta x_i \Delta y_j$$

$$= \left[\frac{(i-1)^2}{n^2} + \frac{(j-1)^2}{m^2}\right]\frac{1}{n}\frac{1}{m}$$

$$= \frac{1}{mn}\left[\frac{(i-1)^2}{n^2} + \frac{(j-1)^2}{m^2}\right]. \tag{1}$$

Correspondingly,

$$\overline{\Delta M}_{ij} = \frac{1}{mn}\left[\frac{i^2}{n^2} + \frac{j^2}{m^2}\right]. \tag{2}$$

Therefore,

$$M = \sum_{i=1}^{n}\sum_{j=1}^{m}\Delta M_{ij} \rightarrow$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{1}{mn}\left[\frac{(i-1)^2}{n^2} + \frac{(j-1)^2}{m^2}\right] < M < \sum_{i=1}^{n}\sum_{j=1}^{m}\frac{1}{mn}\left[\frac{i^2}{n^2} + \frac{j^2}{m^2}\right]. \tag{3}$$

Since m and n are fixed integers, $\frac{1}{mn}$ is a constant, hence, by Exercise 5.1.3(L), part (a),

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{1}{mn}\left[\frac{i^2}{n^2} + \frac{j^2}{m^2}\right] = \frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m}\left[\frac{i^2}{n^2} + \frac{j^2}{m^2}\right]. \tag{4}$$

On the other hand, by part (d) of the same exercise,

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\left[\frac{i^2}{n^2} + \frac{j^2}{m^2}\right] = m\sum_{i=1}^{n}\frac{i^2}{n^2} + n\sum_{j=1}^{m}\frac{j^2}{m^2} \tag{5}$$

so putting (5) into (4) yields

5.1.5(L) continued

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \frac{1}{mn}\left[\frac{i^2}{n^2} + \frac{j^2}{m^2}\right] = \frac{1}{mn}\left[m \sum_{i=1}^{n} \frac{i^2}{n^2} + n \sum_{j=1}^{m} \frac{j^2}{m^2}\right]$$

$$= \frac{1}{n} \sum_{i=1}^{n} \frac{i^2}{n^2} + \frac{1}{m} \sum_{j=1}^{m} \frac{j^2}{m^2}$$

$$= \frac{1}{n^3} \sum_{i=1}^{n} i^2 + \frac{1}{m^3} \sum_{j=1}^{m} j^2. \qquad (6)$$

In Part 1 of our course, we showed that the sum of the first k squares was $\frac{k(k+1)(2k+1)}{6}$, and with this information (6) becomes

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \frac{1}{mn}\left[\frac{i^2}{n^2} + \frac{j^2}{m^2}\right] = \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] + \frac{1}{m^3}\left[\frac{m(m+1)(2m+1)}{6}\right]$$

$$= \frac{1}{6}\left[\frac{n}{n}\frac{(n+1)}{n}\frac{(2n+1)}{n} + \frac{m}{m}\frac{(m+1)}{m}\frac{(2m+1)}{m}\right]$$

$$= \frac{1}{6}\left[\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) + \left(1 + \frac{1}{m}\right)\left(2 + \frac{1}{m}\right)\right]. \qquad (7)$$

In a similar manner [the only difference being that we use the fact that $1^2 + \ldots + (k-1)^2 = \frac{(k-1)(k)(2k-1)}{6}$], we may show that

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \frac{1}{mn}\left[\frac{(i-1)^2}{n^2} + \frac{(j-1)^2}{m^2}\right] = \frac{1}{6}\left[\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) + \left(1 - \frac{1}{m}\right)\left(2 - \frac{1}{m}\right)\right]. \qquad (8)$$

Putting (7) and (8) into (3) yields

$$\frac{1}{6}\left[\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) + \left(1 - \frac{1}{m}\right)\left(2 - \frac{1}{m}\right)\right] < M < \frac{1}{6}\left[\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) + \right.$$

$$\left. + \left(1 + \frac{1}{m}\right)\left(2 + \frac{1}{m}\right)\right]. \qquad (9)$$

5.1.5(L) continued

b. Letting $n = m = 10^6$, statement (9) becomes

$$\frac{1}{6}\left[(1 - 10^{-6})(2 - 10^{-6}) + (1 - 10^{-6})(2 - 10^{-6})\right] < M < \frac{1}{6}\left[(1 + 10^{-6})\right.$$

$$\left.(2 + 10^{-6}) + (1 + 10^{-6})(2 + 10^{-6})\right]$$

or

$$\frac{1}{3}(1 - 10^{-6})(2 - 10^{-6}) < M < \frac{1}{3}(1 + 10^{-6})(2 + 10^{-6}) \qquad (10)$$

Expanding (10) shows that

$$\frac{1}{3}\left[2 - 3(10)^{-6} + 10^{-12}\right] < M < \frac{1}{3}\left[2 + 3(10)^{-6} + 10^{-12}\right]. \qquad (11)$$

Now,

$$2 + 3(10)^{-6} + 10^{-12} = 2 + .000003 + .000000000001$$

$$= 2.000003000001,$$

while

$$2 - 3(10)^{-6} + 10^{-12} = 2 - .000003 + .000000000001$$

$$= 1.999997000001$$

whereupon (11) becomes

$$0.666665666667 < M < 0.666667666667. \qquad (12)$$

Thus, no matter what the exact mass of the plate is, (12) convinces us that to five decimal places M = 0.66667.

c. To find the exact value of M [part (b) probably leads us to expect $M = \frac{2}{3}$], we return to (9) and compute

$$\lim_{\substack{n \to \infty \\ m \to \infty}} \left[\frac{1}{6}\left\{\left(1 \pm \frac{1}{n}\right)\left(2 \pm \frac{1}{n}\right) + \left(1 \pm \frac{1}{m}\right)\left(2 \pm \frac{1}{m}\right)\right\}\right]. \qquad (13)$$

5.1.5(L) continued

One key in evaluating (13) lies in our comments in Exercise
5.1.4(L). That is, (13) represents

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{mn} \left[\frac{i^2}{n^2} + \frac{j^2}{m^2} \right] \quad \text{and} \quad \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{mn} \left[\frac{(i-1)^2}{n^2} + \frac{(j-1)^2}{m^2} \right]$$

and the order of taking limits might affect our answer. The point
is that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$$

when $a_{ij} \geqslant 0$ (which is the case in the present example), so we may
evaluate $\lim_{\substack{n \to \infty \\ m \to \infty}}$ by letting n and m$\to \infty$ separately, in either order.

For example,

$$\lim_{\substack{n \to \infty \\ m \to \infty}} \left[\frac{1}{6} \left\{ \left(1 \pm \frac{1}{n}\right) \left(2 \pm \frac{1}{n}\right) + \left(1 \pm \frac{1}{m}\right) \left(2 \pm \frac{1}{m}\right) \right\} \right] =$$

$$\lim_{n \to \infty} \left\{ \lim_{m \to \infty} \left[\frac{1}{6} \left\{ \left(1 \pm \frac{1}{n}\right) \left(2 \pm \frac{1}{n}\right) + \left(1 \pm \frac{1}{m}\right) \left(2 \pm \frac{1}{m}\right) \right\} \right] \right\} =$$

$$\lim_{n \to \infty} \left\{ \frac{1}{6} \left[\left(1 \pm \frac{1}{n}\right) \left(2 \pm \frac{1}{n}\right) + (1 \pm 0)(2 \pm 0) \right] \right\} =$$

$$\frac{1}{6} \left[(1 \pm 0)(2 \pm 0) + (1 \pm 0)(2 \pm 0) \right] = \frac{2}{3}.$$

Hence, from (9),

$$\frac{2}{3} \leqslant M \leqslant \frac{2}{3}.$$

Therefore,

$$\underline{M = \frac{2}{3}.}$$

5.1.5(L) continued

The main idea, stripped of the computational details, is that our theorems about double sums allow us (just as in the case of the calculus of a single variable), to determine the mass M of the given plate <u>exactly</u>, and that without using limits, we can find an approximation for M accurate to as many decimal places as we may desire.

It should also be noticed that the solution of this problem (again just as in the calculus of a single variable) does not require that we know anything about taking partial derivatives of functions of several (two) variables. To be sure, the arithmetic gets quite complicated. Indeed, in the present exercise, the density function is the relatively simple $x^2 + y^2$, and yet the arithemtic was already on the verge of being overwhelming (and this also happened in our study of a single variable; that is, when we found the area under the curve $y = x^2$, above the x-axis and between the lines $x = 0$ and $x = 1$, the computation of the infinite sum was tedious).

In the next unit, we shall establish a corresponding Fundamental Theorem of Integral Calculus for the calculus of several variables, and find more pleasant ways of computing masses and other related numbers.

d. The answer here is the same as that in the previous part of this exercise, namely $\frac{2}{3}$. The reason for this is that the double infinite sums that we evaluated in part (c) also yield upper and lower bounds for the volume of S. That is, if we now use $\rho(x,y)$ to denote the height of the solid S above the point (x,y) in the xy-plane, an element of volume of S is bounded between $\underline{\rho_{ij}} \Delta x_i \Delta y_j$ and $\overline{\rho_{ij}} \Delta x_i \Delta y_j$.

We shall not belabor the details here (hopefully, they will become clearer as we proceed through the block), but we do want to point out that there are often many different physical examples that lead to the same double infinite sum, and that consequently, evaluating one such sum may yield the answer to several different concrete problems. More importantly, again just as in the case of Part 1 of our course, we should learn to understand the double infinite sum abstractly and to think of the interpretations given in parts (c) and (d) of this exercise as simply two rather common applications for which one is interested in obtaining the value of this sum.

5.1.6

Assuming that for small ΔA_{ij}, $\rho_{ij} \approx$ constant, we may evaluate the mass of PQRS in Figure 1 of the previous exercise by saying

$$\Delta M_{ij} \approx \rho\left(\frac{i}{n},\frac{j}{m}\right) \frac{1}{n} \frac{1}{m}$$

$$= \left[\frac{ij}{mn}\right]\left[\frac{1}{mn}\right]$$

$$= \frac{ij}{(mn)^2}$$

Therefore,

$$M \approx \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{ij}{(mn)^2}$$

$$= \frac{1}{(mn)^2} \sum_{i=1}^{n} \sum_{j=1}^{m} ij,$$

[and by Exercise 5.1.3, part (b)],

$$= \frac{1}{(mn)^2} \left(\sum_{i=1}^{n} i\right)\left(\sum_{j=1}^{m} j\right). \tag{1}$$

Since $\displaystyle\sum_{i=1}^{n} i = 1 + \ldots + n = \frac{n(n+1)}{2}$ and $\displaystyle\sum_{j=1}^{m} j = \frac{m(m+1)}{2}$, we obtain from (1) that

$$M \approx \frac{1}{(mn)^2} \left[\frac{n(n+1)}{2}\right]\left[\frac{m(m+1)}{2}\right] = \frac{(n+1)(m+1)}{4nm} \tag{2}$$

or, more suggestively,

5.1.6 continued

$$M \approx \frac{1}{4} \left(\frac{n+1}{n}\right)\left(\frac{m+1}{m}\right) = \frac{1}{4} \left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{m}\right) \tag{3}$$

and taking the limit in (3) as both m and n approach infinity, we conclude that

$$M = \frac{1}{4}.$$

In concluding this exercise, we should point out that we have de- liberately taken certain liberties in order to emphasize how we may arrive at the result without all of the computational details of the previous exercise. Unless ample theory is known, however, notice that equation (3) leaves a gap in our information that was not present in the previous exercise. For example, in obtaining the estimate for M given in (3), we do not have both an upper and a lower bound for the error in determining M. Rather we have assumed that all the error is "squeezed out" as the size of our "mesh" goes to zero. The validity of this result lies in a theorem (which is the counterpart of the one used in our study of calculus of a single variable) that if the density function is continuous, the value of M can be found by picking any point in an incremental rectangle. That is, while picking the point of minimum density and the point of maximum density gives us a good way to estimate M by obtaining upper and lower bounds, the exact value of M does not depend on the point we choose.

As far as this exercise is concerned, we should point out that this problem is very much like the previous one, even though the density function is different, in the sense that for positive values of x and y, xy is minimum when both x and y are minimum, and maximum when both x and y are maximum. In other words, if we again refer to Figure 1 of the previous exercise, notice that on each element of area, the point of least density still occurs at the lower left hand corner of the rectangle, and the point of maximum density occurs at the upper right hand corner of the rectangle. Thus, in this example, it is very easy to compute the upper and lower approximations of M as a function of m and n and then take the limit as both m and n approach infinity. These details are left for the interested reader, but it is easily checked that this pro- cedure "validifies" our technique of using (3) to deduce the exact value of M.

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