Unit 4: Polar Coordinates I

2.4.1(L)

Hopefully, it is clear by now that what we <u>don't</u> mean is

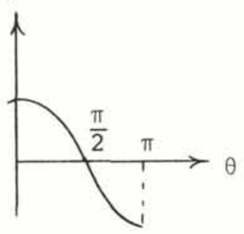


Figure 1

What Figure 1 does tell us is how r varies with θ, but this is not what is meant by the curve whose <u>polar</u> equation is $r = \cos\theta$, $0 \leqslant \theta \leqslant \pi$. In fact, Figure 1 represents a curve in <u>Cartesian</u> coordinates, where the x and y-axes have simply been <u>renamed</u> as the θ and r-axes, respectively.

Recalling that θ determines the line from the origin to a point and r the distance (either in the direction of θ or in the opposite sense) from the origin to the point, we can begin our plot at the pre-calculus level (just as in the Cartesian case) by locating specific points which must belong to the curve. In chart form, we might have:

θ	r (= cos θ)
0	$1 = 1.00$
$\frac{\pi}{6}$	$\frac{1}{2}\sqrt{3} \approx .87$
$\frac{\pi}{4}$	$\frac{1}{2}\sqrt{2} \approx .71$
$\frac{\pi}{3}$	$\frac{1}{2} = .50$
$\frac{\pi}{2}$	$0 = .00$
$\frac{2\pi}{3}$	$-\frac{1}{2}$
$\frac{3\pi}{4}$	$-\frac{1}{2}\sqrt{2}$
$\frac{5\pi}{6}$	$-\frac{1}{2}\sqrt{3}$
π	-1

2.4.1(L) continued

Therefore,

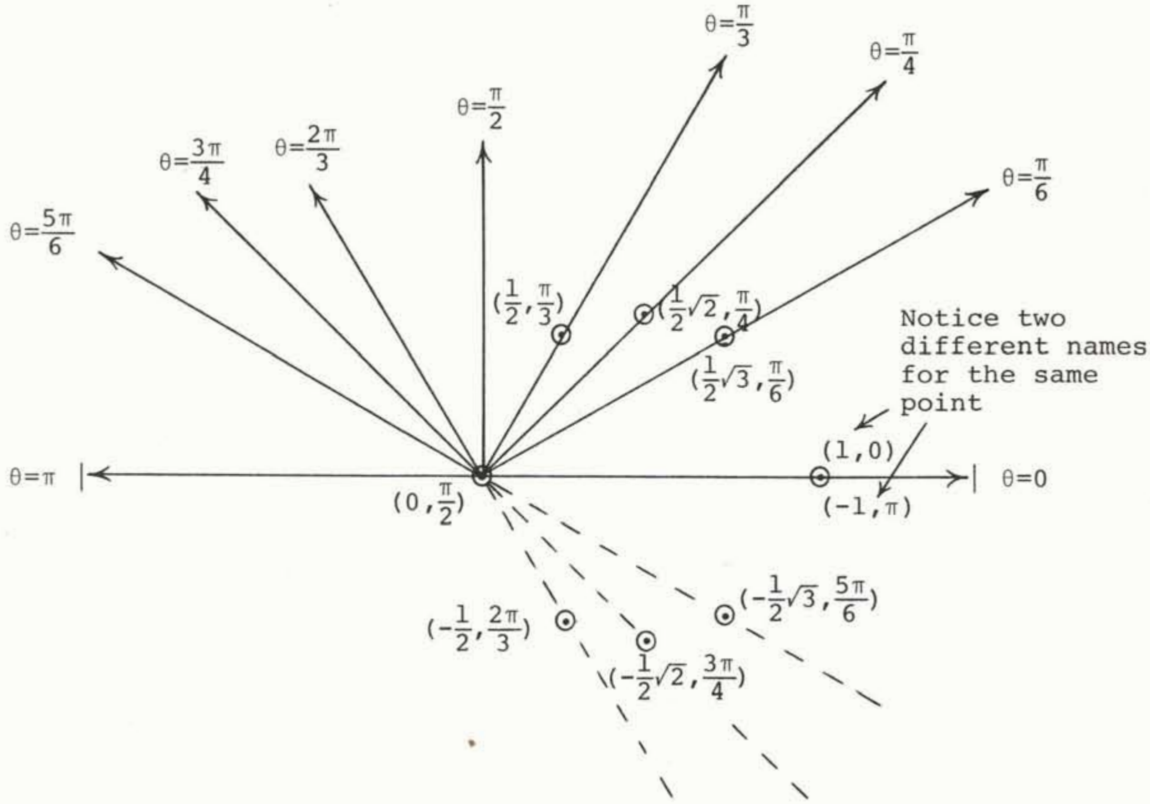
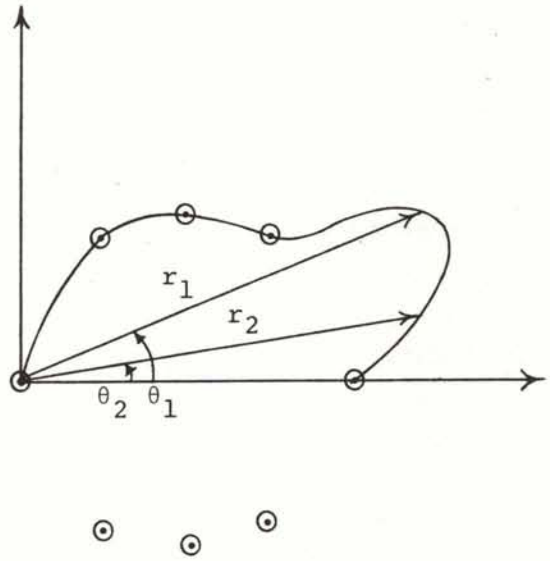


Figure 2

Notice that Figure 2 gives us a collection of discrete (isolated) points that belong to the curve. If we did not have more knowledge at our disposal, the best we could do would be to approximate C by locating more and more points on C and then sketching our curve through these points.

It should be noted now that, although Figure 1 is an incorrect sketch of C, it offers us vital information concerning r as a function of θ that can be used in Figure 1. For example, we can see at a glance from Figure 1 that as θ moves continuously from 0 to $\frac{\pi}{2}$, r steadily decreases. This excludes any curve such as

2.4.1(L) continued



since $r_1 > r_2$ even though $\theta_1 > \theta_2$. In other words, it is important to know such features as "the greater θ, the less r."

Similarly, such devices as symmetry help us. For example, notice that the points (r,θ) and $(r,-\theta)$ are symmetrically located with respect to the x-axis. (Actually, there is no need to use any language connected with Cartesian coordinates. The union of the rays $\theta = 0$ and $\theta = \pi$ says the same thing as the x-axis. As we shall soon see, the fact that we are so familiar with Cartesian coordinates often [but not always] makes it helpful to convert polar equations to Cartesian equations, but it must nevertheless be understood that polar coordinates exist in their own right, independent of whether Cartesian coordinates were ever invented.) Hence, if $f(\theta) = f(-\theta)$ for all values of θ, the curve whose polar equation is $r = f(\theta)$ is symmetric with respect to the x-axis.

In our present example, since $r = \cos\theta$ and $\cos\theta = \cos(-\theta)$, our curve must have symmetry with respect to the x-axis.

We shall pursue these ideas in later problems. For now, let us observe that translating the equation of C into Cartesian coordinates may make us feel more at home. Knowing that $r^2 = x^2 + y^2$ and $x = r\cos\theta$, we are tempted to multiply

$$r = \cos\theta \tag{1}$$

2.4.1(L) continued

by r to obtain

$$r^2 = r \cos \theta. \tag{2}$$

The only time (1) and (2) <u>may</u> represent different curves is when $r = 0$. (As usual, we must guard against multiplying both sides of an equation by zero.) Notice that $r = 0$ corresponds to the single point, the origin (since for any other point $r \neq 0$). Since the origin belongs to (1), i.e., $(0,\frac{\pi}{2})$* satisfies (1), equations (1) and (2) represent the same curve. Equation (2), however, converts very simply into Cartesian coordinates. Namely,

$$x^2 + y^2 = x. \tag{3}$$

Equation (3) may be transformed as

$$x^2 - x + y^2 = 0$$

whereupon, completing the square yields

$$(x^2 - x + \tfrac{1}{4}) + y^2 = \tfrac{1}{4}$$

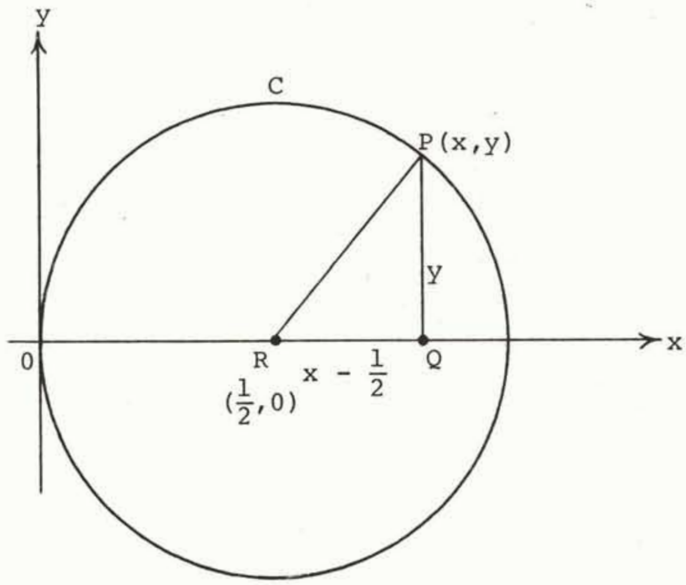
or

$$(x - \tfrac{1}{2})^2 + y^2 = (\tfrac{1}{2})^2 \tag{4}$$

and we recognize (4) as the Cartesian equation of the circle centered at $(\frac{1}{2},0)$ with radius $\frac{1}{2}$. That is,

*As we shall see in more detail later, the origin belongs to a curve as soon as there is a value of θ which allows r to equal 0. This value of θ need not itself be 0. That is, $(0,\theta)$ is the origin for any value of θ.

2.4.1(L) continued

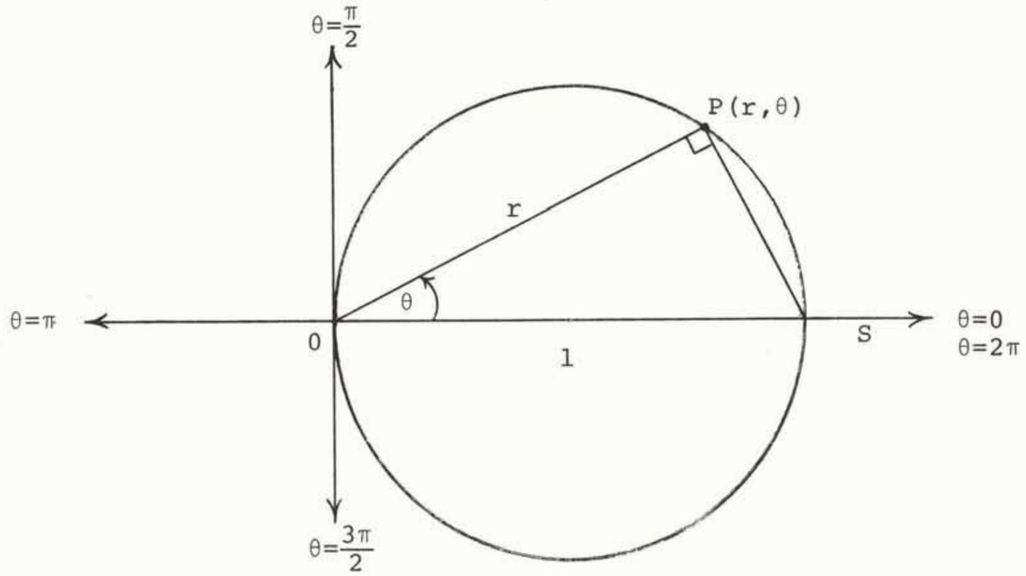


Equation (4) follows from the Pythagorean Theorem applied to ΔPQR

Figure 3

A quick check shows that the discrete points located in Figure 2 satisfy C as drawn in Figure 3. Moreover, C, as shown in Figure 3, is the desired curve since equations (4) and (1) name the <u>same</u> curve (in different coordinate systems), and Figure 3 is the curve named by equation (4).

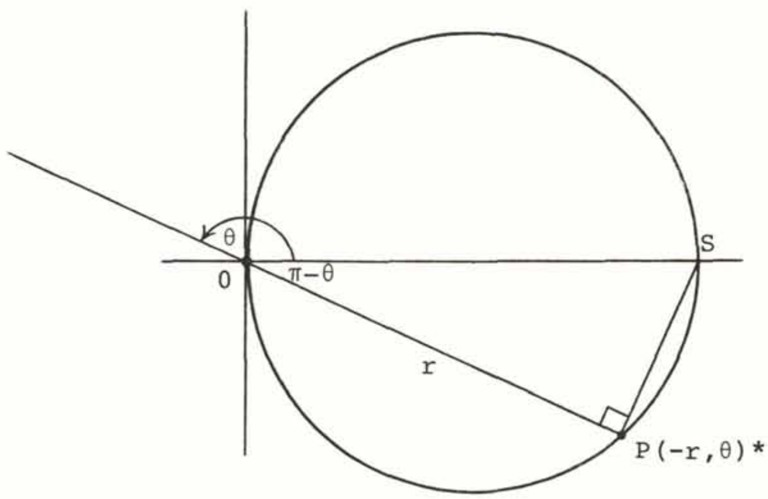
Also, in accord with an earlier remark, we should observe that the curve C as shown in Figure 3 can be verified without any reference to Cartesian coordinates. Namely,



2.4.1(L) continued

In ΔOPS, since $\overline{OS} = 1$, we have immediately that $r = \cos\theta$, since $\overline{OP} = \overline{OS}\cos\theta$.

Notice, however, that not only have we chosen P to lie in the first quadrant, but our diagram also seems to suggest that for the particular value of θ, r is non-negative. For example, P could have been chosen in the fourth quadrant, in which case it would have been on the curve for a second-quadrant angle θ and a negative value of r. In this case our diagram would have been



From ΔOPS we have

$\overline{OP} = \overline{OS}\cos(\pi-\theta)$.

Therefore,

$$r = 1\cos(\pi-\theta)$$

$$= \cos\pi\cos\theta + \sin\pi\sin\theta$$

$$= -\cos\theta.$$

*The point is that we could not have named $\measuredangle$ SOP θ. Among other things, θ was defined to lie between 0 and π, thus making θ a first or second quadrant angle. $\measuredangle$ SOP, on the other hand, lies in the fourth quadrant.

2.4.1(L) continued

Therefore,

$-r = \cos\theta$

and P is $(\cos\theta, \theta)$, which shows that P satisfies the equation $r = \cos\theta$.

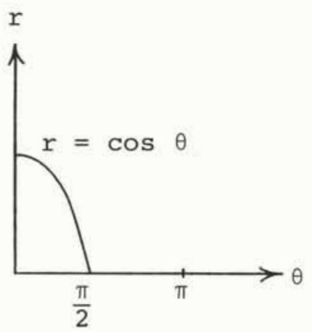
In summary, the fact that we allow r to be negative complicates our analysis of a polar curve, either analytically or pictorially.

As a final note about our decision to allow r to be negative, notice that had we restricted r to being non-negative, C would have been the upper half of the circle since the lower half comes from $r = \cos\theta$ with θ between $\frac{\pi}{2}$ and π and r <u>being negative</u>.

In other words, if r could never be negative, equation (1) would have the additional constraint.

$$r = \cos\theta,\ 0 \leqslant \theta \leqslant \pi,\ \underline{r \geqslant 0}. \tag{1'}$$

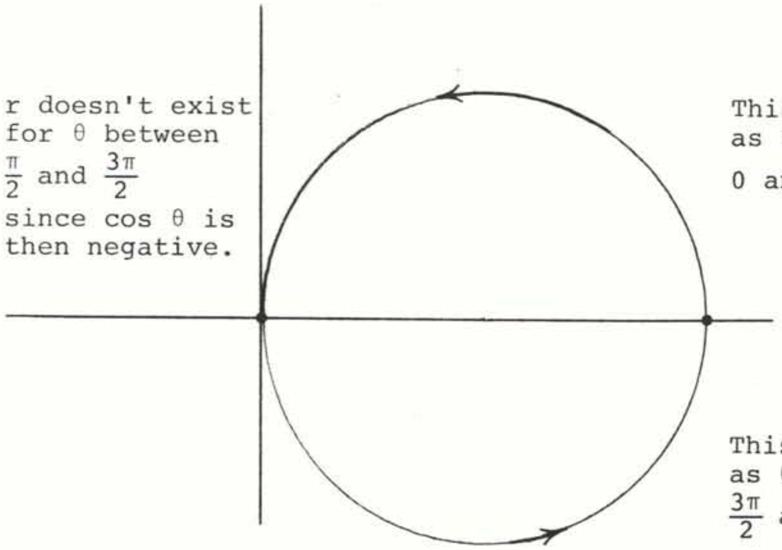
In this case, the Cartesian graph of r versus θ would be



since $r < 0$ is forbidden.

In fact, to get the entire circle, if r were restricted to being non-negative, we would have to let θ vary from 0 to 2π, noticing that the lower half of the circle would come from θ varying between $\frac{3\pi}{2}$ and 2π since r would then be non-negative. In summary, to obtain the whole circle, subject to $r \geqslant 0$, we have

2.4.1(L) continued



To think of things in a "smoother" way, imagine that for those values of θ for which r is negative the curve remains fixed at the last "real" point. For example, in this exercise, we might think of the curve as follows:

$$\begin{cases} r = \cos\theta & 0 \leqslant \theta \leqslant \frac{\pi}{2} \\ r = 0 & \frac{\pi}{2} < \theta < \frac{3\pi}{2} \\ r = \cos\theta & \frac{3\pi}{2} \leqslant \theta \leqslant 2\pi. \end{cases}$$

On the other hand, if r can be negative, $r = \cos\theta$, $0 \leqslant \theta \leqslant 2\pi$, traces out the circle twice, once as θ goes from 0 to π and again when θ goes from π to 2π.

2.4.2

We have $\frac{1}{r^2} = 4\cos^2\theta + 9\sin^2\theta$ $(r \neq 0)$. Multiplying through by r^2 yields

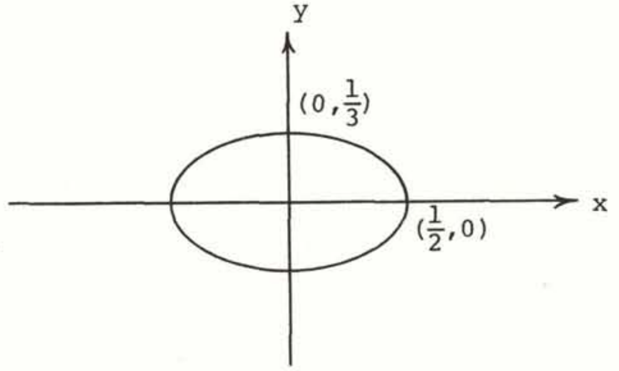
$$1 = 4r^2\cos^2\theta + 9r^2\sin^2\theta. \qquad (1)$$

2.4.2 continued

Recalling that $x = r \cos \theta$ and $y = r \sin \theta$, (1) becomes

$$1 = 4x^2 + 9y^2 \tag{2}$$

and we recognize (2) as the ellipse:*



2.4.3(L)

This exercise was "rigged" to show once and for all that there are times when one coordinate system has a definite advantage over another.

a. $r = \sec \theta$ would probably look more familiar to us in the form

$$r = \frac{1}{\cos \theta}. \tag{1}$$

Observing that $\cos \theta = 0$ when $\theta = -\frac{\pi}{2}$ or $\frac{\pi}{2}$, we recognize trouble with equation (1) when $\theta = \pm\frac{\pi}{2}$, and thus that $r(\frac{\pi}{2})$ means $\lim_{\theta \to \frac{\pi}{2}} \frac{1}{\cos \theta}$.

*Even had we not recognized the ellipse, we could have plotted it. Namely, $1 = 4x^2 + 9y^2$ implies that our curve is symmetric in all quadrants; hence, we would only have to sketch $3y = \sqrt{1 - 4x^2}$ in the first quadrant. We should also recognize without calculus that $|x| \leqslant \frac{1}{2}$ and $|y| \leqslant \frac{1}{3}$ since, for example, $|x| > \frac{1}{2}$ implies that $4x^2 > 1$ and hence that $4x^2 + 9y^2 > 1$ (since $9y^2$ is non-negative). For further review of the conic sections, see Thomas, Chapter 10.

2.4.3(L) continued

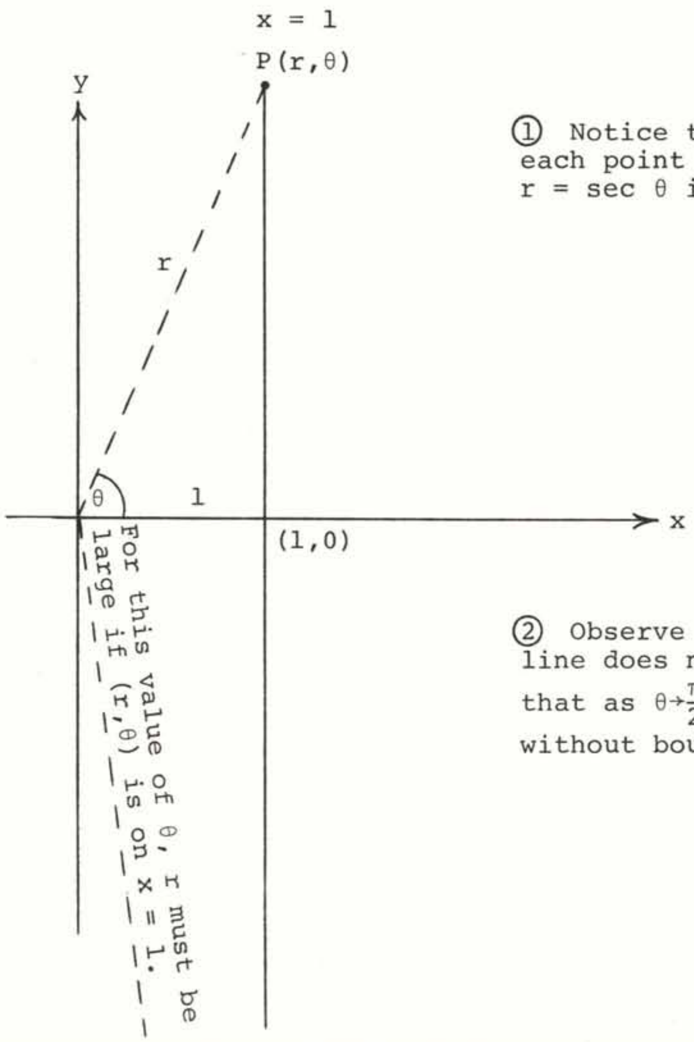
We will bypass the traumatic experiences that the novice can encounter if he tries to graph (1) in terms of r and θ, and, instead, we will rewrite (1) in Cartesian form. As long as $\theta \neq \pm\frac{\pi}{2}$, we may multiply both sides of (1) by $\cos\theta$ to obtain

$$r \cos\theta = 1,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}. \tag{2}$$

Since $x = r\cos\theta$, (2) becomes

$$x = 1$$

which we recognize at once as the line parallel to the y-axis, passing through (1,0). That is,



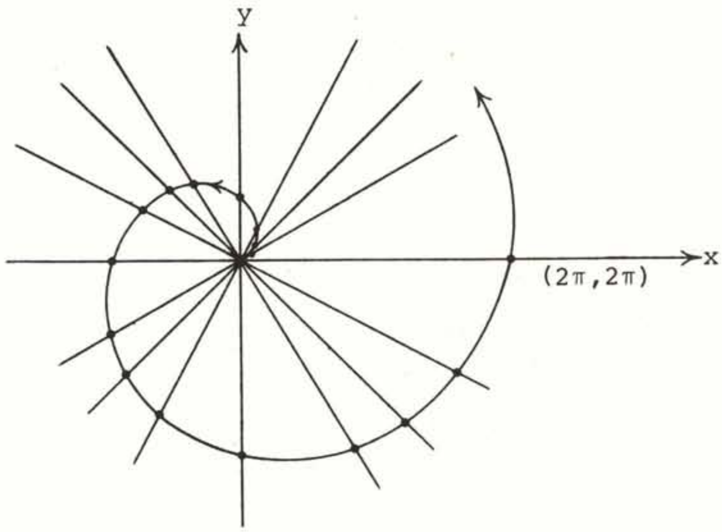
① Notice that in polar coordinates each point $P(r,\theta)$ for which $r = \sec\theta$ is on the line $x = 1$.

② Observe that our "nice" straight line does not contradict the fact that as $\theta \to \frac{\pi}{2}$ (or $-\frac{\pi}{2}$), r increases without bound!

2.4.3(L) continued

The key point we are making is that $x = 1$ is computationally more desirable than the form $r = \sec\theta$. So, at least in this example, Cartesian coordinates have an advantage over polar coordinates.

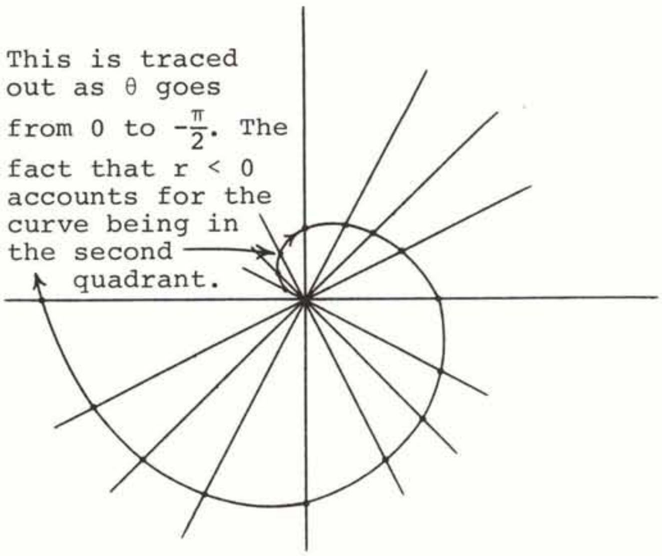
b. The polar curve $r = \theta$ is rather easy to sketch. First of all, if $\theta \geqslant 0$, we have the "spiral"



① Note the need here for radian measure since r is not an angle.

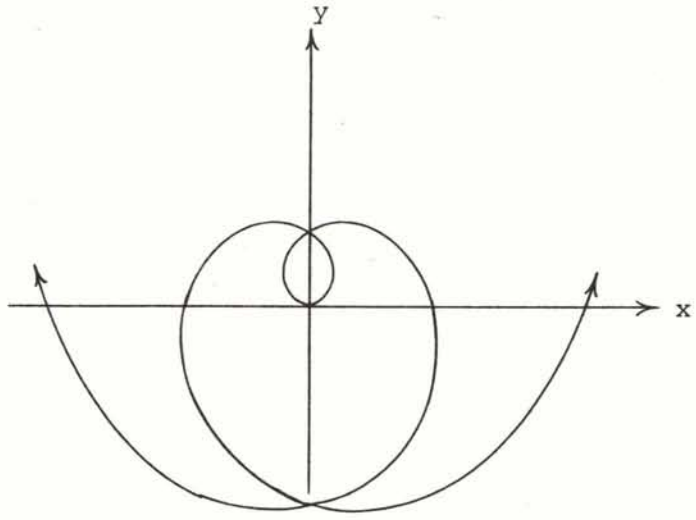
② Notice also that our curve is multi-valued with respect to x and y, but single valued - in fact - (the curve never crosses itself) with respect to r and θ.

Of course, θ is allowed to be negative (i.e. measured clockwise). In this case, $r = \theta$ means $r < 0$, but we have also agreed to permit this. The graph now is



③ Notice that we do not get the same curve when θ exceeds 2π even though $\theta = \theta_o$ and $\theta = \theta_o + 2\pi$ are the same ray. Their value is θ_o on one ray but $\theta_o + 2\pi$ on the other. In other words, our curve spirals on "indefinitely."

2.4.3(L) continued



Thus, as shown above, $r = \theta$ plots as a "double" spiral.

If we convert $r = \theta$ into Cartesian form, we have $r^2 = x^2 + y^2$, whence $r = \pm\sqrt{x^2 + y^2}$ and $\tan \theta = \frac{y}{x}$ or $\theta = \tan^{-1} \frac{y}{x}$ provided we view $\tan^{-1} \frac{y}{x}$ as a multi-valued function of which the principal value is but one branch.

That is, in Cartesian coordinates, our curve is named by

$$\pm\sqrt{x^2 + y^2} = \tan^{-1} \frac{y}{x} + 2\pi k \text{ (where } -\frac{\pi}{2} \leqslant \tan^{-1} \frac{y}{x} \leqslant \frac{\pi}{2}\text{); } k = 0, \pm 1, \pm 2, \ldots .$$

It should be clear at a glance that in this case the polar form is to be desired over the multi-valued Cartesian form.

We also wanted to use this exercise as an excuse to warn against too loose an interpretation of the <u>principal values</u> of the inverse trigonometric functions.

The point is that unless a convention is made to the contrary

$\tan y = x$

for a given x allows infinitely many correct values for y. It is <u>only</u> when we want to talk about an inverse function, so that $y = \tan^{-1} x$ is the same as $\tan y = x$, that we had to pick a 1-1 branch, etc.

2.4.3(L) continued

In polar coordinates, when we write $\tan\theta = \frac{y}{x}$, we are assuming that θ can exist in any quadrant it, can exceed 2π, etc. In this case, if we were to rewrite $\tan\theta = \frac{y}{x}$ as $\theta = \tan^{-1}\frac{y}{x}$, we would mean that $\tan^{-1}$ was a multi-valued function. In summary, if $\tan^{-1}$ is required to mean the principal branch, then $\tan\theta = \frac{y}{x}$ must be interpreted as

$$\theta = \tan^{-1}\frac{y}{x} + 2\pi k,\ \text{where } k = 0, \pm 1, \pm 2, \ldots\ .$$

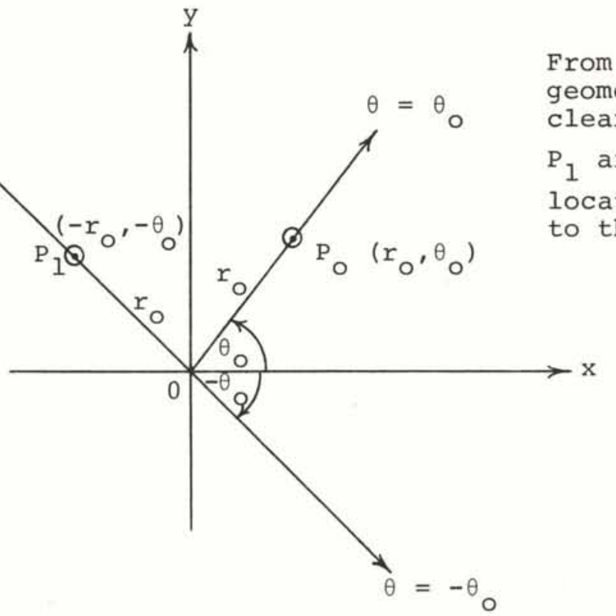
2.4.4(L)

a. While the obvious aim of this exercise is to give you a supplementary list of conditions for symmetry when curves are given in polar form, another aim is to provide an "excuse" to obtain further experience with what is meant by $-r$, $-\theta$, etc. We make no attempt to be any more rigorous than a simple diagram depicting θ in the first quadrant. The interested reader can extend these results to other quadrants.

(1) We have that the curve C is given by $r = f(\theta)$ and that (r_o, θ_o) is on C. We are told that this implies that $(-r_o, -\theta_o)$ is also on C. Drawing a diagram, we see

(Case 1, $r_o \geqslant 0$)

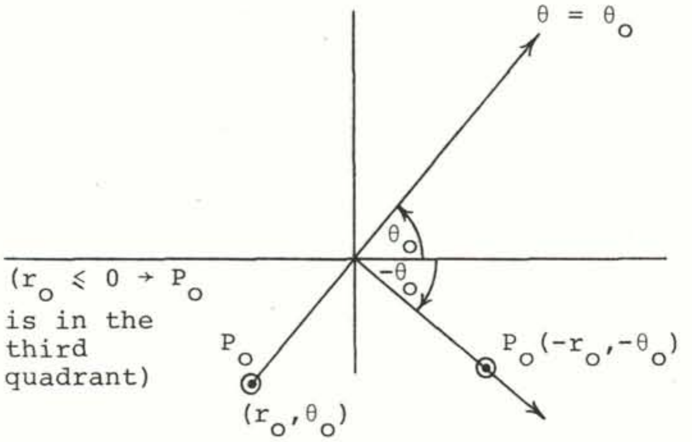
Notice that P_1 is in the 2nd quadrant even though $-\theta_o$ is in the 4th quadrant. This is the effect of $-r_o$ being negative.



From elementary geometry, it is clear that P_o and P_1 are symmetrically located with respect to the y-axis.

2.4.4(L) continued

(Case 2, $r_o \leqslant 0$)



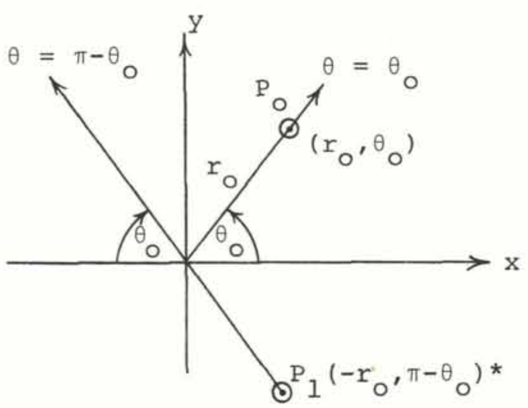
① Notice that $-r_o$ and r_o have the same magnitude, and also that since r_o is negative, $-r_o$ is positive.

② Since $-r_o$ is positive, $(-r_o,-\theta_o)$ is in the direction of $\theta = -\theta_o$.

③ Again, it is clear that P_o and P_1 are symmetric with respect to the y-axis.

④ Just a final note of caution: One tends to read (r,θ) as if $r \geqslant 0$. The point is that r can be negative, in which case, -r is positive.

(2) Here we present only the case in which $r_o > 0$ and $0 \leqslant \theta_o \leqslant \frac{\pi}{2}$.



Again it should be clear from elementary considerations that P_o and P_1 are symmetric with respect to the x-axis.

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*It is clear in this diagram that P_o could also have been named by $(r_o,-\theta_o)$. In this form, the check for symmetry with respect to the x-axis is exactly as given in the text. The major problem, and this will be exploited in the next few exercises, is that given a particular polar equation for a curve C, say, $r = f(\theta)$, it is possible that $(-r_o,\pi-\theta_o)$ will satisfy the equation while $(r_o,-\theta_o)$ won't - even though both name the <u>same</u> point.

2.4.4(L) continued

b. Given that the equation of C is

$$r = \sin 2\theta \tag{1}$$

we test for symmetry with respect to the y-axis by replacing (r,θ) in (1) by $(-r,-\theta)$. This yields

$$-r = \sin 2(-\theta) = \sin(-2\theta) = -\sin 2\theta$$

or

$$r = \sin 2\theta.$$

Since the equation remains unchanged by this substitution, we conclude from part (a) that our curve is symmetric with respect to the y-axis.

We then test for symmetry with respect to the x-axis by replacing (r,θ) in (1) by $(-r,\pi-\theta)$. This leads to

$$\begin{aligned} -r &= \sin 2(\pi-\theta) = \sin(2\pi-2\theta) \\ &= \sin 2\pi \cos 2\theta - \cos 2\pi \sin 2\theta \\ &= 0 - \sin 2\theta \\ &= -\sin 2\theta. \end{aligned}$$

Therefore,

$$r = \sin 2\theta,$$

which is the same as (1). Therefore, again by part (a), we conclude that C is symmetric with respect to the x-axis.

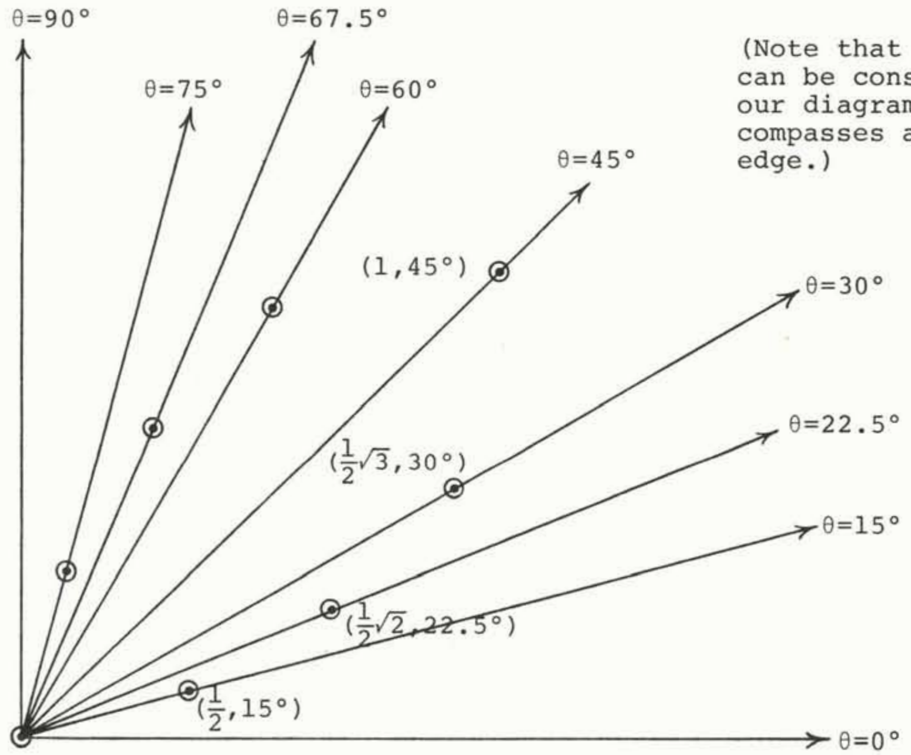
Now, just as in our use of Cartesian coordinates, we may use symmetry to simplify our graphing problem. In this example, we need only sketch the curve in <u>one</u> quadrant, whereupon the remainder of the curve is obtained by symmetry.

2.4.4(L) continued

Since $0 \leqslant \theta \leqslant \frac{\pi}{2}$ implies that $0 \leqslant 2\theta \leqslant \pi$, which in turn implies that $r\ (= \sin 2\theta) \geqslant 0$ in the first quadrant. In other words, in this example, as θ sweeps the first quadrant, so also does our curve. Hence, it is convenient to graph the curve in the first quadrant.

To this end, noticing that 0°, 30°, 45°, 60°, 90°, 120°, 135°, 150°, and 180° are convenient angles for finding sine, and noticing that $r = \sin 2\theta$ means that we are looking at the sine of twice θ, we conclude that the values θ = 0°, 15°, 22.5°, 30°, 45°, 60°, 67.5°, 75°, and 90° would give us a nice collection of points on C in the first quadrant.

Leaving the details to the reader, we obtain



(Note that all rays can be constructed in our diagram by use of compasses and straight edge.)

(We use degrees here rather than radians simply to reinforce the idea that when it is clear we are dealing with angles, the unit of measurement is irrelevant - besides you might feel better seeing degrees once in a while in this course.)

Figure 1

2.4.4(L) continued

If we now want to bring in the information contained in $\frac{dr}{d\theta}$ (which tells us whether r is increasing or decreasing as θ changes) we have, from (1), that

$$\frac{dr}{d\theta} = 2 \cos 2\theta. \qquad (2)$$

Now, $\cos 2\theta \geqslant 0$ if $0 \leqslant 2\theta \leqslant 90°$, or, $0 \leqslant \theta \leqslant 45°$. Thus, r should increase (the point moves further from the origin) as θ varies continuously from 0 to 45°.

Similarly, $\cos 2\theta \leqslant 0$ if $90° \leqslant 2\theta \leqslant 180°$, or, $45° \leqslant \theta \leqslant 90°$. So, as θ varies from 45° to 90°, the point moves in closer to the origin.

The fact that $\frac{dr}{d\theta} = 0$ when $\theta = 45°$ means that r is stationary at that instant [i.e., the curve at (1,45°) "looks like" the circle r = 1. It does not mean that (1,45°) is a high or low point of C, as is clear from Figure 1]. Anyway, combining this discussion with Figure 1, we see that a sketch of C in the first quadrant is given by

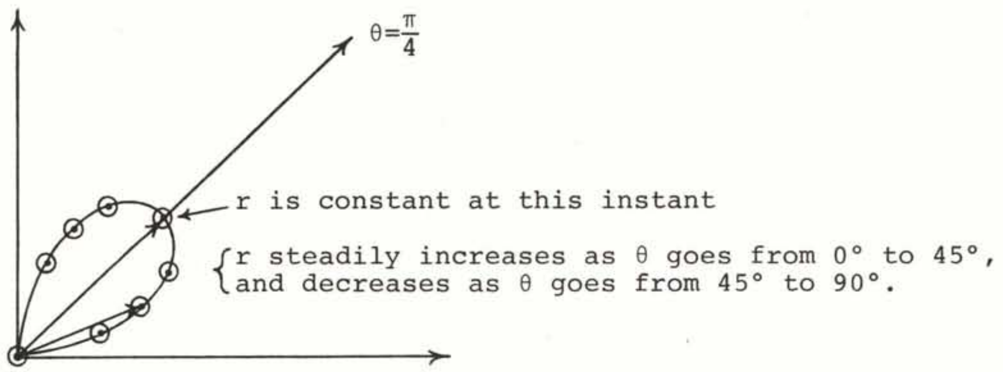
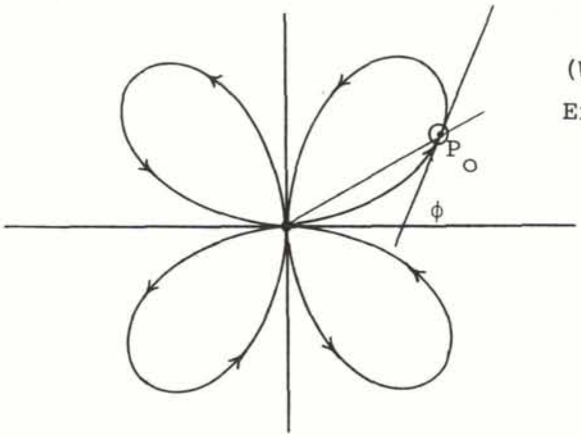


Figure 2

The remainder of the graph of C follows by symmetry from Figure 2. Namely,

2.4.4(L) continued



(We shall refer to P_o and ϕ in Exercise 2.5.2.)

Figure 3

As a final note on Figure 3, it should be clear that we were not forced to use symmetry to sketch C. We could have let θ vary continuously from 0° to 360° and traced out the curve in this way. We have placed arrows on our curve C in Figure 3 to indicate the path which would have been traced had we let θ vary continuously from 0° to 360°. For example when θ is in the second quadrant, we have $90° \leqslant \theta \leqslant 180°$ whence $180° \leqslant 2\theta \leqslant 360°$. Since $\sin 2\theta$ is negative then, it means that our curve appears in the 4th quadrant when θ is in the second quadrant. The interested reader may check the correctness of our arrows in the remaining cases.

c. Since $r \cos \theta = x$ and $r \sin \theta = y$, we might be tempted to write

$$r = \sin 2\theta$$

in the form

$$r = 2 \sin \theta \cos \theta$$

and then multiply by r^2 on both sides to obtain

$$r^3 = 2r^2 \sin \theta \cos \theta$$

or

$$r^3 = 2\,(r \cos \theta)(r \sin \theta)$$

2.4.4(L) continued

whence

$$\sqrt{x^2 + y^2}^{\,3} = 2xy. \tag{3}$$

Again, a word of caution concerning (3). When we made the convention that $\sqrt{x^2}$ meant $|x|$ rather than $\pm x$, it was because we wanted to have single-valued functions. The point here is that if we write

$$r = \sqrt{x^2 + y^2} \tag{4}$$

we must allow both square roots, since r may be negative.

To avoid this problem, it is safer to replace (4) by

$$r^2 = x^2 + y^2.$$

In this context, we should also eliminate the square root in (3) so that we are not tempted to forget about the negative roots. Thus, we should square both sides of (3) to obtain

$$(x^2 + y^2)^3 = 4x^2 y^2. \tag{5}$$

As a final note, Figure 3 is also the graph of (5). That is, don't lose sight of the fact that equations (1) and (5) describe the <u>same</u> curve but with respect to different coordinate systems.

2.4.5(L)

a. Solving

$$\begin{cases} r = \cos\theta + 1 \\ r = \cos\theta - 1 \end{cases}$$

simultaneously, we may subtract the bottom equation from the top to obtain

$$0 = 2. \tag{1}$$

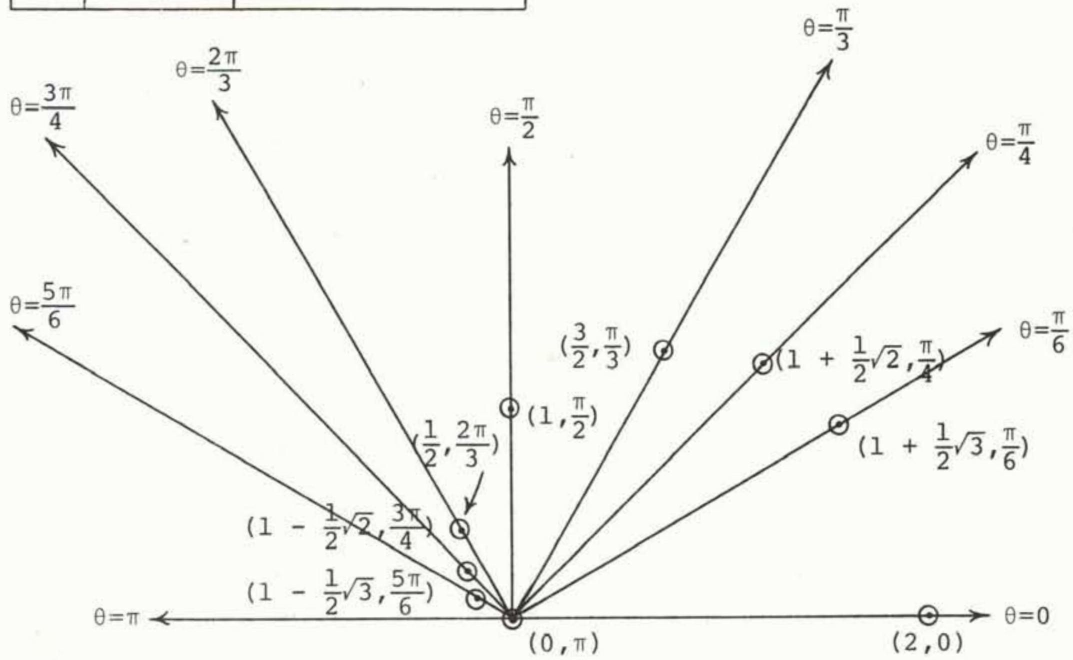
2.4.5(L) continued

Equation (1) is a contradiction, from which we conclude that curves C_1 and C_2 have no simultaneous solutions.

b. Using a plot of specific points on C_1, we obtain

θ	$\cos\theta$	$r = \cos\theta + 1$
0	1	2
$\frac{\pi}{6}$	$\frac{1}{2}\sqrt{3}$	$1 + \frac{1}{2}\sqrt{3} \approx 1.87$
$\frac{\pi}{4}$	$\frac{1}{2}\sqrt{2}$	$1 + \frac{1}{2}\sqrt{2} \approx 1.71$
$\frac{\pi}{3}$	$\frac{1}{2}$	$1 + \frac{1}{2} = 1.50$
$\frac{\pi}{2}$	0	$1 + 0 = 1.00$
$\frac{2\pi}{3}$	$-\frac{1}{2}$	$1 - \frac{1}{2} = 0.50$
$\frac{3\pi}{4}$	$-\frac{1}{2}\sqrt{2}$	$1 - \frac{1}{2}\sqrt{2} \approx 0.29$
$\frac{5\pi}{6}$	$-\frac{1}{2}\sqrt{3}$	$1 - \frac{1}{2}\sqrt{3} \approx 0.13$
π	-1	$1 - 1 = 0$

Since $\cos\theta = \cos(-\theta)$, C_1 is symmetric with respect to the x-axis.



2.4.5(L) continued

Using $\frac{dr}{d\theta}$ to help sketch a smooth curve through the given points and reflecting this about the x-axis yields

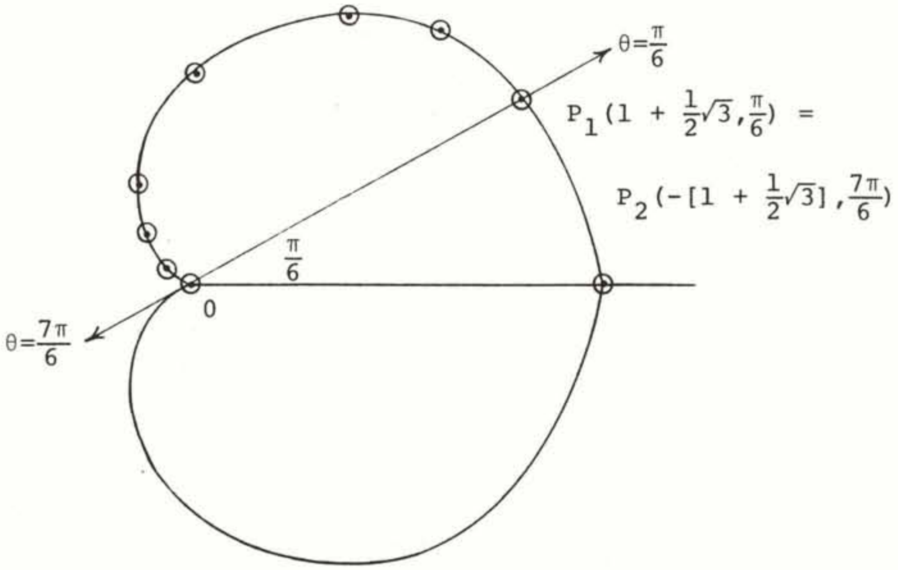


Figure 1

If we now do the same to sketch C_2, we see that we obtain what appears to be the same curve as in Figure 1 but with a difference in phase with respect to θ. For example, when $\theta = \frac{7\pi}{6}$, $\cos\theta - 1 = -\frac{1}{2}\sqrt{3} - 1 = -(1 + \frac{1}{2}\sqrt{3})$. Hence, $P_2(-[1 + \frac{1}{2}\sqrt{3}], \frac{7\pi}{6})$ belongs to C_2. But this means that $P_1 = P_2$!

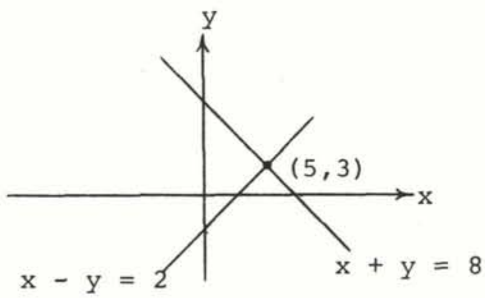
Thus, C_1 and C_2 are <u>different equations</u> of the <u>same curve</u>. A point P belongs to C_1 for one value of θ and to C_2 for another value of θ.

c. The main aim of this part of the exercise is to emphasize the difference between solutions to equations and <u>simultaneous</u> solutions. This type of distinction did not exist in Cartesian coordinates. Namely, in that case, we knew that if $P_1(x_1,y_1) = P_2(x_2,y_2)$ then it had to be that $x_1 = x_2$ and $y_1 = y_2$.

Perhaps at this time a parenthetical note is in order. The usual meaning of "simultaneous" implies the idea of things happening at the same <u>time</u>. Clearly, from its use in elementary mathematics, "simultaneous" does not imply time. For example, when we

2.4.5(L) continued

asked for the simultaneous solutions of the pair of linear equations $x + y = 8$ and $x - y = 2$, there is no mention of time. Geometrically, the simultaneous solution of this pair of equations is the point at which the two lines intersect. That is,



The complication sets in when we think of particles traversing these curves, since it is of course possible that two particles, each traversing a different one of the above curves, need never be at the point (5,3) at the same time.

More generally, from a mathematical point of view if u and v are any two variables, and we have the two equations $f(u,v) = b$ and $g(u,v) = c$, where b and c are given constants, we refer to a simultaneous solution as being one for which the <u>same</u> values for u and v satisfy both equations. That is, $u = u_o$, $v = v_o$ is called a simultaneous solution if $f(u_o,v_o) = b$ <u>and</u> $g(u_o,v_o) = c$.

With respect to Cartesian coordinates, (x_o,y_o) and (x_1,y_1) <u>cannot</u> name the same point in the plane <u>unless</u> both $x_o = x_1$ <u>and</u> $y_o = y_1$. This need not be true for other coordinate systems. In particular, it is <u>not</u> true for polar coordinates. For example, since the domain of θ is unrestricted, it may, in particular, exceed 2π. Thus, $(\frac{1}{2},\frac{\pi}{6}) = (\frac{1}{2},\frac{13\pi}{6})$ even though $\frac{\pi}{6} \neq \frac{13\pi}{6}$.*

This is not the worst of the problem, unfortunately. The fact that we have accepted the convention that r may be negative causes

*Beware here of confusing the fact that $f(\frac{\pi}{6}) = f(\frac{13\pi}{6})$ (where f denotes any circular function) with the fact that $\frac{\pi}{6} = \frac{13\pi}{6}$. As numbers, $\frac{13\pi}{6}$ exceeds $\frac{\pi}{6}$ by 2π. Remember, f is not 1-1; hence, it should be no surprise that $f(\theta_1) = f(\theta_2)$ even though $\theta_1 \neq \theta_2$.

2.4.5(L) continued

a less obvious but, in many ways, a more serious problem. Notice that by our convention (r,θ) and $(-r,\theta+\pi)$ also name the same point.

What is even more unfortunate is that the same point P may satisfy a polar equation for one of its "names" but not for the other. Again, by way of illustration, $P(\frac{1}{2},\frac{\pi}{3})$ (satisfies the polar equation $r = 1 - \cos\theta$ since $\frac{1}{2} = 1 - \cos\frac{\pi}{3}$.) Now, another name for P is $(-\frac{1}{2},\frac{4\pi}{3})$. Notice that, with respect to the "new" name, P no longer satisfies the equation $r = 1 - \cos\theta$. Namely, $-\frac{1}{2} = 1 - \cos\frac{4\pi}{3}$ is false since $1 - \cos\frac{4\pi}{3} = 1 - (-\frac{1}{2}) = \frac{3}{2}$. Clearly, either P is on the curve or it isn't. Yet, for a given form of the equation of the curve, P satisfies the equation with one name, but not with the other. The key point is that, while in Cartesian coordinates a curve had a unique equation, in polar coordinates the same curve may be represented by different polar equations. This means, awkward as it may seem, that given two curves that intersect at a point P, P, with a chosen name, may satisfy neither, one, or both of the particular equations that were chosen to represent the curves. [As a trivial example, the point $(0,30°)$ belongs to both $r = \sin\theta$ and $r = \cos\theta$, yet it satisfies neither equation.]

This exercise is meant to shed some light on this problem.

In particular, to see why (a) and (b) are not contradictory, notice first that (r,θ) and $(-r,\theta+\pi)$ are two different names for the same point. In this respect, the equations for C_1 and C_2 are much more strongly related than may at first meet the eye. Beginning with the equation for C_1, namely

$$r = \cos\theta + 1,$$

we replace r by $-r$ and θ by $\theta+\pi$. In effect, this leaves the point P unchanged but does change its name as described above. At any rate, with this change, we obtain the new equation

$$-r = \cos(\theta+\pi) + 1$$

2.4.5(L) continued

or

$-r = -\cos\theta + 1$

or

$r = \cos\theta - 1$

which happens to be the equation for C_2.

In other words, by construction of C_2, the fact that (r,θ) satisfies C_1 guarantees that $(-r,\theta+\pi)$ satisfies C_2. From a different emphasis, we are saying that if (r,θ) satisfies C_1, it cannot satisfy C_2.

2.4.6(L)

The main aim of this exercise is to extend the results of the previous one in terms of how we solve pairs of polar equations to find all points of intersection of two curves. From Exercise 2.4.5 and our lecture, it should be clear that points of intersection must be of one of the following types. In what follows, we assume that the equation of C_1 is $r = f_1(\theta)$ while the equation of C_2 is $r = f_2(\theta)$.

(1) Since the origin is characterized by $r = 0$, independent of the value of θ, we simply check whether each of the equations $f_1(\theta) = 0$ and $f_2(\theta) = 0$ have solutions. If each has a solution, then $r = 0$ satisfies each equation and, therefore, the origin will be a point of intersection. If it happens that $f_1(\theta) = f_2(\theta) = 0$ for the same value of θ, the solution will be called simultaneous; otherwise, not.

(2) To take into account the fact that (r_o,θ_o) and $(r_o,\theta_o + 2\pi k)$ are different polar names for the same point (where k is any integer), we must find all solutions of the equation:

2.4.6(L) continued

$f_1(\theta) = f_2(\theta + 2\pi k)$.*

If it turns out that $f_2(\theta + 2\pi k) = f_2(\theta)$, the point of intersection will be a simultaneous one; otherwise, not.

(3) To take into account that (r_o,θ_o) and $(-r_o,\theta_o + \pi + 2\pi k)$ name the same point [and remembering that $r_o = f_1(\theta_o)$ or $f_2(\theta_o)$ depending on whether we are talking about C_1 or C_2], we must also find all solutions of

$f_1(\theta) = -f_2(\theta+\pi + 2\pi k)$.

Here no solution can be simultaneous (unless $r = 0$, which was treated above) since if $r \neq 0$, $r \neq -r$. Hence, (r,θ) is not the same <u>number</u> <u>pair</u> as $(-r,\theta+\pi)$.

Notice that, while it is unfortunate that there are so many ways to name the same point in polar coordinates, the three categories described above exhaust all possibilities for finding all the different names the same point can have in polar coordinates. For example, if we are told that $(r_o,\theta_o) = (r_1,\theta_1)$ then it must be that $r_o = \pm r_1$, since as soon as $|r_o| \neq |r_1|$, we must have two different points since they are at different distances from the origin. In a similar way, except at the origin (which we have, therefore, considered as a special case) unless either $\theta_1 = \theta_o + 2\pi k$ or $\theta_1 = \theta_o + \pi + 2\pi k$, the two points have different directions (including sense) and hence must be different.

We have that C_1 is given by $r = \cos 2\theta$, while C_2 is given by $r = 1 + \cos \theta$.

---

*Notice here that it is irrelevant which curve is labeled f_1 and which is f_2. That is, we could just as logically solve the equation $f_2(\theta) = f_1(\theta + 2\pi k)$. Indeed, from an analytical point of view, let $\phi = \theta + 2\pi k$. Then, $\theta = \phi - 2\pi k$, whence $f_1(\theta) = f_2(\theta + 2\pi k)$ implies $f_1(\phi - 2\pi k) = f_2(\phi)$. Since k is an integer, so also is -k, and our last equation has the desired form.

2.4.6(L) continued

Step 1

The simplest test is to see whether the origin is a point of intersection. To this end, we need only check that each of the equations $\cos 2\theta = 0$ and $1 + \cos\theta = 0$ has a solution. Now, $\cos 2\theta = 0$ when $\theta = \frac{\pi}{4}$ (or $\theta = \frac{3\pi}{4}$). Hence, $(0,\frac{\pi}{4})$ [and $(0,\frac{3\pi}{4})$] belong to C_1. And, $1 + \cos\theta = 0$ if $\theta = \pi$. Hence, $(0,\pi)$ belongs to C_2. The intersection, however, is not simultaneous because $\theta_1 \neq \theta_2$, i.e. $\frac{\pi}{4} \neq \pi$. Therefore, the origin belongs to both curves (for C_1, $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$; for C_2, $\theta = \pi$).

Step 2

Letting $f_1(\theta) = \cos 2\theta$ and $f_2(\theta) = 1 + \cos\theta$, we now check for intersections of the form

$$f_1(\theta) = f_2(\theta + 2\pi k). \tag{4}$$

In this case, we get the equation

$$\cos 2\theta = 1 + \cos(\theta + 2\pi k), \tag{5}$$

and, since $\cos(\theta + 2\pi k) = \cos\theta$, equation (5) becomes

$$\cos 2\theta = 1 + \cos\theta.\text{*} \tag{6}$$

Equation (6) becomes easier to solve if we express $\cos 2\theta$ in terms of $\cos^2\theta$, thus giving us a quadratic equation in $\cos\theta$. To this end,

*Notice that (6) could have been obtained had we neglected to include $2\pi k$ in (4). In many cases, as in this example, every time θ appears, it happens that $f(\theta) = f(\theta + 2\pi k)$. However, as we shall show in 2.4.7, do not automatically exclude $2\pi k$ in (4), because it can make a difference. As a good rule of thumb, always use (4) and then see in each case whether $2\pi k$ can be neglected. Where it can be neglected, we will obtain simultaneous solutions.

2.4.6(L) continued

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \cos^2\theta - (1 - \cos^2\theta) = 2\cos^2\theta - 1.$$

Then, (6) may be rewritten as

$$2\cos^2\theta - 1 = 1 + \cos\theta$$

or

$$2\cos^2\theta - \cos\theta - 2 = 0. \qquad (7)$$

From (7), we obtain

$$\cos\theta = \frac{1 \pm \sqrt{1 + 16}}{4} = \frac{1 \pm \sqrt{17}}{4}. \qquad (8)$$

We may discard the plus sign in (8) since $\cos\theta$ cannot exceed 1. That is, $\frac{1 + \sqrt{17}}{4} > \frac{1 + 4}{4} > 1$. Hence,

$$\cos\theta = \frac{1 - \sqrt{17}}{4}. \qquad (9)$$

To find r, we need only put (9) into the equation of C_1, $r = \cos 2\theta$,* and obtain

*Technically speaking, we must be very cautious here. For example, it might have seemed quite natural to substitute (9) into $r = 1 + \cos\theta$ (i.e., C_2) since then the computation would have been given more simply by

$$r = 1 + \cos\theta = 1 + \left(\frac{1 - \sqrt{17}}{4}\right) = \frac{5 - \sqrt{17}}{4}.$$

The point is that when we wrote $f_1(\theta) = f_2(\theta + 2\pi k)$ we were implying that (r,θ) denoted the generic point on $C_1(r = \cos 2\theta)$ while the corresponding name for the same point on $C_2(r = 1 + \cos\theta)$ was $(r,\theta + 2\pi k)$. That is, once θ was determined from (9), r should have been computed from the equation of C_1. In this case, however, it simply happened by coincidence that $f_2(\theta) = f_2(\theta + 2\pi k)$, so that every solution of this type is a simultaneous solution. For this reason, we were free to compute r from either of the two equations. In general, however, as we shall soon see, when the solutions are not simultaneous, it is crucial that we keep track of which curve is being represented pointwise in the form (r,θ).

2.4.6(L) continued

$$r = \cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 2\left[\frac{1 - 2\sqrt{17} + 17}{16}\right] - 1$$

or

$$r = \frac{5 - \sqrt{17}}{4}. \tag{10}$$

From (9) and (10) we see that the simultaneous intersections are the points

$$\left(\frac{5 - \sqrt{17}}{4},\ \cos^{-1} \frac{1 - \sqrt{17}}{4}\right)$$

and

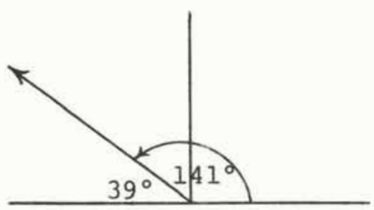
$$\left(\frac{5 - \sqrt{17}}{4},\ -\cos^{-1} \frac{1 - \sqrt{17}}{4}\right) \tag{11}$$

where we are honoring the convention that the domain of $\cos^{-1}$ is $[0,\pi]$. In this context, $\cos^{-1}\left(\frac{1 - \sqrt{17}}{4}\right)$ refers to the second quadrant angle, while its negative is the third quadrant angle. The fact that $\frac{1 - \sqrt{17}}{4}$ is negative means that the angle must be in either the second or third quadrant since its cosine is negative.

To get a more concrete feeling for (11), observe that $\sqrt{17} \approx 4.1$; hence, $\frac{5 - \sqrt{17}}{4} \approx \frac{.9}{4} = 0.225$. Similarly, $\frac{1 - \sqrt{17}}{4} \approx \frac{1 - 4.1}{4} = -\frac{3.1}{4} = -(0.775)$. From tables we then see that

$\cos 39° \approx 0.775.$

Thus,



Recall how we measure 2nd quadrant angles. The 39° angle is our reference angle.

2.4.6(L) continued

Thus, numerical approximations for the points found in (11) are

(0.23,141°)

and

(0.23,219°) $\left(= (0.23,-141°)\right)$

or, more compactly,

(0.23,±141°).

Step 3

What remains now are those intersections which come from the equation

$$f_1(\theta) = -f_2(\theta + \pi + 2\pi k). \tag{12}$$

In this case, (12) yields

$$\cos 2\theta = -[1 + \cos(\theta + \pi + 2\pi k)]. \tag{13}$$

Since $\cos(\theta + \pi + 2\pi k) = \cos(\theta + \pi) = \cos\theta \cos\pi - \sin\theta \sin\pi = -\cos\theta$, equation (13) becomes

$$\cos 2\theta = -[1 - \cos\theta] = \cos\theta - 1.$$

Therefore,

$$\cos 2\theta - \cos\theta + 1 = 0, \tag{14}$$

and again letting $\cos 2\theta = 2\cos^2\theta - 1$, (14) becomes

$$2\cos^2\theta - \cos\theta = 0 \tag{15}$$

whence

$$\cos\theta\,(2\cos\theta - 1) = 0.$$

2.4.6(L) continued

Therefore,

$\cos\theta = 0$ or $\cos\theta = \frac{1}{2}$.

Therefore,

$$\theta = \pm\frac{\pi}{2} \quad \text{or} \quad \theta = \pm\frac{\pi}{3}. \tag{16}$$

Now, however, we must be extremely careful which curve we use with (16) to find r since our solutions cannot be simultaneous! Recall that in our notation (r,θ) was identified with $r = f_1(\theta)$ while $(-r,\theta + \pi)$ was identified with $r = f_2(\theta)$. Since we have let $f_1(\theta) = \cos 2\theta$ [see equations (12) and (13)], we must obtain values for r using $r = \cos 2\theta$. (Later we shall do the problem incorrectly so that you can see the difference.)

At any rate,

$$\theta = \frac{\pi}{2} \rightarrow 2\theta = \pi \rightarrow \cos 2\theta = -1 \rightarrow r = -1.$$

Therefore, $(-1,\frac{\pi}{2})$ is one of our points.

Similarly,

$$\theta = -\frac{\pi}{2} \rightarrow 2\theta = -\pi \rightarrow \cos 2\theta = -1 \rightarrow r = -1.$$

Therefore, $(-1,-\frac{\pi}{2})$ is another point.

$$\theta = \frac{\pi}{3} \rightarrow 2\theta = \frac{2\pi}{3} \rightarrow \cos 2\theta = -\frac{1}{2} \rightarrow r = -\frac{1}{2}$$

Therefore, $(-\frac{1}{2},\frac{\pi}{3})$ is another point.

$$\theta = -\frac{\pi}{3} \rightarrow 2\theta = -\frac{2\pi}{3} \rightarrow \cos 2\theta = -\frac{1}{2} \rightarrow r = -\frac{1}{2}$$

Therefore, $(-\frac{1}{2},-\frac{\pi}{3})$ is another point.

Summed up, then, four points of intersection, with names chosen to obey $r = \cos 2\theta$, are:

2.4.6(L) continued

$$(-1,\pm\frac{\pi}{2}) \quad \text{and} \quad (-\frac{1}{2},\pm\frac{\pi}{3}). \tag{17}$$

Notice that not one of the points listed in (17) can satisfy $r = 1 + \cos\theta$, for if this were the case the point would be a simultaneous solution, and we found all of these in Step 2, and none of the points in (17) was obtained as a solution in Step 2.

For example, with $r = -1$ and $\theta = \frac{\pi}{2}$, $r = 1 + \cos\theta$ becomes $-1 = 1 + 0$, which, trivially, is false.

What is true, however, from our general theory is that if (r,θ) satisfies C_1 then $(-r,\theta + \pi)$ will satisfy C_2. With this in mind, each point in (17) becomes a point which satisfies C_2 (i.e., $r = 1 + \cos\theta$) as soon as we replace r by $-r$ and θ by $\theta + \pi$.

Therefore, from (17), we obtain

$$(1,\pi \pm \frac{\pi}{2}) \quad \text{and} \quad (\frac{1}{2},\pi \pm \frac{\pi}{3}) \tag{18}$$

satisfy $r = 1 + \cos\theta$.

For example, with $r = \frac{1}{2}$ and $\theta = \pi - \frac{\pi}{3}$, $r = 1 + \cos\theta$ becomes

$$\frac{1}{2} = 1 + \cos(\frac{2\pi}{3}) = 1 - \frac{1}{2} = \frac{1}{2},$$

which checks. Notice that (17) and (18) name the <u>same</u> four points, but (17) is in the form which satisfies $r = \cos 2\theta$, while (18) satisfies $r = 1 + \cos\theta$.

To get back to an earlier remark, had we used (16) together with $r = 1 + \cos\theta$ to find r, we would have obtained the four points

$$(1,\pm\frac{\pi}{2}) \quad \text{and} \quad (\frac{3}{2},\pm\frac{\pi}{3}). \tag{19}$$

While these points belong to C_2, we get into trouble now when we replace r by $-r$ and θ by $\theta + \pi$.

For in this case, the points in (19) become

2.4.6(L) continued

$(-1, \pi \pm \frac{\pi}{2})$ and $(\frac{3}{2}, \pi \pm \frac{\pi}{3})$.

Among other things, it is impossible for $(\frac{3}{2}, \pi \pm \frac{\pi}{3})$ to be points on C_1 since from $r = \cos 2\theta$ we see at once that $|r| \leqslant 1$. Thus, $(\frac{3}{2}, \pi \pm \frac{\pi}{3})$ lies "outside" $r = \cos 2\theta$, since $\frac{3}{2} > 1$.

Hopefully, the case we have taken in this example allows you to see the various pitfalls.

As a pictorial summary, we have

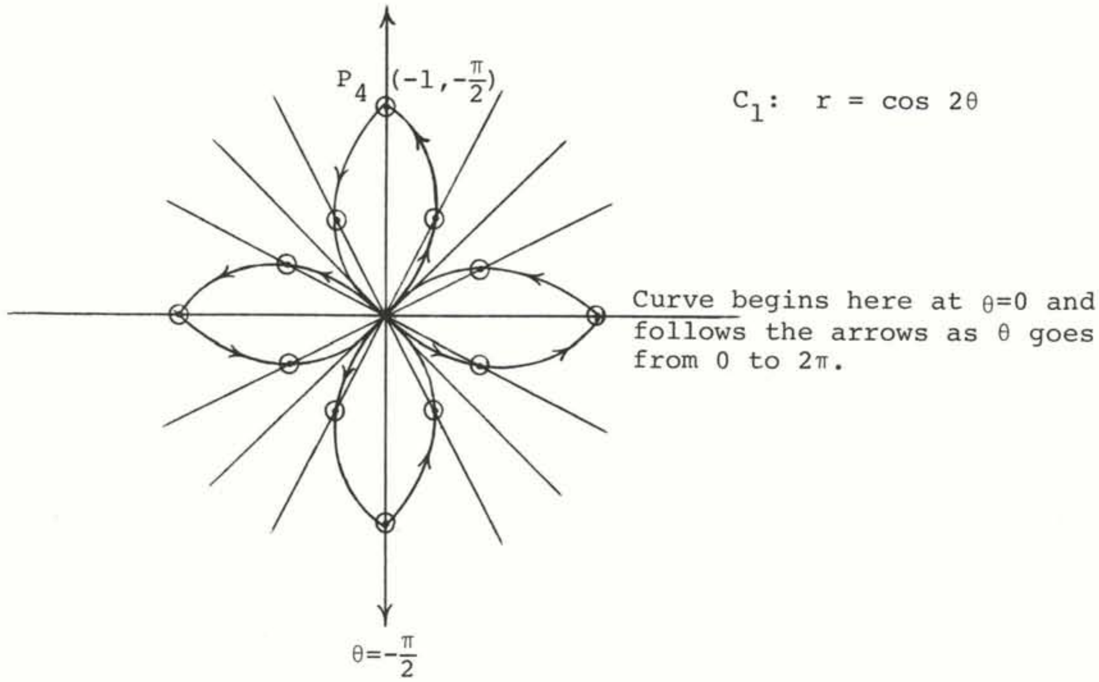


Figure (i)

The graph of $r = 1 + \cos \theta$ is given by

2.4.6(L) continued

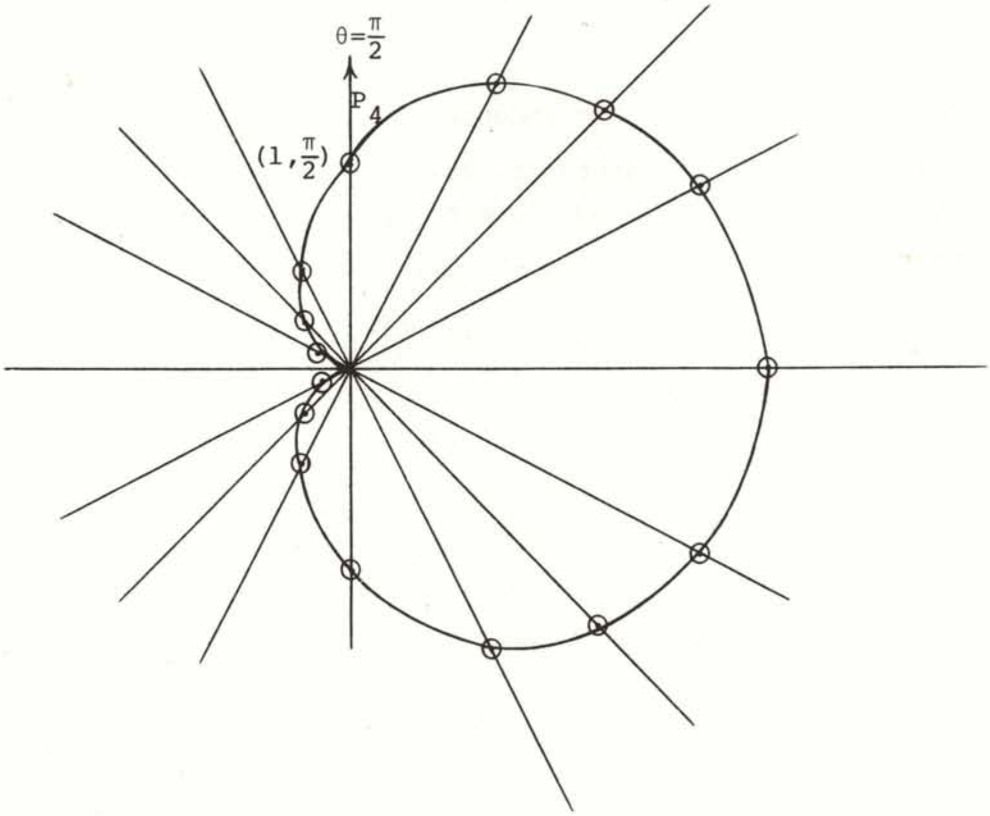


Figure (ii)

And, if we now superimpose (i) and (ii)

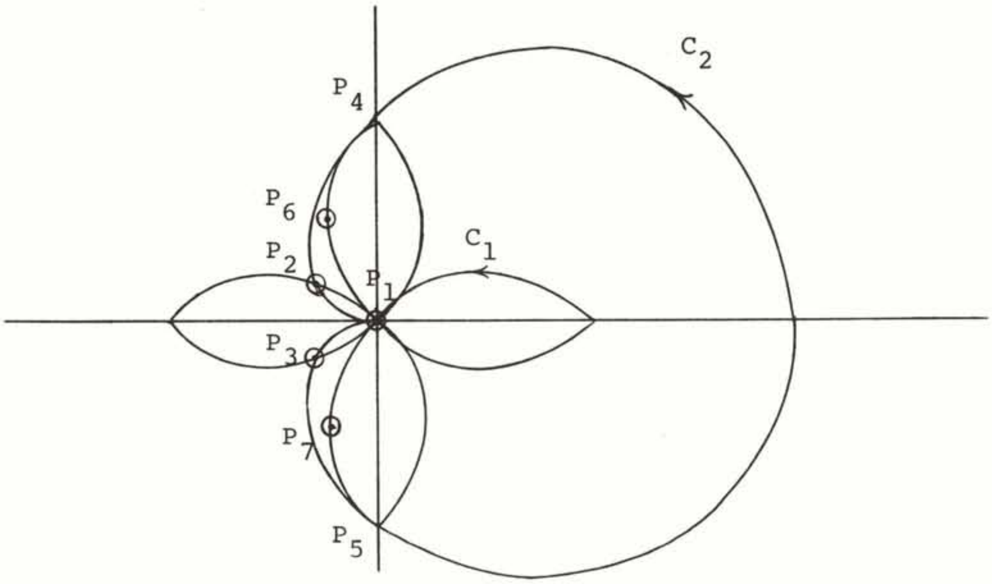


Figure (iii)

2.4.6(L) continued

Summary of (iii):

(1) P_1 is the origin, which belongs to both curves.

(2) P_2 and P_3 are the points (0.23,141°), (0.23,-141°) respectively which are simultaneous intersections.

(3) P_4, P_5, P_6, and P_7 are the "nastier" intersections. That is, under the name $(-1,-\frac{\pi}{2})$, P_4 is on C_1 while it is on C_2 under the name $(1,\frac{\pi}{2})$. [See (i) and (ii).]

Similarly, $P_5 = (-1,\frac{\pi}{2}) = (1,\frac{3\pi}{2})$; $P_6 = (-\frac{1}{2},-\frac{\pi}{3}) = (\frac{1}{2},\frac{2\pi}{3})$;
$P_7 = (-\frac{1}{2},\frac{\pi}{3}) = (\frac{1}{2},\frac{4\pi}{3})$.

Note:

It is our feeling that Step 3 is probably the most difficult of the three to grasp. For this reason, it might be advantageous to revisit Exercise 2.4.5(L) from this point of view. If we let C_1 be denoted by $r = \cos\theta + 1$ and C_2 by $r = \cos\theta - 1$, letting $f_1(\theta) = -f_2(\theta + \pi + 2\pi k)$, we obtain

$$\begin{aligned}\cos\theta + 1 &= -[\cos(\theta + \pi + 2\pi k) - 1]\\ &= -[\cos(\theta + \pi) - 1]\\ &= -[-\cos\theta - 1]\\ &= \cos\theta + 1.\end{aligned}$$

This last equation is an identity! In other words, in terms of the language of sets, if we let

$$\{\theta: \cos\theta + 1 = -[\cos(\theta + \pi) - 1]\},$$

then this set includes every real number (angle).

This, in turn, means that if $(\cos\theta + 1, \theta)$ [i.e., (r,θ)] denotes any point on C_1, then $(-r,\theta + \pi)$ denotes the name that this same point has on C_2.

2.4.6(L) continued

This just happens to be an extreme case. In general, when we look at

$\{\theta: f_1(\theta) = -f_2(\theta + \pi + 2\pi k)\}$

we obtain a finite number of values for θ (including the possibility that the set is empty). For each value of θ thus obtained, we compute the corresponding r from the equation for C_1. Then (r,θ) denotes an intersection as it is named on C_1. We then look (quite mechanically) at $(-r,\theta + \pi + 2\pi k)$ and check that this names the same point as denoted with respect to C_2.

2.4.7(L)

Our main aim here is to present an exercise for which $f(\theta) \neq f(\theta + 2\pi)$. In particular, notice that since

$$f(\theta) = \sin\frac{\theta}{4} \tag{1}$$

then

$$\begin{aligned} f(\theta + 2\pi) &= \sin\left(\frac{\theta + 2\pi}{4}\right) \\ &= \sin\left(\frac{\theta}{4} + \frac{\pi}{2}\right) = \sin\frac{\theta}{4}\cos\frac{\pi}{2} + \cos\frac{\theta}{4}\sin\frac{\pi}{2} \\ &= \cos\frac{\theta}{4}. \end{aligned} \tag{2}$$

A comparison of (1) and (2) is enough to convince us that $f(\theta + 2\pi) \not\equiv f(\theta)$.

In any case, if we let $r = \frac{1}{2}$ and $\theta = 240°$, then in $r = \sin\frac{\theta}{4}$, we obtain

$$\frac{1}{2} = \sin\frac{240°}{4} = \sin 60° = \frac{1}{2}\sqrt{3}$$

which is false. Hence, P, at least with its given name, does not belong to C.

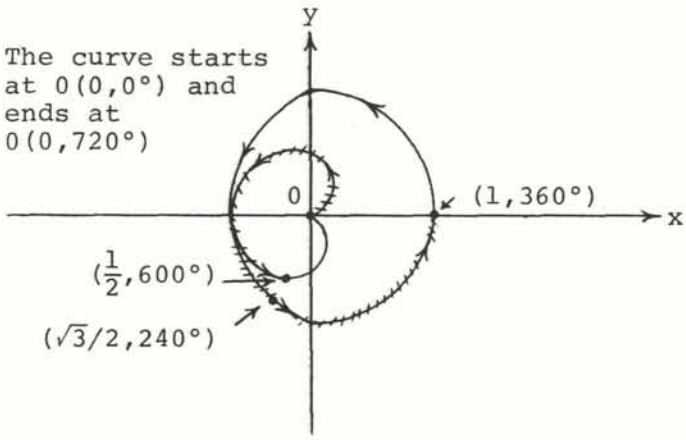
2.4.7(L) continued

On the other hand, another name for P is $(\frac{1}{2}, 240° + 360°)$, or $(\frac{1}{2}, 600°)$. Letting $r = \frac{1}{2}$ and $\theta = 600°$ in the equation $r = \sin\frac{\theta}{4}$, we see that

$$\frac{1}{2} = \sin\frac{600°}{4} = \sin 150° = \frac{1}{2}.$$

Hence, P belongs to C but under the name $(\frac{1}{2}, 600°)$ rather than $(\frac{1}{2}, 240°)$.

In fact, from (1) and (2), we see that the curve crosses itself when $\sin\frac{\theta}{4} = \cos\frac{\theta}{4}$. This, in turn, means that $\tan\frac{\theta}{4} = 1$, whence, $\frac{\theta}{4} = \frac{\pi}{4}$ or $\frac{5\pi}{4}$. Hence, the curve crosses itself at $\theta = \pi$ and $\theta = 5\pi$. Pictorially,



(Hatched portion indicates the part of the curve which corresponds to the range $0° \leqslant \theta \leqslant 360°$. Other portion corresponds to $360° \leqslant \theta \leqslant 720°$).

2.4.8

We have $f_1(\theta) = 1 + \cos\theta$, $f_2(\theta) = 1 + \sin\theta$, and C_1 is defined by $r = f_1(\theta)$ while C_2 is defined by $r = f_2(\theta)$.

Step 1

$1 + \cos\theta = 0$, if $\theta = \pi$. Therefore, the origin belongs to C_1 in the form $(0,\pi)$.

$1 + \sin\theta = 0$, if $\theta = \frac{3\pi}{2}$. Hence, the origin belongs to C_2 in the form $(0,\frac{3\pi}{2})$.

Therefore, the origin is a point of intersection.

2.4.8 continued

Step 2

We solve $f_1(\theta) = f_2(\theta + 2\pi k)$. This yields

$$1 + \cos\theta = 1 + \sin(\theta + 2\pi k)$$

or

$$1 + \cos\theta = 1 + \sin\theta$$

Therefore,

$$\sin\theta = \cos\theta$$

or

$$\tan\theta = 1$$

Therefore,

$$\theta = \frac{\pi}{4} \quad \text{or} \quad \theta = \frac{5\pi}{4}.$$

These values of θ lead to the points $(1 + \frac{1}{2}\sqrt{2}, \frac{\pi}{4})$ and $(1 - \frac{1}{2}\sqrt{2}, \frac{5\pi}{4})$ as simultaneous intersections (where r is found for a given θ from either $r = 1 + \cos\theta$ or $r = 1 + \sin\theta$, since the intersection is simultaneous).

Step 3

We solve $f_1(\theta) = -f_2(\theta + \pi + 2\pi k)$ and obtain

$$1 + \cos\theta = -(1 + \sin[\theta + \pi + 2\pi k])$$

or

$$\begin{aligned}\underline{1 + \cos\theta} &= -(1 + \sin[\theta + \pi]) \\ &= -1 - \sin(\theta + \pi) \\ &= \underline{-1 + \sin\theta}. \qquad (1)\end{aligned}$$

2.4.8 continued

Since $1 + \cos\theta$ can never by negative while $-1 + \sin\theta$ can never be positive, (1) can have no solution unless both $1 + \cos\theta$ and $-1 + \sin\theta$ were 0, but clearly any value of θ which makes $1 + \cos\theta = 0$, does not make $-1 + \sin\theta = 0$. Hence, (1) has no solutions, and, therefore, Step 3 yields no additional points.

In summary, all points of intersection are given by

(1) the origin (P_o)

(2) $(1 + \frac{1}{2}\sqrt{2}, \frac{\pi}{4})$ (P_1)

(3) $(1 - \frac{1}{2}\sqrt{2}, \frac{5\pi}{4})$ (P_2)

A rough pictorial summary shows