1.1.9 continued

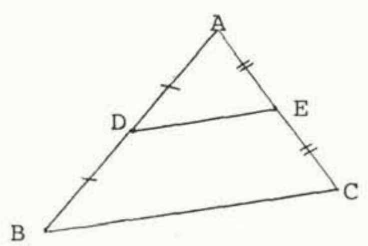
the "fact" that if $a \neq 0$ there exists a number a^{-1} such that $a \times a^{-1} = 1$. As we saw in part (c), (2), this need not happen in modular-6 arithmetic (in particular, when $a = 3$). Since the rule does not apply here, neither need any inescapable consequence of the rule apply.

Unit 2: Structure of Vector Arithmetic

1.2.1(L)

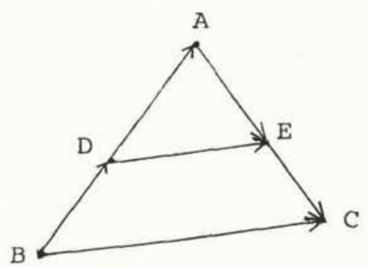
Aside from reviewing the basic vector operations, this exercise affords us the opportunity to see how vector arithmetic can be applied to plane geometry by the rather neat device of "arrowizing" (or "vectorizing") lines.

In terms of the usual plane geometry notation, we have



and we want to prove that DE||BC and $\overline{DE} = \frac{1}{2}\,\overline{BC}$*.

If we now "arrowize" the triangle we obtain:



In the language of vectors we are being asked to prove that:

$$\vec{DE} = \frac{1}{2}\,\vec{BC} \qquad (1)$$

Notice the beauty of our vector notation. Namely, equation (1)

*Our convention is to write AB, say, when we are <u>referring</u> to the line which passes through A and B, but to write $\overline{AB}$ when we are referring to the <u>length</u> of the line segment which extends from A to B.

1.2.1 continued

tells the whole story. That is, the fact that DE and BC are parallel is equivalent to saying that the vectors $\vec{DE}$ and $\vec{BC}$* are scalar multiples of one another; while the fact that the length of DE is half the length of BC is equivalent to saying in vector language that the scalar factor is $\frac{1}{2}$. In other words, using our notation for magnitude, $\vec{DE} = \frac{1}{2}\vec{BC}$ implies, aside from DE and BC being parallel, that $|\vec{DE}| = \frac{1}{2}|\vec{BC}|$. Notice now that a connection between line segments and vectors is that for any line segment AB, $\overline{AB} = |\vec{AB}|$. Thus $\vec{DE} = \frac{1}{2}\vec{BC}$ also says that the length of DE is half the length of BC.

To prove equation (1) we could employ vector arithmetic as follows:

(1) $\vec{DE} = \vec{DA} + \vec{AE}$** (by the definition of vector addition).

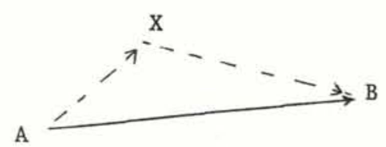
(2) $\vec{DA} = \frac{1}{2}\vec{BA}$, $\vec{AE} = \frac{1}{2}\vec{AC}$ (by the definition of scalar multiplication, that is, DA anb BA are parallel, and $\overline{DA} = \frac{1}{2}\overline{BA}$ since D is the midpoint of AB).

(3) $\vec{DE} = \frac{1}{2}\vec{BA} + \frac{1}{2}\vec{AC}$ (by substituting (2) into (1)).

---

*Our convention is to write $\vec{AB}$ if we wish to denote the arrow (vector) which originates at A (i.e., its "tail" is at A) and terminates at B (i.e., its "head" is at B).

**Notice, again, the interesting result that if A, B, and X are any three points in space then $\vec{AB} = \vec{AX} + \vec{XB}$. In terms of a "recipe," our definition of vector addition allows us, if given $\vec{AB}$, to "pull apart" A and B and insert a third point X after A and before B. Pictorially:



$\vec{AB} = \vec{AX} + \vec{XB}$

1.2.1 continued

(4) $\frac{1}{2}\vec{BA} + \frac{1}{2}\vec{AC} = \frac{1}{2}(\vec{BA}+\vec{AC})$ (by the distributive property of scalar multiplication).

(5) Therefore, $\vec{DE} = \frac{1}{2}(\vec{BA}+\vec{AC})$ (by substituting (4) into (3)).

(6) $\vec{BA} + \vec{AC} = \vec{BC}$ * (by definition of vector addition)

(7) Therefore, $\vec{DE} = \frac{1}{2}\vec{BC}$ (by substituting (6) into (5)). <u>(7) is the desired result</u>.

If we wish we can rewrite steps (1) through (7) in the statement-reason format of plane geometry without reference to any picture - and in this way we perhaps emphasize even more the structure of vector arithmetic.

In this context our problem and solution could be stated as follows:

<u>Given</u>: $\vec{DA} = \frac{1}{2}\vec{BA}$ and $\vec{AE} = \frac{1}{2}\vec{AC}$

<u>Prove</u>: $\vec{DE} = \frac{1}{2}\vec{BC}$

<u>Proof</u>:

Statement	Reason
1. $\vec{DE} = \vec{DA}+\vec{AE}$	1. Definition of Vector addition**
2. $\vec{DA} = \frac{1}{2}\vec{BA}$, $\vec{AE} = \frac{1}{2}\vec{AC}$	2. Given
3. $\vec{DE} = \frac{1}{2}\vec{BA} + \frac{1}{2}\vec{AC}$	3. Substitution
4. $\frac{1}{2}\vec{BA} + \frac{1}{2}\vec{AC} = \frac{1}{2}(\vec{BA}+\vec{AC})$	4. Distributive property of scalar multiplication
5. $\vec{DE} = \frac{1}{2}(\vec{BA}+\vec{AC})$	5. Substitution
6. $\vec{BA}+\vec{AC} = \vec{BC}$	6. Definition of vector addition
7. $\vec{DE} = \frac{1}{2}\vec{BC}$	7. Substitution q.e.d.

*This is an inverse, of sorts, of a previous footnote. In the same way, that given $\vec{AB}$, we can separate A and B to insert X to form $\vec{AB} = \vec{AX}+\vec{XB}$, given $\vec{AX}+\vec{XB}$ we can eliminate X and "squeeze" A and B together to form $\vec{AB}$. In still other words, $\vec{AX}+\vec{XB} = \vec{AB}$ may be obtained by reading $\vec{AB} = \vec{AX}+\vec{XB}$ from <u>right to left</u>.

**Notice that we may <u>define</u> $\vec{AX}+\vec{XB} = \vec{AB}$ without reference to any diagram. To be sure, this definition is motivated by what our pictorial intuition tells us, but this is certainly "legal."

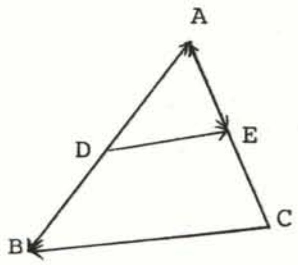
1.2.1 continued

Once the statement-reason format is fully understood and appreciated, it is conventional to "abbreviate" the proofs. For example, we might simply write

$$\begin{aligned} \vec{DE} &= \vec{DA} + \vec{AE} \\ &= \frac{1}{2}\vec{BA} + \frac{1}{2}\vec{AC} \\ &= \frac{1}{2}(\vec{BA} + \vec{AC}) \\ &= \frac{1}{2}\vec{BC} \end{aligned}$$

but we must be prepared at all times to support any statement of equality by a suitable definition, rule, hypothesis, or theorem if called upon to do so.

As a final note, let us observe that in "vectorizing" lines there is no set convention as to how we should do it. For example, had we wished, we could certainly have assigned the sense of each vector as follows:

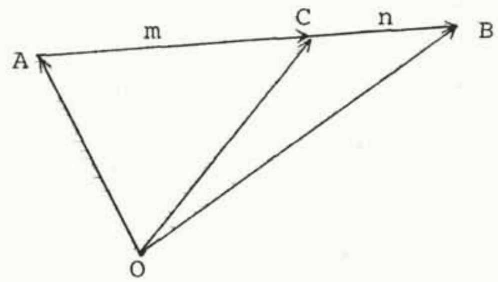


We would then have had to write things differently in terms of this diagram. For example, as drawn, $\vec{DE}$ and $\vec{CB}$ have opposite sense and our proof would be to show that $\vec{DE} = -\frac{1}{2}\vec{BC}$ etc. The point is that no vectors appear in the original problem; hence, we are free to "vectorize" at our own convenience. Accordingly, we will, to our own taste, pick a scheme which seems most advantageous for the particular problem under consideration.

As a final note, please notice that what we call "vector methods" is more than ordinary plane geometry in disguise. Our vector methods are much more algebraic than geometric in application.

1.2.2(L)

Using a picture as a mental-aid we have



a. $\vec{OC} = \vec{OA} + \vec{AC}$ (1)

The length of AC is $\frac{m}{m+n}$ of the length of AB. Hence, vectorially, $\vec{AC} = \frac{m}{m+n}\vec{AB}$.

Therefore

$$\vec{OC} = \vec{OA} + \frac{m}{m+n}\vec{AB} \quad (2)$$

The only thing "wrong" with (2) is that we do not want our answer expressed in terms of $\vec{AB}$ but rather in terms of $\vec{OA}$ and $\vec{OB}$. With this in mind, we rewrite $\vec{AB}$ as $\vec{AO} + \vec{OB}$ (which is the definition of vector addition) and we then note that $\vec{AO} = -\vec{OA}$ since $\vec{AO}$ and $\vec{OA}$ have the same magnitude and direction, but the opposite sense.

Hence, (2) may be rewritten as

$$\begin{aligned}\vec{OC} &= \vec{OA} + \frac{m}{m+n}(\vec{AO} + \vec{OB}) \\ &= \vec{OA} + \frac{m}{m+n}(-\vec{OA} + \vec{OB}) \\ &= \vec{OA} - \frac{m}{m+n}\vec{OA} + \frac{m}{m+n}\vec{OB} \\ &= \left(1 - \frac{m}{m+n}\right)\vec{OA} + \frac{m}{m+n}\vec{OB}\end{aligned}$$

Therefore

$$\vec{OC} = \frac{n}{m+n}\vec{OA} + \frac{m}{m+n}\vec{OB}$$

1.2.2 continued

Notice that our string of equalities seems to be derived as if we were using ordinary "algebra" applied to numbers. In terms of our "game" idea, this occurs because vectors and numbers obey the same rules or properties (i.e., they have the same structure) with respect to the given operations. In still other words, notice that while we may have used a picture to help us visualize what was happening, we could have, once again, proceeded in a purely algebraic statement-reason format to deduce our result <u>without reference to any geometry</u>.

b. Here, for the first time, we come to grips with the role of a specific coordinate system. That is, it is reasonable for us to assume that while the sum of two vectors depends only on the vectors and not on the coordinate system being employed, the actual arithmetic computations may very well depend on the coordinate system.

The beauty of Cartesian coordinates is that the arithmetic is done in a particularly convenient way. In particular if A is the point given in Cartesian coordinates by (a_1,a_2) and B by (b_1,b_2) then the vector $\vec{AB}$ is simply

$$(b_1-a_1)\vec{i} + (b_2-a_2)\vec{j}$$

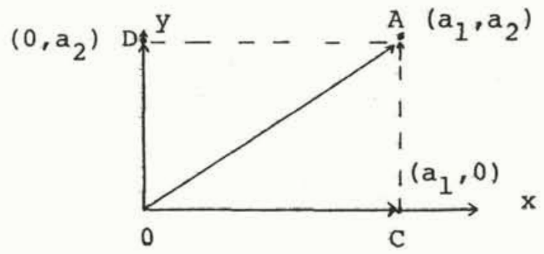
That is, we vectorize the line segment from A to B, <u>in Cartesian coordinates</u>, by substracting the coordinates of A from those of B to obtain the components of $\vec{AB}$.* The proof of this fact

---

*This is a generalization of our approach to the arithmetic of the x-axis. For example, if P and Q denoted numbers on the x-axis we saw that Q-P was the <u>directed</u> length of the line which went <u>from</u> P to Q. At that time we were not (at least consciously) thinking about vectors, but in effect we were forming the vector $\vec{PQ}$ by subtracting P from Q.

1.2.2 continued

may be obtained by observing that, in terms of $\vec{i}$ and $\vec{j}$ components, the vector $a_1\vec{i} + a_2\vec{j}$ and the point (a_1,a_2) are related by the fact that if $a_1\vec{i} + a_2\vec{j}$ originates at (0,0) then it terminates at (a_1,a_2)*. Pictorially,



$$\vec{OA} = \vec{OC} + \vec{CA}$$

$$= \vec{OC} + \vec{OD}$$

$$= a_1\vec{i} + a_2\vec{j}$$

Then,

$$\vec{AB} = \vec{AO} + \vec{OB}$$

$$= -\vec{OA} + \vec{OB}$$

$$= \vec{OB} - \vec{OA} \qquad (3)$$

Now, by our previous observation, if O is the origin, then since $B = (b_1,b_2)$, $\vec{OB} = b_1\vec{i} + b_2\vec{j}$. In a similar way, $\vec{OA} = a_1\vec{i} + a_2\vec{j}$.

*Recall that our definition of equality for vectors allows us to assume that we may place a vector at the origin without changing it provided that we do not change either its magnitude, direction, or sense.

1.2.2 continued

Accordingly,

$$\vec{OB} - \vec{OA} = (b_1\vec{i} + b_2\vec{j}) - (a_1\vec{i} + a_2\vec{j}) \qquad (4)$$

Combining (3) and (4) we see that

$$\vec{AB} = (b_1\vec{i} + b_2\vec{j}) - (a_1\vec{i} + a_2\vec{j}) \qquad (5)$$

Since vector addition and scalar multiplication have the same structure as for ordinary arithmetic, we may manipulate (5) in the "usual" way to obtain

$$\vec{AB} = (b_1-a_1)\vec{i} + (b_2-a_2)\vec{j}\ ^*$$

Thus, for example, (where we are using Cartesian coordinates) if A = (2,5) and B = (3,1) then $\vec{AB} = (3-2,\ 1-5) = (1,-4) = \vec{i} - 4\vec{j}$.

That this is not true for other coordinate systems may be seen as follows. Let us invent a type of "radar-coordinates" (later to be called polar coordinates) in the nautical tradition of giving the range-and-bearing of an object. Specifically, we pick a point O, called our origin or pole. Then given a point P we let r denote the distance between O and P and we let θ denote the angle OP makes with the positive x-axis. Pictorially,

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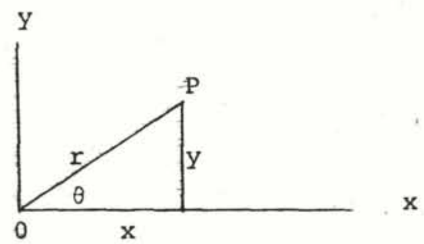
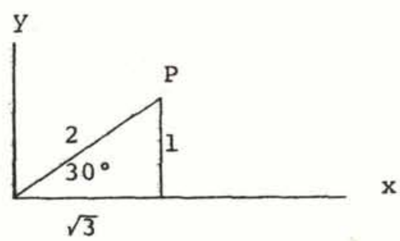
*Since the vector $x\vec{i} + y\vec{j}$ terminates at (x,y) if it originates at (0,0) it is customary to abbreviate $x\vec{i} + y\vec{j}$ by (x,y), leaving it to context to determine whether (x,y) means a vector or a point. In this context, one might write $\vec{AB}$ as $(b_1-a_1,\ b_2-a_2)$. We shall do this on some occasions but not on others; sometimes because one notation is more convenient than the other, and sometimes only to help you get used to either notation.

1.2.2 continued



Thus while P is the <u>same</u> point regardless of whether we denote it by Cartesian or polar coordinates, it would be labeled (x,y) in Cartesian coordinates but as (r,θ) in polar coordinates. In terms of a more concrete application, $(\sqrt{3}, 1)$ labels the same point in Cartesian coordinates that $(2,30°)$* labels in polar coordinates. Again, pictorially,



In any event consider the following situation (merely as a typical example). Let A and B be denoted in polar coordinates by $(\sqrt{3}, 0°)$ and $(2, 30°)$ respectively. To be sure we could <u>quite mechanically</u> form the difference

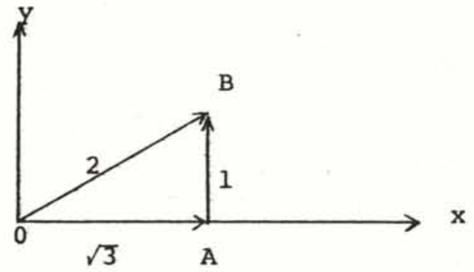
$$(2 - \sqrt{3}, 30° - 0°) = (2 - \sqrt{3}, 30°)$$

<u>but</u> this would <u>not</u> represent the <u>vector $\vec{AB}$</u>. More specifically,

*Since we are specifically dealing with angular measure here, degrees are as permissible as radians. If we so wished, however, we could have written

$$(2, \tfrac{\pi}{6} \text{ radians}).$$

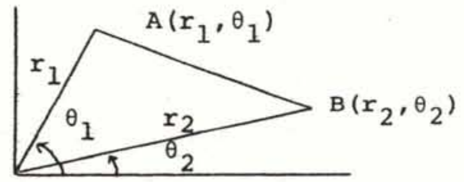
1.2.2 continued



$\vec{AB} = \vec{OB} - \vec{OA}$ (This is true independent of any coordinate system).

If we now place $\vec{AB}$ so that it originates at 0, it will terminate at (0,1) in Cartesian coordinates or (1, 90°) in polar coordinates. Obviously $(1, 90°) \neq (2 - \sqrt{3}, 30°)$.

In general, if $A = (r_1, \theta_1)$ and $B = (r_2, \theta_2)$, in polar coordinates, the length of $\vec{AB}$ is not simply $r_2 - r_1$. Rather, we must use the law of cosines to obtain:



$$\underline{|\vec{AB}|^2 = r_1^{\,2} + r_2^{\,2} - 2r_1r_2\cos(\theta_1 - \theta_2)}$$

Again, it is crucial to note that, once the points A and B are specified, the vector $\vec{AB}$ is <u>uniquely</u> determined. Only the arithmetic is affected by the coordinate system. Thus, to find $|\vec{AB}|$ from $A = (r_1, \theta_1)$ and $B = (r_2, \theta_2)$ we would have to compute

1.2.2 continued

$\sqrt{r_1^{\ 2} + r_2^{\ 2} - 2r_1r_2 \cos(\theta_1-\theta_2)}$ while if we had $A = (x_1,y_1)$ and $B = (x_2,y_2)$ then

$$|\vec{AB}| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

In either case the numerical value of $|\vec{AB}|$ is the same regardless of the formula employed.

With this lengthly aside in mind, we handle (b) as follows:

From (a) we have

$$\vec{OC} = \frac{n}{m+n}\ \vec{OA} + \frac{m}{m+n}\ \vec{OB}$$

Therefore,

$$\vec{OC} = \frac{n}{m+n}\ (a_1\vec{i} + a_2\vec{j}) + \frac{m}{m+n}\ (b_1\vec{i} + b_2\vec{j})$$

$$= \left(\frac{na_1}{m+n} + \frac{mb_1}{m+n}\right)\vec{i} + \left(\frac{na_2}{m+n} + \frac{mb_2}{m+n}\right)\vec{j}$$

$$= \left(\frac{na_1 + mb_1}{m+n}\right)\vec{i} + \left(\frac{na_2 + mb_2}{m+n}\right)\vec{j}$$

Finally, since O denotes the origin and we are dealing with Cartesian coordinates, the vector OC terminates at the point C where the coordinates of C equal the components of $\vec{OC}$. In other words:

$$C = \left(\frac{na_1 + mb_1}{m+n},\ \frac{na_2 + mb_2}{m+n}\right)$$

1.2.2 continued

c. This is a "concrete" application of (b). We have:

$$\vec{OC} = \vec{OA} + \vec{AC}$$

$$= \vec{OA} + \frac{3}{5}\vec{AB}$$

$$= (1,2) + \frac{3}{5}(3-1,\ 5-2)$$ (where we are using (a,b) to denote $a\vec{i} + b\vec{j}$)

$$= (1,2) + \frac{3}{5}(2,3)$$

$$= (1,2) + \left(\frac{6}{5}, \frac{9}{5}\right)$$

$$= \left(1 + \frac{6}{5},\ 2 + \frac{9}{5}\right)$$

$$= \left(\frac{11}{5}, \frac{19}{5}\right)$$

This, in turn, tells us that C, in Cartesian coordinates, is given by $\left(\frac{11}{5}, \frac{19}{5}\right)$.

As a check, we observe that the answer to (b) applies with $a_1 = 1$, $a_2 = 2$, $b_1 = 3$, $b_2 = 5$, $m = 3$ and $n = 2$ (since $\frac{3}{5}$ implies 3:2). Then:

$$\frac{na_1 + mb_1}{m+n} = \frac{2(1) + 3(3)}{3+2} = \frac{11}{5}$$

$$\frac{na_2 + mb_2}{m+n} = \frac{2(2) + 3(5)}{3+2} = \frac{19}{5}$$

Notice once again that we have "vectorized" a non-vector problem. That is, we have used vector methods to solve a problem which can be formulated without reference to vectors.

1.2.3

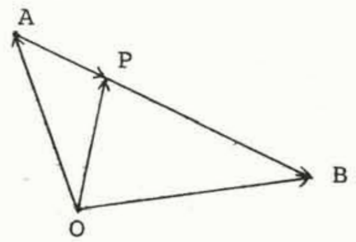
This is another version of Exercise 1.2.2(L). Notice in that exercise we showed that

$$\vec{OP} = \frac{n}{m+n}\ \vec{OA} + \frac{m}{m+n}\ \vec{OB}$$

Letting $t = \frac{m}{m+n}$ it follows that $1 - t = 1 - \frac{m}{m+n} = \frac{n}{m+n}$; whence

$$\vec{OP} = (1-t)\ \vec{OA} + t\ \vec{OB}$$

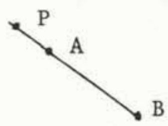
Had we wished to show this result directly without reference to Exercise 1.2.2(L), we might have proceeded as follows:



Assume P is on AB.

Then since $\vec{AP}$ and $\vec{AB}$ have the same direction* it follows that $\vec{AP} = t\ \vec{AB}$ where t is a scalar which depends on the position of P.

*In our diagram it appears tha $\vec{AP}$ has the same sense as $\vec{AB}$ since that is how we drew the picture. Actually, the line determined by A and B has infinite extent and we could have chosen P so that $\vec{AP}$ and $\vec{AB}$ had different sense. For example



Since t is any scalar, not necessarily positive, $\vec{AP} = t\ \vec{AB}$ takes care of either possibility.

1.2.3 continued

Then:

$$\begin{aligned}\vec{OP} &= \vec{OA} + \vec{AP}\\ &= \vec{OA} + t\,\vec{AB}\\ &= \vec{OA} + t(\vec{AO} + \vec{OB})\\ &= \vec{OA} + t(\vec{OB} - \vec{OA})\\ &= \vec{OA} + t\,\vec{OB} - t\,\vec{OA}\\ &= \underline{(1-t)\vec{OA} + t\,\vec{OB}}\end{aligned}$$

Conversely, if all we know is that

$\vec{OP} = (1-t)\,\vec{OA} + t\,\vec{OB}$,

it follows that

$$\begin{aligned}\vec{OP} &= \vec{OA} - t\,\vec{OA} + t\,\vec{OB}\\ &= \vec{OA} + t(\vec{OB} - \vec{OA})\end{aligned}$$

Therefore,

$$\vec{OP} - \vec{OA} = t(\vec{OB} - \vec{OA}) \qquad (1)$$

Now,

$$\begin{aligned}\vec{OP} - \vec{OA} &= \vec{OP} + \vec{AO}\\ &= \vec{AO} + \vec{OP}\\ &= \vec{AP} \qquad (2)\end{aligned}$$

while

1.2.3 continued

$$\vec{OB} - \vec{OA} = \vec{AO} + \vec{OB} = \vec{AB} \tag{3}$$

Putting (2) and (3) into (1) yields

$$\vec{AP} = t\,\vec{AB} \tag{4}$$

Equation (4) says that AP || AB, or that P is on the line determined by A and B.

As a final note, when O denotes the origin, it is customary to abbreviate $\vec{OQ}$ by $\vec{Q}$ where Q is any point in the plane. In this form, the equation of the line determined by A and B in vector form is

$$\underline{\vec{P} = (1-t)\vec{A} + t\,\vec{B}}$$

where t tells us where on AB the point P is located.

1.2.4

Letting A = (1,2) and B = (3,5), we have that $\vec{OA} = (1,2) = \vec{i}+2\vec{j}$ and $\vec{OB} = (3,5) = 3\vec{i} + 5\vec{j}$.

Hence,

$$(1-t)\vec{OA} + t\,\vec{OB} = \left[(1-t)\vec{i} + 2(1-t)\vec{j}\right] + 3t\vec{i} + 5t\vec{j}$$

$$= (1+2t)\vec{i} + (2+3t)\vec{j} \tag{1}$$

Letting P = (x,y), $\vec{OP} = x\vec{i} + y\vec{j}$ (2)

Since we already know that $\vec{OP} = (1-t)\vec{OA} + t\,\vec{OB}$, we may combine (1) and (2) to obtain:

$$\underline{x\vec{i} + y\vec{j} = (1 + 2t)\vec{i} + (2 + 3t)\vec{j}} \tag{3}$$

1.2.4 continued

In Cartesian coordinates, for two vectors to be equal they must be equal component by component. Hence, from (3) we obtain:

$$\left.\begin{aligned} x &= 1 + 2t \\ y &= 2 + 3t \end{aligned}\right\} \qquad (4)$$

In (4) if we solve for x and y in terms of t we obtain:

$$\left.\begin{aligned} \frac{x-1}{2} &= t \\ \frac{y-2}{3} &= t \end{aligned}\right\} \qquad (5)$$

and from (5) it follows that

$$\frac{x-1}{2} = \frac{y-2}{3} \text{ or } 3x - 2y + 1 = 0$$

Without using vectors, we have that

$$\frac{y-2}{x-1} = m = \frac{5-2}{3-1}$$

or

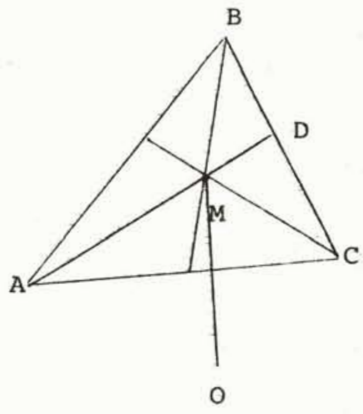
$$\frac{y-2}{x-1} = \frac{3}{2}$$

or

$$\frac{x-1}{2} = \frac{y-2}{3}$$

which agrees with the vector result.

1.2.5



a. $\vec{OM} = \vec{OA} + \vec{AM}$ (1)

$$= \vec{OA} + \frac{2}{3}\vec{AD}$$

$$= \vec{OA} + \frac{2}{3}(\vec{AB} + \vec{BD})$$

$$= \vec{OA} + \frac{2}{3}\left(\vec{AB} + \frac{1}{2}\vec{BC}\right) \quad (2)$$

(Up to this point, our main aim was to express $\vec{OM}$ in terms of O, A, B, and C, since these are the only points mentioned in the answer.)

We now recall that $\vec{AB} = \vec{AO} + \vec{OB} = -\vec{OA} + \vec{OB}$ and $\vec{BC} = \vec{BO}+\vec{OC} = -\vec{OB}+\vec{OC}$. Putting these results into (2) yields

$$\vec{OM} = \vec{OA} + \frac{2}{3}\left([\vec{OB} - \vec{OA}] + \frac{1}{2}[\vec{OC} - \vec{OB}]\right)$$

$$= \vec{OA} + \frac{2}{3}(\vec{OB} - \vec{OA}) + \frac{2}{3}\left[\frac{1}{2}(\vec{OC} - \vec{OB})\right]$$

$$= \vec{OA} + \frac{2}{3}\vec{OB} - \frac{2}{3}\vec{OA} + \frac{1}{3}\vec{OC} - \frac{1}{3}\vec{OB}$$

$$= \frac{1}{3}\vec{OA} + \frac{1}{3}\vec{OB} + \frac{1}{3}\vec{OC} = \frac{1}{3}(\vec{OA} + \vec{OB} + \vec{OC})$$

1.2.5 continued

(Notice that there are several alternative approaches. For Example: $\vec{OM} = \vec{OB} + \vec{BM}$ etc.)

b. We now apply the result of (a) to Cartesian coordinates where $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$ and $O = (0,0)$.* Then,

$$\vec{OA} = a_1\vec{i} + a_2\vec{j}$$

$$\vec{OB} = b_1\vec{i} + b_2\vec{j}$$

$$\vec{OC} = c_1\vec{i} + c_2\vec{j}$$

Therefore,

$$\vec{OA} + \vec{OB} + \vec{OC} = (a_1 + b_1 + c_1)\vec{i} + (a_2 + b_2 + c_2)\vec{j}$$

Therefore,

$$\vec{OM} = \left(\frac{a_1 + b_1 + c_1}{3}\right)\vec{i} + \left(\frac{a_2 + b_2 + c_2}{3}\right)\vec{j}$$

But the components of $\vec{OM}$ are the coordinates of M. Consequently,

$$M = \left(\frac{a_1 + b_1 + c_1}{3}, \frac{a_2 + b_2 + c_2}{3}\right)$$

c. Merely applying (b) we have that the medians of the triangle meet at

$$\left(\frac{1 + 3 + 4}{3}, \frac{2 + 5 + 9}{3}\right) = \left(\frac{8}{3}, \frac{16}{3}\right)$$

*Here, again, we see how we manipulate according to what best suits our purpose. Notice that in the statement of part (a) of this exercise, 0 is any point in the plane. By specifying 0 as the origin, the vectors $\vec{OA}$, $\vec{OB}$, and $\vec{OC}$ may be identified with (a_1,a_2), (b_1,b_2), (c_1,c_2). I.e., $\vec{OA} = a_1\vec{i} + a_2\vec{j}$ etc. Had 0 been another point (d_1,d_2), then $\vec{OA} = (a_1-d_1)\vec{i} + (a_2-d_2)\vec{j}$ etc., and the arithmetic may have been "nastier."

1.2.5 continued

(Again, notice how we vectorize a problem in order to apply vector techniques to a non-vector problem.)

1.2.6

a. The slope of $y = x^2$ at (3,9) is 6 since $\frac{dy}{dx} = 2x$. In terms of $\vec{i}$ and $\vec{j}$ components the slope of a vector is the quotient of the $\vec{j}$-component divided by the $\vec{i}$-component. Hence, the vector $\vec{i} + 6\vec{j}$, originating at (3,9) is a vector tangent to $y = x^2$ at (3,9). Since its magnitude is

$$\sqrt{1^2 + 6^2} = \sqrt{37} ,$$

$$\frac{1}{\sqrt{37}} \vec{i} + \frac{6}{\sqrt{37}} \vec{j} = \frac{1}{37} (\sqrt{37}\, \vec{i} + 6\sqrt{37}\, \vec{j})$$

is a unit tangent vector to the curve $y = x^2$ at (3,9). If we reverse the sense of this vector it is still a unit tangent vector. Hence

$$-\frac{1}{37} (\sqrt{37}\, \vec{i} + 6\sqrt{37}\, \vec{j})$$

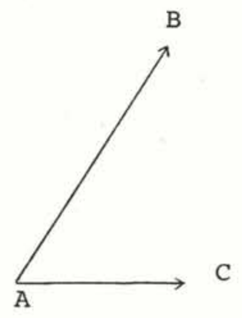
is also a correct answer.

b. To be perpendicular to $\vec{i} + 6\vec{j}$ (whose slope is 6), a vector must have slope $-\frac{1}{6}$. Hence, $6\vec{i} - \vec{j}$ is one such vector. Thus, all unit normal vectors to $y = x^2$ at (3,9) are given by

$$\pm \frac{1}{\sqrt{37}} (6\vec{i} - \vec{j})$$

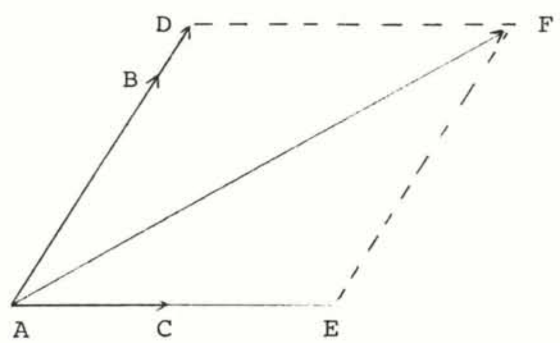
1.2.7(L)

a.



The vector $\vec{AC} + \vec{AB}$ is a diagonal of the parallelogram determined by $\vec{AB}$ and $\vec{AC}$. Such a diagonal need not bisect $\angle$ BAC. However, it will bisect the angle if $\vec{AB}$ and $\vec{AC}$ have equal magnitudes.

Obviously, this need not be the case. However, since changing only the magnitude of $\vec{AB}$ and $\vec{AC}$ will not change $\angle$ BAC, we may (scalar) multiply $\vec{AB}$ by the magnitude of $\vec{AC}$ and $\vec{AC}$ by the magnitude of $\vec{AB}$. That is, we replace $\vec{AB}$ by $\vec{AD} = |\vec{AC}|\ \vec{AB}$ and $\vec{AC}$ by $\vec{AE} = |\vec{AB}|\ \vec{AC}$. This makes $\vec{AD}$ and $\vec{AE}$ equal in magnitude, and, accordingly, $\vec{AF} = \vec{AD} + \vec{AE}$ is an angle bisector of $\angle$ DAE and hence also of $\angle$ BAC. Pictorially,



$|\vec{AD}| = |\vec{AE}|$

Therefore, $\vec{AF}$ bisects $\angle$ DAE which equals $\angle$ BAC

1.2.7(L) continued

Notice, of course, that there are other ways of "scaling" the lengths of $\vec{AB}$ and $\vec{AC}$ to make them equal. For example, we could have scaled them to unit vectors by replacing $\vec{AB}$ by $\frac{1}{|\vec{AB}|}\ \vec{AB}$ and $\vec{AC}$ by $\frac{1}{|\vec{AC}|}\ \vec{AC}$. In this case a bisecting vector would be

$$\frac{1}{|\vec{AB}|}\ \vec{AB} + \frac{1}{|\vec{AC}|}\ \vec{AC}$$

Clearly (we hope!),

$$\frac{1}{|\vec{AB}|}\ \vec{AB} + \frac{1}{|\vec{AC}|}\ \vec{AC}$$

and

$$|\vec{AC}|\ \vec{AB} + |\vec{AB}|\ \vec{AC}$$

are scalar multiples of one another. Indeed:

$$|\vec{AB}|\,|\vec{AC}|\left(\frac{1}{|\vec{AB}|}\ \vec{AB} + \frac{1}{|\vec{AC}|}\ \vec{AC}\right) = |\vec{AC}|\vec{AB} + |\vec{AB}|\vec{AC}$$

In fact, there are <u>infinitely many</u> bisecting vectors, each a scalar multiple of $|\vec{AC}|\vec{AB} + |\vec{AB}|\vec{AC}$. (Geometrically, all we are saying is that once we find one angle bisector we may obtain any other one simply by altering the <u>length</u> of the given one.)

b. Since A = (1,1), B = (4,5), and C = (6,13), we have that

$$\vec{AB} = 3\vec{i} + 4\vec{j},\ \vec{AC} = 5\vec{i} + 12\vec{j},\ |\vec{AB}| = \sqrt{3^2+4^2} = 5,\ |\vec{AC}| = \sqrt{5^2+12^2} = 13$$

Therefore, applying the theory of part (a), we have that

1.2.7(L) continued

$13\vec{AB} + 5\vec{AC}$ bisects $\measuredangle$ BAC. Now,

$$\begin{aligned} 13\vec{AB} + 5\vec{AC} &= 13(3\vec{i} + 4\vec{j}) + 5(5\vec{i} + 12\vec{j}) \\ &= 39\vec{i} + 52\vec{j} + 25\vec{i} + 60\vec{j} \\ &= 64\vec{i} + 112\vec{j} \end{aligned}$$

In other words, the vector $64\vec{i} + 112\vec{j}$ bisects $\measuredangle$ BAC if its tail is placed at A.

Of course any scalar multiple of $64\vec{i} + 112\vec{j}$ would do as well. Hence since $64\vec{i} + 112\vec{j} = 16(4\vec{i} + 7\vec{j})$, we have that

$$4\vec{i} + 7\vec{j}$$

is also a bisecting vector.

c. Here we merely apply vector techniques to a geometry problem again. To find the equation of a line we already know it is sufficient to know a point on the line and the slope of the line. Since $4\vec{i} + 7\vec{j}$ is an angle bisector and its slope is $\frac{7}{4}$, the slope of the required line is $\frac{7}{4}$. Moreover since the line passes through A, (1,1) is on the line. Hence its equation is:

$$\frac{y-1}{x-1} = \frac{7}{4}$$

or

$$7x - 4y = 3$$

1.2.8(L)

In a manner of speaking, it is not clear whether this problem is sufficiently difficult to deserve the designation of a learning exercise since we have solved a similar problem in the previous

1.2.8(L) continued

unit. We wish, however, to take this opportunity to once again stress the concept of <u>structure</u> of vector arithmetic.

Notice that when we proved that for any number <u>a</u>, a0 = 0, the properties of numbers that we needed for this proof were also "rules of the game" for vector arithmetic. Hence, the proof in the last unit should carry through <u>verbatim</u>, once we make the appropriate changes of notation. To this end, we rewrite the proof of the previous unit, making only the appropriate language changes. Thus,

$$a(\vec{0} + \vec{0}) = a\vec{0} + a\vec{0} \tag{1}$$

and

$$a(\vec{0} + \vec{0}) = a\vec{0} \tag{2}$$

Comparing (1) and (2) shows us that

$$a\vec{0} = a\vec{0} + a\vec{0} \tag{3}$$

Since $a\vec{0}$ is a vector, let us denote it by $\vec{b}$.

Then (3) becomes

$$\vec{b} = \vec{b} + \vec{b} \tag{4}$$

Since $\vec{b} = \vec{b} + \vec{0}$, (4) becomes

$$\vec{b} + \vec{0} = \vec{b} + \vec{b} \tag{5}$$

Finally, since the cancellation law for vector addition holds (we proved this in the supplementary notes), we may deduce from (5) that $\vec{b} = 0$, as asserted.

The main point we are trying to make is that while we quite possibly do not feel as at home with vector arithmetic as we do with "ordinary" arithmetic, their structures are remarkably alike in

1.2.8(L) continued

many respects. When this happens the recipes and their applications are precisely the same in both situations. The object of this entire section is to make you more at home with vector arithmetic in particular and the arithmetic of abstract systems in general.

1.2.9

Again, we mimic a previous procedure (although in all fairness we should point out that we could proceed from scratch without reference to a previous proof).

In either event, let us suppose that $a \neq 0$; we will prove that $\vec{v} = 0$. Now, if $a \neq 0$ then $a^{-1}(a\vec{v}) = a^{-1}\vec{0}$. In the previous exercise, however, we proved that $a^{-1}\vec{0} = \vec{0}$ (since, in particular a^{-1} is a number if $a \neq 0$). Thus, we may conclude that

$$a^{-1}(a\vec{v}) = \vec{0} \tag{1}$$

But by our rules for scalar multiplication

$$a^{-1}(a\vec{v}) = (a^{-1}a)\vec{v} = 1\vec{v} = \vec{v} \tag{2}$$

Substituting (2) into (1) yields

$\vec{v} = \vec{0}$

We have, therefore, shown that if $a\vec{v} = 0$ and a is not 0, then $\vec{v}$ is $\vec{0}$. In other words:

If $a\vec{v} = \vec{0}$, then either (meaning, at least one) $a = 0$ or $\vec{v} = \vec{0}$. (Notice the notation of distinguishing between 0 and $\vec{0}$. Conceptually, a denotes a number; hence, $a = 0$ refers to the number 0. $\vec{v}$ denotes a vector; hence, $\vec{v} = \vec{0}$ refers to the vector $\vec{0}$.)