

Unit 5: More on Polar Coordinates

1. Overview

In Unit 3 we discussed the role of the Jacobian determinant in evaluating double integrals involving a change of variables. This technique generalizes very nicely to triple integrals. In fact, except that the diagrams are harder to visualize in 3-dimensional space, the theory is precisely the same as in the 2-dimensional case.

In particular, if R is now a 3-dimensional region and $\underline{f}: E^3 \rightarrow E^3$ such that \underline{f} is 1-1 and onto on R , then if \underline{f} is defined by $\underline{f}(x, y, z) = (u, v, w)$, we have that

$$\iiint_R dv_R = \iiint_{S = \underline{f}(R)} \frac{\partial(x, y, z)}{\partial(u, v, w)} dv_S. \quad (1)$$

One very common application of (1) is in the case of "3-dimensional polar coordinates" (which we shall treat more rigorously very soon) where, just as in the 2-dimensional case, we can replace equation (1) by a rather convenient geometric argument (which is the procedure used in Sections 16.6 and 16.8 of the text).

Before talking about this extension of polar coordinates, notice that the analytic proof of equation (1) can be made without recourse to a drawing; hence, its validity carries over to any higher order dimensional space.

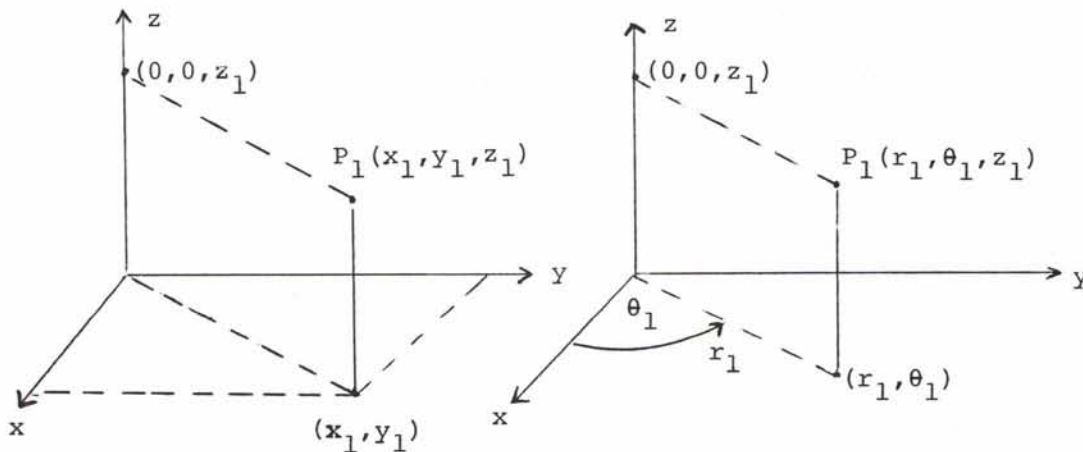
For example, if R is a region of E^n and $\underline{f}: E^n \rightarrow E^n$ and \underline{f} is 1-1 and onto on R , then if $S = \underline{f}(R)$ and $\underline{f}(x_1, \dots, x_n) = (u_1, \dots, u_n)$ we have that

$$\begin{aligned} \int \int \dots \int dv_R &= \int \int \dots \int dx_1 \dots dx_n \\ &= \int \int \dots \int_S \frac{\partial(x_1, \dots, x_n)}{\partial(u_1, \dots, u_n)} dv_S. \end{aligned} \quad (1')$$

The difficult part of (1') is writing the right hand side as an iterated integral with respect to u_1, \dots, u_n . In other words, we must express u_1 in terms of u_2, \dots, u_n ; u_2 in terms of u_3, \dots, u_n , etc., and this may be very complicated, the degree of complication depending upon how u_1, \dots, u_n are related in terms of x_1, \dots, x_n .

For our purposes, there is no need to delve beyond triple integrals, and this shall be the pursuit of this unit.

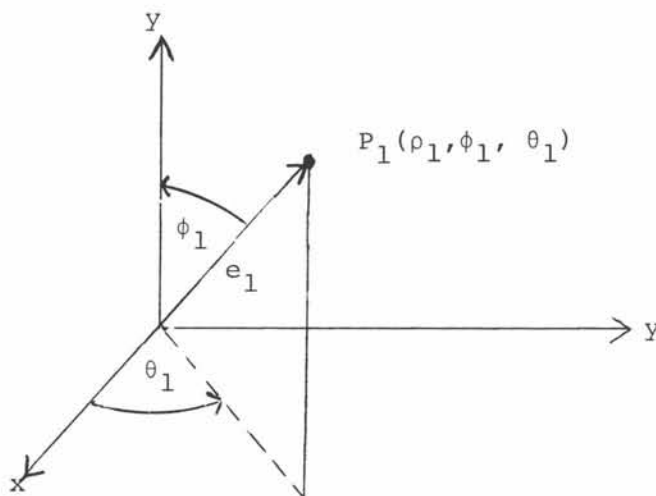
Now for a few words about 3-dimensional polar coordinates. When we decided to extend 2-dimensional Cartesian coordinates to 3-dimensional Cartesian coordinates, it seemed rather natural to augment the xy -plane by the z -axis and locate points by the triplet (x, y, z) rather than by the doublet (x, y) . Thus, it would seem natural that to extend polar coordinates to 3-dimensional space, we would need only augment the r and θ coordinates by a z -coordinate. In this context, then (r, θ, z) would refer to a point z units above the xy -plane and the projection of this point onto the xy -plane would be named by (r, θ) in polar coordinates. Pictorially,



When polar coordinates are extended to 3-dimensional space in this way, the resulting coordinate system is known under the name of cylindrical coordinates.

There is another way, however, that one could have elected to extend the concept of polar coordinates to 3-dimensional space. We could have decided that the important idea was that of a position (or radius) vector. In other words, it might be that

one of the coordinates should measure the distance of the point in space from the origin. This distance is usually denoted by ρ and it means in 3-space what r means in 2-space; namely, the distance from the origin to the point. In any event, if we decide to extend polar coordinates in this way, it is conventional that we use the angle ϕ to measure the angle between the line from the origin to the point and the positive z-axis. We then let θ , just as in the case of 2-dimensional polar coordinates, measure the angle between the positive x-axis and the line that joins the origin to the projection of the point in the xy-plane. Again, pictorially,



When this system is used, the resulting coordinate system is known as spherical coordinates. Why we use the names cylindrical and spherical will become clearer in the development of the unit, but the important point is that both of these systems, while they are quite different, are nevertheless both natural extensions of polar coordinates to 3-dimensional space.

2. Read: Thomas; Section 12.4.
3. Do Exercises 5.5.1, 5.5.2, 5.5.3.
4. Read: Thomas; Sections 16.6, 16.8 (as an optional assignment you may also read 16.7 which supplies some practical motivations for the use of polar coordinates).
5. Do the remaining exercises.

5.5.1(L)

Describe explicitly the set of all points

$$S = \left\{ (\rho, \phi, \theta) : \rho = 6, \phi = \frac{\pi}{4}, \theta = \frac{\pi}{3} \right\}.$$

5.5.2

Describe the set S if

$$S = \left\{ (r, \theta, z) : r = a \right\}.$$

5.5.3

Describe each of the following sets S.

a. $S = \left\{ (\rho, \phi, \theta) : \rho = 3 \cap (\rho, \phi, \theta) : \theta = \frac{\pi}{4} \right\}$
[i.e., $S = \left\{ (\rho, \phi, \theta) : \rho = 3 \text{ and } \theta = \frac{\pi}{4} \right\}$]

b. $S = \left\{ (\rho, \phi, \theta) : \phi = \theta = \frac{\pi}{4} \right\}$

5.5.4(L)

The solid sphere $S = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$ has a variable density ρ . In fact at any point $(x, y, z) \in S$, $\rho(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ (i.e., ρ is equal to the distance from the point to the origin). Use spherical coordinates to compute the mass of S.

5.5.5

R is the portion of the cylinder $x^2 + y^2 = 4$ bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $z = 5$. Use cylindrical coordinates to compute $\int \int_R xz \, dz \, dy \, dx$.

5.5.6

Let R be the region contained between the two spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$ ($a < b$). Compute $\int \int_R z^2 \, dz \, dy \, dx$.

5.5.7(L)

Compute $\iint_R x^2 y^2 dy dx$ where R is the elliptic region

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1.$$

Use a transformation (change of variables) which results in a simpler double integral.

5.5.8

Compute $\iiint_R xyz dz dy dx$ if R is the portion of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$$

which lies in the first octant.

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Resource: Calculus Revisited: Multivariable Calculus
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