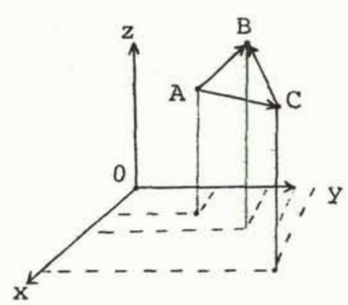
Unit 3: Applications to 3-Dimensional Space

1.3.1(L)

Note: The study of lines and planes comes up later in our course after we have studied the additional concepts of the dot and the cross products for vectors. The point is that we already have all the ingredients which are necessary to handle lines and planes, even though we lack a few more powerful computational tools. Our feeling is that the equations of lines and planes play such a vital role in the study of real functions of two variables that we want the student to have as much familiarity with these equations as is possible. It is our thought that equations of lines and planes introduced here will make their later study much easier to grasp when they are reintroduced later in the course.

a. The main aim of this exercise is to point out that the structure of vector arithmetic, as developed in the first two units of this Block, is independent of whether we think in terms of two dimensional space or three dimensional space. For example, in the previous section we saw that if A, B, and C were any three points in the (Cartesian) plane then $\vec{AB} = \vec{AC} + \vec{CB}$. The key point now is to notice that this same result is true even if A, B, and C are considered as points in 3-space. Pictorially,



Notice that A, B, and C determine a plane, so at least $\vec{AB}$, $\vec{AC}$, and $\vec{BC}$ are in the same plane.

With this in mind, we have

$$\vec{OP} = \vec{OA} + \vec{AP} \tag{1}$$

and since P is on AB, we have that $\vec{AP} = t\,\vec{AB}$, where t is a scalar variable which depends on the position of P along the line which joins A and B.

1.3.1(L) continued

Putting this into (1), we obtain

$$\vec{OP} = \vec{OA} + t\,\vec{AB}. \tag{2}$$

Again recalling that $\vec{AB} = \vec{AO} + \vec{OB}$ and that $\vec{AO} = -\vec{OA}$, equation (2) becomes

$$\begin{aligned}\vec{OP} &= \vec{OA} + t(-\vec{OA} + \vec{OB})\\ &= \vec{OA} - t\,\vec{OA} + t\,\vec{OB}\\ &= (1 - t)\vec{OA} + t\,\vec{OB}. \qquad (3)\end{aligned}$$

We hasten to point out that it is almost as important to recognize that (3) looks exactly as it did when we solved the same problem in the last unit (when we were restricted to the xy-plane rather than to xyz-space) as it is to see that (3) is the correct answer.

Now that we have solved part a. of this exercise, we observe that the plan of attack on part b. again mimics what we did in the last section. Now, however, we are in three dimensional space rather than in two dimensional space. More specifically,

b. If we let $A = (a_1,a_2,a_3)$ and $B = (b_1,b_2,b_3)$, then we know that if O denotes the origin, the vector $\vec{OA}$ has components equal to a_1, a_2, and a_3, respectively. That is,

$$\vec{OA} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

and, in a similar way,

$$\vec{OB} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}.$$

Since we still multiply component by component to find scalar multiples of a vector, we see that

$$(1 - t)\vec{OA} = (1 - t)a_1\vec{i} + (1 - t)a_2\vec{j} + (1 - t)a_3\vec{k}$$

while

$$t\,\vec{OB} = t\,b_1\vec{i} + tb_2\vec{j} + t\,b_3\vec{k}.$$

1.3.1(L) continued

Adding component by component (which still holds in 3-dimensional Cartesian coordinates), our result of part a. becomes

$$\vec{OP} = (a_1 - ta_1 + tb_1)\vec{i} + (a_2 - ta_2 + tb_2)\vec{j} + (a_3 - ta_3 + tb_3)\vec{k} \, .$$

If we now assume that P is represented as the point (x, y, z), we have that $\vec{OP} = x\vec{i} + y\vec{j} + z\vec{k}$, and since in Cartesian coordinates two vectors are equal if and only if they are equal component by component, the fact that $\vec{OP} = (1 - t)\vec{OA} + t\vec{OB}$ allows us to conclude that

$$\left.\begin{aligned} x &= a_1 - ta_1 + tb_1 \\ y &= a_2 - ta_2 + tb_2 \\ z &= a_3 - t_3 + tb_3 \, . \end{aligned}\right\} \qquad (4)$$

Notice in (4) that the a's and b's are given constants (since they are the coordinates of two fixed points) and that t is the only "unknown" on the right side of the equations. Since t appears only as a linear factor, we can easily solve each of the three equations in (4) for t, to obtain:

$$\left.\begin{aligned} \frac{x - a_1}{b_1 - a_1} &= t \\ \frac{y - a_2}{b_2 - a_2} &= t \\ \frac{z - a_3}{b_3 - a_3} &= t \, . \end{aligned}\right\} \qquad (5)$$

And (5) can be written on one line as:

$$\frac{x - a_1}{b_1 - a_1} = \frac{y - a_2}{b_2 - a_2} = \frac{z - a_3}{b_3 - a_3} \qquad (= t) \quad . \qquad (6)$$

1.3.1(L) continued

While we have now found the answer to b. there are a few remarks worth making about equation (6). For one thing, get used to the form of (6) so that you see at a glance that it is really three equations in three unknowns (that is, x, y, and z are the unknowns while the a's and b's denote coordinates of fixed points) and that it has but one degree of freedom. That is, as soon as we specify any one of the three coordinates of the point, equation (6) uniquely determines the other two coordinates if the point is to be on the given line. For example, as soon as we specify a value for x, $(x - a_1)/(b_1 - a_1)$ becomes a fixed number (constant) and equating this number to each of the expressions $(y - a_2)/(b_2 - a_2)$ and $(z - a_2)/(b_3 - a_3)$ gives us unique values for y and z.

Secondly, notice that, in Cartesian coordinates, if $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ then $\vec{AB}$ has as its components $b_1 - a_1$, $b_2 - a_2$, and $b_3 - a_3$. That is,

$$\vec{AB} = (b_1 - a_1)\vec{i} + (b_2 - a_2)\vec{j} + (b_3 - a_3)\vec{k} .$$

This, in turn, shows us that our denominators in (6) are the components of the vector $\vec{AB}$ (see the note at the end of this exercise for an additional remark).

Since this discussion may seem a bit abstract, we illustrate our above remarks in terms of a specific example. Namely,

c. We have A = (2, -3, 5) and B = (5, 4, -1). Therefore, if we wish to use the result of (6) mechanically we have

$$a_1 = 2,\ a_2 = -3,\ a_3 = 5$$

$$b_1 = 5,\ b_2 = 4,\ b_3 = -1$$

$$b_1 - a_1 = 3,\ b_2 - a_2 = 7,\ b_3 - a_3 = -6 .$$

Hence

$$\frac{x - 2}{3} = \frac{y + 3}{7} = \frac{z - 5}{-6} . \qquad (1)$$

As a check that (1) is correct, observe that we know that (2, -3, 5)

1.3.1(L) continued

and (5, 4, -1) must belong to the required line. Consequently, equation (1) must be satisfied by the sets of values $\{x = 2, y = -3, z = 5\}$ and $\{x = 5, y = 4, z = -1\}$.

A quick check shows that the first set yields

$$\frac{2 - 2}{3} = \frac{-3 + 3}{7} = \frac{5 - 5}{-6} \text{ or } 0 = 0 = 0, \text{ which checks,}$$

and the second set yields

$$\frac{5 - 2}{3} = \frac{4 + 3}{7} = \frac{-1 - 5}{-6} \text{ or } 1 = 1 = 1, \text{ which also checks.}$$

Had we wished to proceed from scratch without reference to b. we could have let $P(x, y, z)$ denote any point on AB. Then

$$\begin{aligned}
\vec{OP} &= \vec{OA} + \vec{AP} \\
&= \vec{OA} + t\vec{AB} \\
&= \vec{OA} + t(\vec{AO} + \vec{OB}) \\
&= \vec{OA} + t(-\vec{OA} + \vec{OB}) \\
&= \vec{OA} - t\vec{OA} + t\vec{OB} \\
&= (1 - t)(2\vec{i} - 3\vec{j} + 5\vec{k}) + t(5\vec{i} + 4\vec{j} - \vec{k}) \\
&= (2 - 2t + 5t)\vec{i} + (-3 + 3t + 4t)\vec{j} + (5 - 5t - t)\vec{k}, \text{ or:}
\end{aligned}$$

$$x\vec{i} + y\vec{j} + z\vec{k} = (2 + 3t)\vec{i} + (7t - 3)\vec{j} + (5 - 6t)\vec{k}$$

Therefore, $\left\{\begin{array}{l} x = 2 + 3t \text{ or } \frac{x - 2}{3} = t \\ y = 7t - 3 \text{ or } \frac{y + 3}{7} = t \\ z = 5 - 6t \text{ or } \frac{z - 5}{-6} = t. \end{array}\right\}$

Therefore, $\frac{x - 2}{3} = \frac{y + 3}{7} = \frac{z - 5}{-6}$.

1.3.1(L) continued

We shall talk more about equations of lines later, but, for now, it is our main aim to make sure that the concept of vector arithmetic is becoming more familiar. In this example, we are again trying to show that the structure of vector arithmetic is independent of whether we are dealing in 2-space or 3-space.

A Note on Equations of Lines

We have just seen that the equation of the line which passes through (a_1, a_2, a_3) and (b_1, b_2, b_3) is given by

$$\frac{x - a_1}{b_1 - a_1} = \frac{y - a_2}{b_2 - a_2} = \frac{z - a_3}{b_3 - a_3} \quad (= t). \tag{1}$$

While we may not be used to the equations of lines in 3-space the fact is that this equation is a "natural extension" of equations of lines in 2-space.

For example, if $A = (a_1, a_2)$ and $B = (b_1, b_2)$ with $a_1 \neq b_1$, we have that the slope of our line is

$$\frac{b_2 - a_2}{b_1 - a_1}.$$

Then if (x, y) is any point on our line, we have

$$\frac{y - a_2}{x - a_1} = \frac{b_2 - a_2}{b_1 - a_1},$$

which is the special case of (1) in the event $a_3 = b_3 = 0$.

1.3.2

a. $A = (3, 5, 1)$, $B = (7, 2, 4)$

Therefore, $\vec{AB} = (7 - 3)\vec{i} + (2 - 5)\vec{j} + (4 - 1)\vec{k}$

$$= 4\vec{i} - 3\vec{j} + 3\vec{k}.$$

Therefore, $\frac{x - 3}{4} = \frac{y - 5}{-3} = \frac{z - 1}{3}$. (1)

1.3.2 continued

b. The point (x, y, z) at which the line intersects the xy-plane is characterized by the fact that z = 0. With z = 0, (1) becomes:

$$\frac{x-3}{4} = \frac{y-5}{-3} = \frac{0-1}{3}.$$

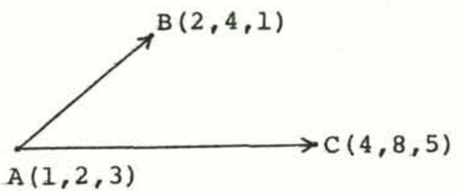
But $\frac{0-1}{3} = \frac{-1}{3}$.

Therefore, we must have

$$\left.\begin{array}{l} \frac{x-3}{4} = \frac{-1}{3} \text{ or } 3x - 9 = -4 \text{ therefore } x = \frac{5}{3} \\ \\ \frac{y-5}{-3} = \frac{-1}{3} \text{ or } 3y - 15 = 3 \text{ therefore } y = 6\ . \end{array}\right\}$$

Therefore, the line meets the xy-plane at $(\frac{5}{3}, 6, 0)$.

1.3.3



(Notice that there is no need to try to draw things to scale since our vector methods work with the <u>values</u> of the coordinates of the points, not with the geometry.)

We saw in the previous unit that a required vector would be

$$|\vec{AC}|\ \vec{AB} + |\vec{AB}|\ \vec{AC} \tag{1}$$

and that any other solution would be a scalar multiple of (1).

The point is that only the computation, not the theory, is affected by a switch to 3-space from 2-space.

Noticing that

1.3.2 continued

$$\vec{AB} = (2 - 1)\vec{i} + (4 - 2)\vec{j} + (1 - 3)\vec{k} = \vec{i} + 2\vec{j} - 2\vec{k}$$

$$\vec{AC} = (4 - 1)\vec{i} + (8 - 2)\vec{j} + (5 - 3)\vec{k} = 3\vec{i} + 6\vec{j} + 2\vec{k},$$

we have:

$$|\vec{AB}| = \sqrt{1^2 + 2^2 + (-2)^2} = \sqrt{1 + 4 + 4} = 3$$

$$|\vec{AC}| = \sqrt{3^2 + 6^2 + 2^2} = \sqrt{9 + 36 + 4} = 7 .$$

Putting this into (1), we see that one correct vector is:

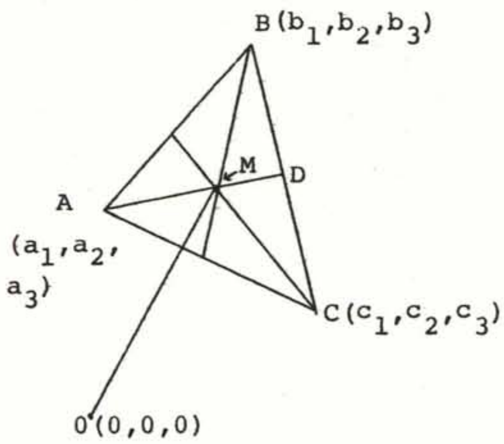
$$7(\vec{i} + 2\vec{j} - 2\vec{k}) + 3(3\vec{i} + 6\vec{j} + 2\vec{k}) = 7\vec{i} + 14\vec{j} - 14\vec{k} + 9\vec{i} + 18\vec{j} + 6\vec{k}$$

$$= 16\vec{i} + 32\vec{j} - 8\vec{k} = 8(2\vec{i} + 4\vec{j} - \vec{k}).$$

Therefore, any scalar multiple of $2\vec{i} + 4\vec{j} - \vec{k}$ is a solution.

1.3.4

a. Again, every step that we used in Exercise 1.2.5 applies verbatim.



$$\vec{OM} = \vec{OA} + \vec{AM}$$

$$= \vec{OA} + \frac{2}{3}\vec{AD}$$

$$= \vec{OA} + \frac{2}{3}(\vec{AB} + \vec{BD})$$

1.3.2 continued

$$= \vec{OA} + \frac{2}{3}\vec{AB} + \frac{2}{3}\vec{BD}$$

$$= \vec{OA} + \frac{2}{3}(\vec{AO} + \vec{OB}) + \frac{2}{3}(\frac{1}{2}\vec{BC})$$

$$= \vec{OA} + \frac{2}{3}\vec{AO} + \frac{2}{3}\vec{OB} + \frac{1}{3}\vec{BC}$$

$$= \vec{OA} - \frac{2}{3}\vec{OA} + \frac{2}{3}\vec{OB} + \frac{1}{3}(\vec{BO} + \vec{OC})$$

$$= \vec{OA} - \frac{2}{3}\vec{OA} + \frac{2}{3}\vec{OB} + \frac{1}{3}\vec{BO} + \frac{1}{3}\vec{OC}$$

$$= \vec{OA} - \frac{2}{3}\vec{OA} + \frac{2}{3}\vec{OB} - \frac{1}{3}\vec{OB} + \frac{1}{3}\vec{OC}$$

$$= \frac{1}{3}\vec{OA} + \frac{1}{3}\vec{OB} + \frac{1}{3}\vec{OC}$$

$$= \frac{1}{3}(\vec{OA} + \vec{OB} + \vec{OC}).$$

b. Utilizing Cartesian Coordinates and letters $M = (m_1, m_2, m_3)$ we see that:

$$m_1\vec{i} + m_2\vec{j} + m_3\vec{k} = \frac{1}{3}[(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) + (b_1\vec{i} + b_2\vec{j} + b_3\vec{k}) + (c_1\vec{i} + c_2\vec{j} + c_3\vec{k})]$$

$$= \frac{(a_1 + b_1 + c_1)}{3}\vec{i} + \frac{(a_2 + b_2 + c_2)}{3}\vec{j} + \frac{(a_3 + b_3 + c_3)}{3}\vec{k}$$

Therefore $m_1 = \frac{a_1 + b_1 + c_1}{3}$, $m_2 = \frac{a_2 + b_2 + c_2}{3}$, $m_3 = \frac{a_3 + b_3 + c_3}{3}$.

c. The medians of ΔABC meet at

$$\left(\frac{1 + 2 + 4}{3}, \frac{2 + 4 + 8}{3}, \frac{3 + 1 + 5}{3}\right) = \left(\frac{7}{3}, \frac{14}{3}, 3\right).$$

d. If we let C be denoted by (C_1, C_2, C_3), we want:

$$\left(\frac{1 + 2 + C_1}{3}, \frac{2 + 4 + C_2}{3}, \frac{3 + 1 + C_3}{3}\right) = (0, 0, 0)$$

1.3.4. continued

Therefore, $\frac{1 + 2 + c_1}{3} = 0$ or $c_1 = -3$

$\frac{2 + 4 + c_2}{3} = 0$ or $c_2 = -6$

$\frac{3 + 1 + c_3}{3} = 0$ or $c_3 = -4$.

Therefore, if the medians of ΔABC are to meet at the origin, then C must be the point (-3, -6, -4).

1.3.5(L)

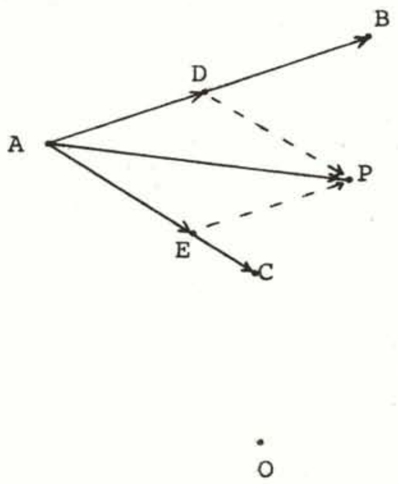
Perhaps the only major difference between this exercise and any of the others is that we are now interested in planes rather than lines. It is also worth noting that the equation of the plane is not given in terms of the Cartesian coordiantes of the points A, B, C, and P. While we might feel more at home with x, y, and z as variables, the fact remains that the form in which we have stated the equation of the plane depends only on the points which determine the plane, and are, thus, free of any particular coordinate system. In particular, this means that our present equation is valid, regardless of the coordinate system under consideration. Now, we know that if a point P is on the line determined by two points A and B, then, vectorially, we have that there exists a scalar t such that

$\vec{AP} = t\,\vec{AB}$.

Suppose now that we are given three points A, B, and C not on the same straight line. In this case, the three points determine a plane. Vectorially, this plane is determined by the vectors $\vec{AB}$ and $\vec{AC}$. Thus, the point P is in this plane if and only if there exist two scalars t_1 and t_2 such that

$\vec{AP} = t_1\vec{AB} + t_2\vec{AC}$ (see diagram on next page).

1.3.5(L) continued



(1) Since A, B, and C are not on the same line, $\vec{AB}$ and $\vec{AC}$ are not parallel, Hence, they determine a plane.

(2) If P is in the plane we may draw two lines through P, one parallel to $\vec{AB}$ and the other parallel to $\vec{AC}$, thus forming the parallelogram ADPE.

(3) $\vec{AP} = \vec{AD} + \vec{DP}$, and since ADPE is a parallelogram, $\vec{DP} = \vec{AE}$. Therefore, $\vec{AP} = \vec{AD} + \vec{AE}$; but $\vec{AD}$ is a scalar multiple of $\vec{AB}$ and $\vec{AE}$ is a scalar multiple of $\vec{AC}$. Hence, by definition, there exist scalars t_1 and t_2 such that $\vec{AD} = t_1 \vec{AB}$ and $\vec{AE} = t_2 \vec{AC}$.

Therefore, $\vec{AP} = t_1 \vec{AB} + t_2 \vec{AC}$.

(4) Notice that t_1 and t_2 are variables that depend on the choice of P. t_1 measures the relative length of the "parallel" projection of $\vec{AP}$ onto $\vec{AB}$ (and in this respect if $\sphericalangle BAP > 90°$ t_1 will be negative) while t_2 does the same for the projection of $\vec{AP}$ onto $\vec{AC}$. It should be clear that if $\vec{AP} = t_1 \vec{AB} + t_2 \vec{AC}$ and $\vec{AP_1} = t_3 \vec{AB} + t_4 \vec{AC}$ then $P = P_1$ if and only if $t_1 = t_3$ and $t_3 = t_4$, for if the points are different at least one of the projected lengths must differ.

1.3.5(L) continued

In this respect then, not only can we relate A, B, C, and P by

$$\vec{AP} = t_1 \vec{AB} + t_2 \vec{AC}$$

but (since $\vec{AB}$ is not parallel to AC) we may also conclude that for a given P, the choices of t_1 and t_2 are unique. (In other words, $t_1 \vec{AB}$ is along the line AB and $t_2 \vec{AC}$ is along the line AC, so the sum of two such vectors lies in the plane determined by $\vec{AB}$ and $\vec{AC}$ since the plane depends on the location of the lines, not on their lengths.) We now proceed systematically by writing

$$\vec{OP} = \vec{OA} + \vec{AP}. \qquad (1)$$

Then, since P is in the plane determined by A,B, and C, we have that there are scalars t_1 and t_2 such that

$$\vec{AP} = t_1 \vec{AB} + t_2 \vec{AC}. \qquad (2)$$

If we then put (2) into (1), we obtain

$$\vec{OP} = \vec{OA} + t_1 \vec{AB} + t_2 \vec{AC}. \qquad (3)$$

Now, as we have already done several times, we write

$$\vec{AB} = \vec{AO} + \vec{OB} = \vec{OB} - \vec{OA}$$

and

$$\vec{AC} = \vec{AO} + \vec{OC} = \vec{OC} - \vec{OA}.$$

Putting these results into (3), we obtain

1.3.5(L) continued

$$\vec{OP} = \vec{OA} + t_1(\vec{OB} - \vec{OA}) + t_2(\vec{OC} - \vec{OA})$$

and we then expand and collect terms (again, just as in ordinary arithmetic since the rules of the games are alike) to obtain the desired result:

$$\vec{OP} = (1 - t_1 - t_2)\vec{OA} + t_1\ \vec{OB} + t_2\ \vec{OC}.$$

We may wish to observe that this recipe seems to be a natural extension of the result for a line, namely

$$\vec{OP} = (1 - t_1)\vec{OA} + t_1\ \vec{OB}$$

where we have written t_1 rather than t simply to emphasize the similarity in the expressions.

1.3.6(L)

a. We have that

$$\vec{OP} = (1 - t_1 - t_2)\vec{OA} + t_1\ \vec{OB} + t_2\ \vec{OC}$$

$$= (1 - t_1 - t_2)(\vec{i} + 2\vec{j} + 3\vec{k}) + t_1(2\vec{i} + 4\vec{j} + 5\vec{k}) + t_2(4\vec{i} + 5\vec{j} + 7\vec{k})$$

$$= (1 - t_1 - t_2 + 2t_1 + 4t_2)\vec{i} + (2 - 2t_1 - 2t_2 + 4t_1 + 5t_2)\vec{j}$$
$$+ (3 - 3t_1 - 3t_2 + 5t_1 + 7t_2)\vec{k}$$

$$= (1 + t_1 + 3t_2)\vec{i} + (2 + 2t_1 + 3t_2)\vec{j} + (3 + 2t_1 + 4t_2)\vec{k}. \quad (1)$$

Letting P = (x,y,z), equation (1) becomes

1.3.6(L) continued

$$x\vec{i} + y\vec{j} + z\vec{k} = (1 + t_1 + 3t_2)\vec{i} + (2 + 2t_1 + 3t_2)\vec{j} + (3 + 2t_1 + 4t_2)\vec{k}.$$

Therefore,

$$\left.\begin{aligned} x &= 1 + t_1 + 3t_2 \\ y &= 2 + 2t_1 + 3t_2 \\ z &= 3 + 2t_1 + 4t_2 \end{aligned}\right\} \qquad (2)$$

where x, y, and z appear as linear combinations of t_1 and t_2, as we expect should happen in a plane.

b. Recall that t_1 and t_2 were defined by

$$\vec{AP} = t_1\,\vec{AB} + t_2\,\vec{AC}.$$

In particular if $P = A$, $\vec{AP} = \vec{AA} = \vec{0}$. Therefore, $\underline{t_1 = t_2 = 0.}$ (Remember in this case $\vec{AP} = \vec{0} = 0\,\vec{AB} + 0\,\vec{AC}$, and as we remarked in 1.3.5, t_1 and t_2 are unique for a given P.)

In this case (2) yields

$$\left.\begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned}\right\} \text{ as required.}$$

If $P = B$, then

$$\vec{AP} = t_1\,\vec{AB} + t_2\,\vec{AC}$$

implies

1.3.6(L) continued

$\vec{AB} = t_1 \vec{AB} + t_2 \vec{AC}$.

One set of choices that will obviously work is $t_1 = 1$, $t_2 = 0$ since then we obtain $\vec{AB} = \vec{AB}$. Since t_1 and t_2 are unique, it must be that $t_1 = 1$ and $t_2 = 0$, whereupon (2) becomes

$$\left.\begin{array}{l} x = 1 + 1 + 0 = 2 \\ y = 2 + 2 + 0 = 4 \\ z = 3 + 2 + 0 = 5 \end{array}\right\} \text{ as required } B = (2,4,5).$$

Finally if $P = C$; $\vec{AP} = t_1 \vec{AB} + t_2 \vec{AC}$ implies that

$\vec{AC} = t_1 \vec{AB} + t_2 \vec{AC}$. Therefore, $\underline{t_1 = 0,\ t_2 = 1.}$

Then (2) becomes

$$\left.\begin{array}{l} x = 1 + 0 + 3(1) = 4 \\ y = 2 + 0 + 3(1) = 5 \\ z = 3 + 0 + 4(1) = 7 \end{array}\right\} \text{ which checks since } C = (4,5,7).$$

c. $t_1 = t_2 = 1$ implies that P is defined by

$\vec{AP} = \vec{AB} + \vec{AC}$.

But $\vec{AB} + \vec{AC}$ is a diagonal of the parallelogram determined by $\vec{AB}$ and $\vec{AC}$. That is, P is the point such that the quadrilateral ABCP is a parallelogram.

From (2) with $t_1 = t_2 = 1$ we see that

$$\left.\begin{array}{l} x = 1 + 1 + 3 = 5 \\ y = 2 + 2 + 3 = 7 \\ z = 3 + 2 + 4 = 9 \end{array}\right\} \text{ therefore } P = (5,7,9).$$

d. We know that whenever (x,y,z) is in the plane

$$\begin{aligned} x &= 1 + t_1 + 3t_2 \\ y &= 2 + 2t_1 + 3t_2 \\ z &= 3 + 2t_1 + 4t_2. \end{aligned}$$

1.3.6(L) continued

Therefore, if $x = 3$ and $y = 4$ we must have:

$$\left.\begin{aligned} 3 &= 1 + t_1 + 3t_2 \\ 4 &= 2 + 2t_1 + 3t_2 \end{aligned}\right\} \text{or} \quad \left.\begin{aligned} t_1 + 3t_2 &= 2 \\ 2t_1 + 3t_2 &= 2 \end{aligned}\right\} \text{therefore, } t_1 = 0 \text{ and } t_2 = \tfrac{2}{3}.$$

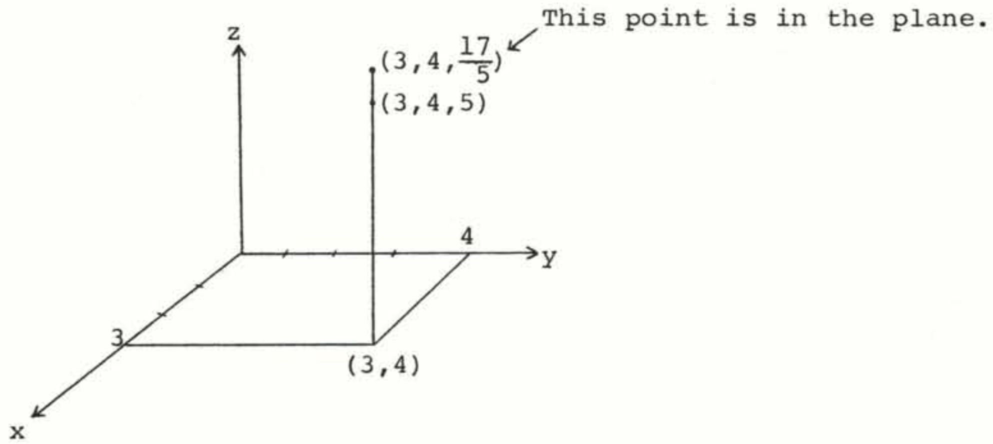
(In other words, if $(3,4,z)$ is in the plane it must lie on $\vec{AC}$ since $\vec{AP} = t_1\,\vec{AB} + t_2\,\vec{AC} \rightarrow \vec{AP} = \frac{2}{3}\,\vec{AC} \rightarrow AP \parallel AC$.)

But, now that we know t_1 and t_2, z is completely determined. Namely,

$$z = 3 + 2t_1 + 4t_2 \rightarrow z = 3 + 2(0) + 4(\tfrac{2}{3}) = 3 + \tfrac{8}{3} = \tfrac{17}{3}.$$

Therefore, $(3,4,\frac{17}{3})$ is in the plane.

Therefore, $(3,4,5)$ is below the plane since $5 < \frac{17}{3}$. That is, both $(3,4,5)$ and $(3,4,\frac{17}{5})$ are on the line parallel to the z-axis which passes through $(3,4)$ in the xy-plane.



1.3.7

a. $\vec{OP} = (1 - t_1 - t_2)\vec{OA} + t_1\vec{OB} + t_2\vec{OC}$.

1.3.7 continued

Therefore, $x\vec{i} + y\vec{j} + z\vec{k} = (1 - t_1 - t_2)(2\vec{i} + 3\vec{j} + 4\vec{k}) + t_1(3\vec{i} + \vec{j} + 2\vec{k}) + t_2(4\vec{i} + 2\vec{j} + 5\vec{k})$

$$= (2 - 2t_1 - 2t_2 + 3t_1 + 4t_2)\vec{i} + (3 - 3t_1 - 3t_2 + t_1 + 2t_2)\vec{j} + (4 - 4t_1 - 4t_2 + 2t_1 + 5t_2)\vec{k}$$

$$= (2 + t_1 + 2t_2)\vec{i} + (3 - 2t_1 - t_2)\vec{j} + (4 - 2t_1 + t_2)\vec{k}.$$

Therefore,
$$\left.\begin{aligned} x &= 2 + t_1 + 2t_2 \\ y &= 3 - 2t_1 - t_2 \\ z &= 4 - 2t_1 + t_2. \end{aligned}\right\} \qquad (1)$$

In (1) if we let $t_1 = t_2 = 0$ we obtain (2,3,4). If we let $t_1 = 1$, $t_2 = 0$, we obtain (3,1,2); and if we let $t_1 = 0$, $t_2 = 1$, we obtain (4,2,5). Thus, the three points we are sure must belong to the plane "check out."

b. From (1) once we know that $x = 5$ and $y = 6$, we must have:

$$\left.\begin{aligned} 5 &= 2 + t_1 + 2t_2 \\ 6 &= 3 - 2t_1 - t_2 \end{aligned}\right\} \text{ or } \left.\begin{aligned} t_1 + 2t_2 &= 3 \\ -2t_1 - t_2 &= 3. \end{aligned}\right\}$$

We may solve simultaneously to obtain

$$\left.\begin{aligned} t_1 + 2t_2 &= 3 \\ -2t_1 - t_2 &= 3 \end{aligned}\right\} \rightarrow \left.\begin{aligned} 2t_1 + 4t_2 &= 6 \\ -2t_1 - t_2 &= 3 \end{aligned}\right\} \rightarrow 3t_2 = 9. \text{ Therefore,}$$

$$\underline{t_2 = 3}, \; t_1 + 2t_2 = 3 \;\rightarrow\; \underline{t_1 = -3}.$$

With $t_1 = -3$ and $t_2 = 3$, $z = 4 - 2t_1 + t_2 \rightarrow z = 4 + 6 + 3 = 13$. Hence, (5,6,13) is in the plane (and is obtained from (1) with $t_1 = -3$, $t_2 = 3$); consequently (5,6,14) is above the plane.

MIT OpenCourseWare
http://ocw.mit.edu

Resource: Calculus Revisited: Multivariable Calculus
Prof. Herbert Gross

The following may not correspond to a particular course on MIT OpenCourseWare, but has been provided by the author as an individual learning resource.

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.