Unit 3: Space Curve and the Bi-Normal Vector (Optional)

2.3.1(L)

The main aim of this exercise is to give us a better feeling about vector calculus applied to curves and, at the same time, by extending our discussion to space curves, to get a better understanding of how the vector concepts we have been discussing transcend the 2-dimensional world.

To begin with, the equation

$$\vec{R}(t) = x(t)\vec{1} + y(t)\vec{j} + z(t)\vec{k} \tag{1}$$

represents the locus of points in space described by the three scalar equations

$$x = x(t),\ y = y(t),\ z = z(t). \tag{2}$$

If we think (simply for the convenience of having a nice physical interpretation) of t as denoting time, it is not hard to see that, in most cases, the domain of $\vec{R}$ is an interval. That is, we are usually interested in studying the path of a particle over some continuous time interval, for example, from $t = t_1$ to $t = t_2$.

A very natural question to ask is whether the locus of points (x,y,z) forms a continuous curve. Perhaps we would feel intuitively that if $x(t)$, $y(t)$, and $z(t)$ were all continuous functions of t, then so is $\vec{R}(t)$, and this is precisely what part a. is asking us to do. Namely,

a. We must first decide what it means for $\vec{R}$ to be a continuous function of t. In terms of our previous strategy, it seems fitting that we take as a first approximation the analogous definition in the scalar case. That is,

$\vec{R}$ is said to be continuous at $t_1 \varepsilon [a,b]$ if and only if $\lim\limits_{t \to t_1} \vec{R}(t) = \vec{R}(t_1)$.

This, in turn, says that given $\varepsilon > 0$ we can find $\delta > 0$ such that

2.3.1 continued

$$0 < |t - t_1| < \delta \rightarrow |\vec{R}(t) - \vec{R}(t_1)| < \varepsilon . \tag{3}$$

Our next step is to verify that this formal definition agrees with our intuition. To this end we observe that our definition says that $\vec{R}$ is defined at each t in the interval and that there are no "gaps" in $\vec{R}$ since, for any t_1 in the interval, $\vec{R}(t)$ can be made arbitrarily nearly equal to $\vec{R}(t_1)$ simply by choosing t sufficiently close to t_1. Since $|\vec{R}|$ measures the distance from the origin to a point on the curve, the fact that there can be no gaps in $\vec{R}$ insures that there can be no gaps in the curve.

Thus, we may take (3) as a practical, yet precise, working definition of continuity. With this in mind,

$$\begin{aligned}
|\vec{R}(t) - \vec{R}(t_1)| &= |[x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}] - [x(t_1)\vec{i} + y(t_1)\vec{j} + z(t_1)\vec{k}]| \\
&= |[x(t) - x(t_1)]\vec{i} + [y(t) - y(t_1)]\vec{j} + [z(t) - z(t_1)\vec{k}| \\
&\leq |[x(t) - x(t_1)]\vec{i}| + |[y(t) - y(t_1)]\vec{j}| + |[z(t) \\
&\quad - z(t_1)]\vec{k}|
\end{aligned}$$

Therefore, $|\vec{R}(t) - \vec{R}(t_1)| \leq |x(t) - x(t_1)| + |y(t) - y(t_1)| + |z(t) - z(t_1)|$. (4)

Since x, y, and z are all continuous <u>scalar</u> functions of t, we may find δ_1, δ_2, δ_3 such that for a given $\varepsilon > 0$,

$$\left.\begin{aligned}
&0 < |t - t_1| < \delta_1 \rightarrow |x(t) - x(t_1)| < \frac{\varepsilon}{3} \\
&0 < |t - t_1| < \delta_2 \rightarrow |y(t) - y(t_1)| < \frac{\varepsilon}{3} \\
&0 < |t - t_1| < \delta_3 \rightarrow |z(t) - z(t_1)| < \frac{\varepsilon}{3} .
\end{aligned}\right\} \tag{5}$$

2.3.1 continued

Hence, if we let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, (5) tells us that

$$0 < |t - t_1| < \delta \rightarrow |x(t) - x(t_1)| + |y(t) - y(t_1)| + |z(t) - z(t_1)| < \varepsilon. \quad (6)$$

Combining (6) with (4) yields

$$0 < |t - t_1| < \delta \rightarrow |\vec{R}(t) - \vec{R}(t_1)| < \varepsilon. \quad (7)$$

What we have thus shown in part a. is that we can define continuity for a space curve in such a way that the definition is independent of our coordinate system. But, if we are dealing with Cartesian coordinates, we have the "luxury" of knowing that we can test for continuity simply by testing each of the scalar components for continuity (and this is something we have already learned to do).

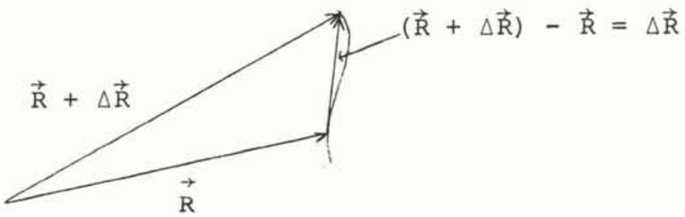
b. (1) Given a continuous space curve, say, $\vec{R} = \vec{R}(t)$, we know that by our rules for differentiation, in Cartesian form, we have

If $\vec{R} = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

then

$$\frac{d\vec{R}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}. \quad (8)$$

Our claim is that the vector defined in (8) is tangent to the space curve. Perhaps the easiest and the safest way to see this is without reference to $\vec{i}$, $\vec{j}$, and $\vec{k}$ components. Quite in general, if $\vec{R}$ and $\vec{R} + \Delta\vec{R}$ are two vectors drawn from the origin to points on the curve, it is easy to see that $\Delta\vec{R}$ is the vector that originates at one of the points and terminates at the other. Thus,



2.3.1 continued

If we now hold one of the points fixed, it is easy to visualize that as $\Delta\vec{R}$ approaches $\vec{0}$, our chord approaches the direction of "the" tangent line to our curve. We have put "the" in quotation marks to highlight a new degree of difficulty that is introduced when we deal with space curves rather than plane curves. In a genuine space curve, the curve does not always stay in the same plane (for if it did it would be a plane curve). Thus, from an intuitive point of view, the curve lies in some sort of instantaneous plane at any given time, and we think of the tangent line as lying in this plane. That is, if we fix the point P_o on our space curve C, and we look at the vector $\overrightarrow{P_oP}$ where P is any other point on C, then if our curve is smooth, we see that as P approaches P_o, the vector $\overrightarrow{P_oP}$ approaches a fixed direction, and it is this vector that we call the tangent vector to the curve.

What really highlights this discussion may be seen from the following 2-dimensional case. Suppose $\vec{T}$ is tangent to the curve C below.

If we now let S denote any plane that intersect C, then by construction any line in S which passes through P_o is tangent to the curve C in the sense that it "touches" C at P_o. In this respect, then, there are infinitely many lines that are "tangent" to C at P_o. (Still another way of looking at this is that to talk about a tangent plane at a point, we should be referring to a surface not a curve.)

The point is that $\frac{d\vec{R}}{dt}$ picks out the most "natural" direction for us to call a tangent to the curve at P_o. That is, if given no other instructions, we could look at a small piece which was "sufficiently small" and we could assume that it was in a plane. The point is that $\frac{d\vec{R}}{dt}$ is a vector in this plane. In other words, the direction of $\frac{d\vec{R}}{dt}$ is the one we would pick intuitively as the direction of the curve at the point.

The major question that remains is that of correlating the sense of the curve C with the sense of the vector $\frac{d\vec{R}}{dt}$. The point is that just as in the planar case, if we let our parameter be arclength (s), then $\frac{d\vec{R}}{ds}$ will be a unit vector and its sense will be the same as that of the curve, no matter how we elect to choose the sense of the curve.

2.3.1 continued

If there is no reason for otherwise choosing a sense for our curve, we simply define $\vec{T}$ by dividing $\frac{d\vec{R}}{dt}$ by its magnitude. In any event, if we let $\vec{R}(t)$ be expressed in Cartesian coordinates, we have:

$$\vec{R}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

whereupon our rules for differentiating vectors yields

$$\frac{d\vec{R}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

and we can now see why it is natural that $\frac{d\vec{R}}{dt}$ be called the velocity vector $(\vec{v})$. Namely, it is tangent to the curve and the components of it are $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dz}{dt}$, which are the components of the speed of the particle.

With $\vec{v}$ now defined to be $\frac{d\vec{R}}{dt}$, we simply define the acceleration to be $\frac{d\vec{v}}{dt}$.

(2) With respect to our example,

$$\vec{R} = \cos 3t\ \vec{i} + \sin 3t\ \vec{j} + t\ \vec{k} \tag{9}$$

$$\vec{v} = -3\sin 3t\ \vec{i} + 3\cos 3t\ \vec{j} + \vec{k} \tag{10}$$

$$\vec{a} = -9\cos 3t\ \vec{i} - 9\sin 3t\ \vec{j} \tag{11}$$

therefore, $|\vec{v}| = \sqrt{(-3\sin 3t)^2 + (3\cos 3t)^2 + 1^2}$

or

$$|\vec{v}| = \sqrt{10}\ . \tag{12}$$

(3) From (12), we see that the particle moves with constant speed. From our earlier discussions, when the speed is constant, the acceleration vector must be at right angles to the velocity vector. As a check, (10) and (11) show that

2.3.1 continued

$\vec{v} \cdot \vec{a} = 0.$

c. Since $\frac{d\vec{R}}{dt}$ is a tangent vector, $\frac{d\vec{R}}{dt} \Big/ \left|\frac{d\vec{R}}{dt}\right|$ will be a unit tangent vector.* In our present example, equation (10) shows that

$\frac{d\vec{R}}{dt}(= \vec{v}) = -3\sin 3t\ \vec{i} + 3\cos t\ \vec{j} + \vec{k}$, while (12) shows that

$$|\vec{v}| = \left|\frac{d\vec{R}}{dt}\right| = \sqrt{10}\ .$$

Thus,

$$\frac{d\vec{R}}{dt} \Big/ \left|\frac{d\vec{R}}{dt}\right| = \frac{-3\sin 3t\ \vec{i} + 3\cos t\ \vec{j} + \vec{k}}{\sqrt{10}} \qquad (13)$$

and since $\frac{d\vec{R}}{dt} \Big/ \left|\frac{d\vec{R}}{dt}\right| = \pm\ \vec{T}$,

$$\vec{T} = \pm\left(\frac{-3\sin 3t\ \vec{i} + 3\cos t\ \vec{j} + \vec{k}}{\sqrt{10}}\right).^{**} \qquad (14)$$

d. Since $\vec{T} \cdot \vec{T} = |\vec{T}|^2 = 1$, $\vec{T} \cdot \frac{d\vec{T}}{ds} = 0$ [c.f., Exercise 2.2.3(L)]. Thus, $\frac{d\vec{T}}{ds} \Big/ \left|\frac{d\vec{T}}{ds}\right|$ is a unit vector perpendicular to $\vec{T}$. Call this vector $\vec{N}$. Then, if we multiply $\frac{d\vec{T}}{ds}$ by the scalar $\left|\frac{d\vec{T}}{ds}\right| \Big/ \left|\frac{d\vec{T}}{ds}\right|$,

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*We say "a" rather than "the" since if the curve has a sense of its own and $\vec{T}$ denotes the unit tangent vector with this sense, then $\frac{d\vec{R}}{dt} \Big/ \frac{d\vec{R}}{dt} = \pm\ \vec{T}$ depending on whether $\frac{d\vec{R}}{dt}$ and $\frac{d\vec{R}}{ds}$ have the same sense (i.e., whether $\frac{ds}{dt}$ is positive or negative).

**Unless the sense is previously imposed on the curve C, we choose its sense so that the positive sign in (14) is used to determine T.

2.3.1 continued

$$\frac{d\vec{T}}{ds} = \left|\frac{d\vec{T}}{ds}\right|\left(\frac{d\vec{T}}{ds}\Big/\left|\frac{d\vec{T}}{ds}\right|\right)$$

$$= \left|\frac{d\vec{T}}{ds}\right| \vec{N}. \tag{15}$$

If we define $\left|\frac{d\vec{T}}{ds}\right|$ to be curvature, κ; we have from (15)

$$\frac{d\vec{T}}{ds} = \kappa\vec{N}. \tag{16}$$

In our present example, from (14), we have

$$\frac{d\vec{T}}{dt} = \frac{1}{\sqrt{10}}(-9\cos 3t\ \vec{i} - 9\sin 3t\ \vec{j}). \tag{17}$$

Therefore, $\frac{d\vec{T}}{ds} = \frac{d\vec{T}}{dt}\Big/\frac{ds}{dt} = \frac{1}{\sqrt{10}}\frac{d\vec{T}}{dt}$.

Therefore, from (17) $\frac{d\vec{T}}{ds} = \frac{9}{10}(-\cos 3t\ \vec{i} - \sin 3t\ \vec{j})$. (18)

Therefore, $\left|\frac{dT}{ds}\right| = \frac{9}{10} = \kappa$.

From (16),

$$\frac{d\vec{T}}{ds} = \kappa\vec{N} = \left|\frac{d\vec{T}}{ds}\right|\left(\frac{d\vec{T}}{ds}\Big/\left|\frac{d\vec{T}}{ds}\right|\right)$$

we have

$$\frac{d\vec{T}}{ds} = \underset{\substack{\uparrow\\ \kappa}}{\frac{9}{10}}\underbrace{(-\cos 3t\ \vec{i} - \sin 3t\ \vec{j})}_{\vec{N}}$$

While we have now solved this part of the exercise, it might still be beneficial to see what the significance of $\vec{N}$ really is. Notice that when we restricted our study to plane curves, once $\vec{T}$ was determined, $\vec{N}$ was determined up to sense. That is, since $\vec{T}$ and $\vec{N}$ had to

2.3.1 continued

be in the same plane, $\vec{N}$ was determined up to sense once we knew that it had to be perpendicular to $\vec{T}$. In 3-space, however, the problem is a bit more complex. The locus of all vectors $\vec{N}$ such that $\vec{T} \cdot \vec{N} = 0$ is now a plane. Thus, we can find an infinity of vectors, $\vec{N}$, each with different directions such that $\vec{T} \cdot \vec{N} = 0$. Intuitively, if we think of the plane which contains C in a small neighborhood of P_0, we may think of $\vec{T}$ as lying in this plane, and in terms of this plane (i.e., from an instantaneous point of view, we are back to the planar case), we expect that $\frac{d\vec{T}}{ds}$ should also lie in this same plane. (In other words, for small increments, we feel that $\vec{T}$ and $\vec{T} + \Delta\vec{T}$ are in the same plane.)

The point is that $\vec{T}$ and $\vec{N}$ (as defined in this exercise) determine a plane to C at P_0 (as would $\vec{T}$ and any other vector), and this particular plane is called the osculating plane to C at P_0. Physically, it means that at any given instant we can assume that the particle is travelling in its osculating plane rather than along the space curve. In this sense, the osculating plane is in 3-dimensions what the osculating circle was in 2-dimensions.

e. With $\vec{T}$ and $\vec{N}$ as defined in the previous parts of this exercise, we may now view a plane moving along the curve from point to point. This plane, as we saw in the previous part of this exercise, is the osculating plane. If we now define $\vec{B}$ by $\vec{B} = \vec{T} \times \vec{N}$, we may view $\vec{B}$ as a unit tangent vector (since it is the cross product of two unit tangent vectors) which is perpendicular to the osculating plane.

Since $\vec{B}$ is a unit vector, we know that it can only change in direction, not magnitude, hence $\frac{d\vec{B}}{ds}$ must measure the change in direction of $\vec{B}$ as we move along the curve. Since $\vec{B}$ is always normal to the osculating plane, the change in direction of $\vec{B}$ also measures the change in direction of the osculating plane. That is, the magnitude of $\frac{d\vec{B}}{ds}$ measures how fast the curve is being "twisted" out of the osculating plane at a given point.*

We also know that since $\vec{B}$ has constant magnitude, $\frac{d\vec{B}}{ds}$ is perpendicular to $\vec{B}$. Hence $\frac{d\vec{B}}{ds}$ is a vector which lies in the osculating plane (since this plane is the locus of all vectors perpendicular to $\vec{B}$).

*In the special case of a plane curve, $\vec{T}$ and $\vec{N}$ always lie in the same plane, say, the xy-plane. In this case $\vec{B} = \vec{k}$ whence $\frac{d\vec{B}}{ds} = 0$. In other words there is no torsion for a plane curve.

2.3.1 continued

More analytically,

Since $\vec{B} = \vec{T} \times \vec{N}$,

$$\frac{d\vec{B}}{ds} = \left(\frac{d\vec{T}}{ds} \times \vec{N}\right) + \left(\vec{T} \times \frac{d\vec{N}}{ds}\right).$$

From d., $\frac{d\vec{T}}{ds} = \kappa\vec{N}$.

Therefore, $\frac{d\vec{B}}{ds} = (\kappa\vec{N} \times \vec{N}) + \left(\vec{T} \times \frac{d\vec{N}}{ds}\right)$, and since $\vec{N} \times \vec{N} = \vec{0}$,

$$\frac{d\vec{B}}{ds} = \vec{T} \times \frac{d\vec{N}}{ds}. \qquad (19)$$

From (19), $\frac{d\vec{B}}{ds}$ is perpendicular to $\vec{T}$ (it is also perpendicular to $\frac{d\vec{N}}{ds}$ but we do not need this for our purpose) and we already know that $\frac{d\vec{B}}{ds}$ is also perpendicular to $\vec{B}$.

Since $\frac{d\vec{B}}{ds}$ is perpendicular to both $\vec{B}$ and $\vec{T}$, $\frac{d\vec{B}}{ds}$ is parallel to $\vec{N}$.

Hence, $\frac{d\vec{B}}{ds} = \tau\vec{N}$ where $|\tau| = \left|\frac{d\vec{B}}{ds}\right|$ since $\vec{N}$ is a unit vector. That is, $\tau = \pm \left|\frac{dB}{ds}\right|$ and is called torsion (a measure of the "twist") of the curve.

2.3.2

a. $\vec{v} = \frac{d\vec{R}}{dt} = \frac{d\vec{R}}{ds}\frac{ds}{dt} = \vec{T}\frac{ds}{dt}$

i.e.,

$$\vec{v} = \frac{ds}{dt}\vec{T}. \qquad (1)$$

Then,

2.3.2 continued

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\vec{T}\right)$$

$$= \frac{ds^2}{dt^2}\vec{T} + \frac{ds}{dt}\frac{d\vec{T}}{dt}$$

$$= \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\frac{d\vec{T}}{ds}\frac{ds}{dt}. \tag{2}$$

Since $\frac{d\vec{T}}{ds} = \kappa\vec{N}$, (2) yields

$$\vec{a} = \frac{d^2s}{dt^2}\vec{T} + \kappa\left(\frac{ds}{dt}\right)^2\vec{N}. \tag{3}$$

[Note that (3) has precisely the same form as in the 2-dimensional case even though $\vec{T}$ and $\vec{N}$ are now space vectors. Notice that (3) agrees perfectly with the 2-dimensional case if we think in terms of the particle being in its osculating plane.]

From (1) and (3), we obtain

$$\vec{v} \times \vec{a} = \frac{ds}{dt}\frac{d^2s}{dt^2}(\vec{T} \times \vec{T}) + \kappa\left(\frac{ds}{dt}\right)^3(\vec{T} \times \vec{N})$$

and since $\vec{T} \times \vec{T} = \vec{0}$ while $\vec{T} \times \vec{N} = \vec{B}$, we have

$$\vec{v} \times \vec{a} = \kappa\left(\frac{ds}{dt}\right)^3\vec{B}. \tag{4}$$

Therefore, $|\vec{v} \times \vec{a}| = |\kappa| \ \left|\frac{ds}{dt}\right|^3 |\vec{B}| = \kappa|\vec{v}|^3$ (since $|\vec{B}| = 1$ and κ is positive).

Not only does (4) reestablish the validity of the recipe

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3} \tag{5}$$

2.3.2 continued

even for 3-dimensions but it also tells us that $\vec{v} \times \vec{a}$ has the same direction as $\vec{B}$ which reaffirms why we call the plane determined by $\vec{T}$ and $\vec{N}$ the osculating plane. Namely, this result is consistent with $\vec{v}$ and $\vec{a}$ both being in the osculating plane.

b. We now use the result of a. as follows.

From the equation for $\vec{R}$, we can find $\vec{v}$ and $\vec{a}$. By forming the cross product of $\vec{v}$ and $\vec{a}$ we get a vector parallel to $\vec{B}$, hence normal to the plane determined by $\vec{T}$ and $\vec{N}$ (the osculating plane). We then write down the equation of the plane. More specifically,

$\vec{R} = t\,\vec{i} + 2t^2\vec{j} + t^3\vec{k}$.

Therefore, $\vec{v} = \vec{i} + 4t\vec{j} + 3t^2\vec{k}$

$\vec{a} = 4\vec{j} + 6t\,\vec{k}$.

Therefore, at the point corresponding to $t = 1$, we have

$\vec{R} = \vec{i} + 2\vec{j} + \vec{k}$

$\vec{v} = \vec{i} + 4\vec{j} + 3\vec{k}$

$\vec{a} = \quad 4\vec{j} + 6\vec{k}$ (6)

Therefore, $\vec{v} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 4 & 3 \\ 0 & 4 & 6 \end{vmatrix} = 12\vec{i} - 6\vec{j} + 4\vec{k}$. (7)

Therefore, $12\vec{i} - 6\vec{j} + 4\vec{k}$ or, equivalently, $6\vec{i} - 3\vec{j} + 2\vec{k}$ is normal to the required plane, while (1,2,1) is a point in the plane (this is the meaning of $\vec{R} = \vec{i} + 2\vec{j} + \vec{k}$).

Therefore, the equation of the plane is

$6(x - 1) - 3(y - 2) + 2(z - 1) = 0$

or

$6x - 3y + 2z = 2$.

2.3.2 continued

c. From (6), $|\vec{v}| = \sqrt{1 + 16 + 9} = \sqrt{26}$. Therefore $|\vec{v}|^3 = 26^{3/2} = 26\sqrt{26}$, while from (7)

$$|\vec{v} \times \vec{a}| = \sqrt{144 + 36 + 16}^* = \sqrt{196} = 14.$$

Therefore, from (5)

$$\kappa = \frac{14}{26\sqrt{26}} = \frac{14\sqrt{26}}{(26)(26)} = \frac{7\sqrt{26}}{338}.$$

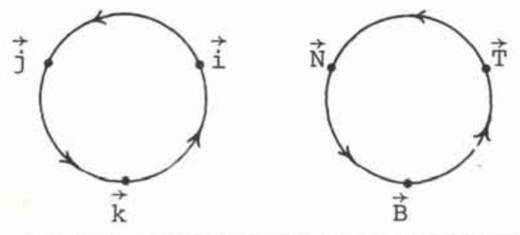
[Note that from a physical point of view the meaning of parts b. and c. is that, at the given instant $t = 1$, we may assume that the particle is at the point $(1,2,1)$ in the plane $6x - 3y + 2z = 2$, moving along the circle in that plane (the osculating circle) whose radius is $\rho = \frac{1}{\kappa} = \frac{13\sqrt{26}}{7}$.]

d. Most of the information we need for this problem is already known. Namely, to all intents and purposes, we know $\vec{T}$, since it is simply $\frac{\vec{v}}{|\vec{v}|}$. We essentially know $\vec{B}$, since we have already shown that

$$\vec{v} \times \vec{a} = \kappa|\vec{v}|^3\,\vec{B} \qquad (8)$$

and we have found κ and $\vec{v}$.

Once $\vec{B}$ and $\vec{T}$ are known, we find $\vec{N}$ from the relation $\vec{N} = \vec{B} \times \vec{T}$. (In the same way that $\vec{T}$ and $\vec{N}$ behave structurally like $\vec{i}$ and $\vec{j}$ at any point on a plane curve; $\vec{T}$, $\vec{N}$, and $\vec{B}$ behave like $\vec{i}$, $\vec{j}$, and $\vec{k}$ at any point on a space curve. Consequently, we may view $\vec{B} = \vec{T} \times \vec{N}$ in the equivalent forms, $\vec{T} = \vec{N} \times \vec{B}$ and $\vec{N} = \vec{B} \times \vec{T}$.) Pictorially,



*Notice here that we must use $12\vec{i} - 6\vec{j} + 4\vec{k}$ not another scalar muliiple such as $6\vec{i} - 3\vec{j} + 2\vec{k}$ since the recipe for κ specifically requires $|\vec{v} \times \vec{a}|$.

2.3.2 continued

More specifically, at t = 1, we saw earlier in this exercise that

$\vec{v} = \vec{i} + 4\vec{j} + 3\vec{k}$ therefore, $|\vec{v}| = \sqrt{26}$

$\vec{a} = \quad 4\vec{j} + 6\vec{k}$

$\vec{v} \times \vec{a} = 12\vec{i} - 6\vec{j} + 4\vec{k}$ therefore $|\vec{v} \times \vec{a}| = 14$.

Therefore, $\kappa = \dfrac{|v \times a|}{|\vec{v}|^3} = \dfrac{14}{26\sqrt{26}}$.

Hence (8) yields

$$12\vec{i} - 6\vec{j} + 4\vec{k} = \frac{14}{26\sqrt{26}}(26\sqrt{26})\,\vec{B} = 14\vec{B}.$$

Therefore, $\vec{B} = \frac{6}{7}\vec{i} - \frac{3}{7}\vec{j} + \frac{2}{7}\vec{k} = \dfrac{6\vec{i} - 3\vec{j} + 2\vec{k}}{7}$.

Moreover,

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \underline{\frac{\vec{i} + 4\vec{j} + 3\vec{k}}{\sqrt{26}}} .$$

Therefore $\vec{N} = \vec{B} \times \vec{T} = \dfrac{1}{7\sqrt{26}} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & -3 & 2 \\ 1 & 4 & 3 \end{vmatrix}$

$$= \frac{1}{7\sqrt{26}}\left[-17\,\vec{i} - 16\,\vec{j} + 27\,\vec{k}\right] .$$

2.3.3

$\vec{N} = \vec{B} \times \vec{T}$ implies that

$$\frac{d\vec{N}}{ds} = \left(\vec{B} \times \frac{d\vec{T}}{ds}\right) + \left(\frac{d\vec{B}}{ds} \times \vec{T}\right). \qquad (1)$$

Since $\dfrac{d\vec{T}}{ds} = \kappa\vec{N}$ and $\dfrac{d\vec{B}}{ds} = \tau\vec{N}$, (1) becomes

2.3.3 continued

$$\frac{d\vec{N}}{ds} = (\vec{B} \times \kappa\vec{N}) + (\tau\vec{N} \times \vec{T})$$

$$= \kappa(\vec{B} \times \vec{N}) + \tau(\vec{N} \times \vec{T}). \qquad (2)$$

From the cyclic orientation of $\vec{T}$, $\vec{N}$, and $\vec{B}$,

$\vec{B} \times \vec{N} = -\vec{T}$ and $\vec{N} \times \vec{T} = -\vec{B}$.

Hence, (2) becomes

$$\frac{d\vec{N}}{ds} = -\kappa\vec{T} - \tau\vec{N}.$$

2.3.4

a. We have already seen that

$$\frac{d\vec{R}}{ds} = \vec{T} \qquad (1)$$

and

$$\frac{d^2\vec{R}}{ds^2} = \frac{d\vec{T}}{ds} = \kappa\vec{N}. \qquad (2)$$

From (2) $\dfrac{d^3\vec{R}}{ds^3} = \dfrac{d}{ds}(\kappa\vec{N})$

Therefore, $\dfrac{d^3\vec{R}}{ds^3} = \kappa\dfrac{d\vec{N}}{ds} + \dfrac{d\kappa}{ds}\vec{N}.$ (3)

From our previous discussion,

$$\frac{d\vec{N}}{ds} = -\kappa\vec{T} - \tau\vec{B}. \qquad (4)$$

Putting (4) into (3) yields

2.3.4 continued

$$\frac{d^3\vec{R}}{ds^3} = \kappa(-\kappa\vec{T} - \tau\vec{B}) + \frac{d\kappa}{ds}\vec{N} .$$

Therefore, $\dfrac{d^3\vec{R}}{ds^3} = -\kappa^2 T + \dfrac{d\kappa}{ds}\vec{N} - \kappa\tau\vec{B}.$ (5)

b. The interesting point here is that $\vec{T}$, $\vec{N}$, and $\vec{B}$ behave structurally just the same as $\vec{i}$, $\vec{j}$, and $\vec{k}$. Thus, if two vectors are written in terms of $\vec{T}$, $\vec{N}$, and $\vec{B}$ components we may compute their crossproduct just as we did in the $\vec{i}$, $\vec{j}$, and $\vec{k}$ case. This is perhaps what might motivate one to compute

$$\frac{d\vec{R}}{ds} \cdot \left(\frac{d^2\vec{R}}{ds^2} \times \frac{d^3\vec{R}}{ds^3}\right).$$

Namely, since the first vector has only a T component and the second only an $\vec{N}$ component, we see that the determinant that yields the triple scalar product is given by:

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & \kappa & 0 \\ -\kappa^2 & \frac{d\kappa}{ds} & -\kappa\tau \end{vmatrix}$$

and expanding along the top row yields

$$1[-\kappa^2\tau - 0\,\frac{d\kappa}{ds}] = -\kappa^2\tau.$$

In other words,

$$-\kappa^2\tau = \frac{d\vec{R}}{ds} \cdot \left(\frac{d^2\vec{R}}{ds^2} \times \frac{d^3\vec{R}}{ds^3}\right). \quad (6)$$

Finally, since $\vec{N} \cdot \vec{N} = 1$, we may use (2) to commute

$$\frac{d^2\vec{R}}{ds^2} \cdot \frac{d^2\vec{R}}{ds^2} = \kappa^2. \quad (7)$$

Putting (7) into (6) yields the desired result:

2.3.4 continued

$$\tau = \frac{-\left(\frac{d\vec{R}}{ds} \cdot \frac{d^2\vec{R}}{ds^2} \times \frac{d^3\vec{R}}{ds^3}\right)}{\frac{d^2\vec{R}}{ds^2} \cdot \frac{d^2\vec{R}}{ds^2}} \tag{8}$$

While we make no attempt to prove it here, it is an interesting fact that knowing κ and τ at each point on the curve is enough to determine the shape of curve uniquely up to position in space. This is analogous in the study of plane curves to saying that knowing the slope of the curve determines the shape of the curve up to position. (In fact, this is precisely what adding an arbitrary constant to the indefinite integral is all about.)

c. In Cartesian coordinates

$\vec{R} = x\vec{i} + y\vec{j} + z\vec{k}.$

Therefore,

$$\left.\begin{aligned} \frac{d\vec{R}}{ds} &= \frac{dx}{ds}\vec{i} + \frac{dy}{ds}\vec{j} + \frac{dz}{ds}\vec{k} \\ \frac{d^2\vec{R}}{ds^2} &= \frac{d^2x}{ds^2}\vec{i} + \frac{d^2y}{ds^2}\vec{j} + \frac{d^2z}{ds^3}\vec{k} \\ \frac{d^3\vec{R}}{ds^3} &= \frac{d^3x\vec{i}}{ds^3} + \frac{d^3y}{ds^3}\vec{j} + \frac{d^3z}{ds^3}\vec{k}\ . \end{aligned}\right\} \tag{9}$$

Therefore $\kappa^2 = \frac{d^2\vec{R}}{ds^2} \cdot \frac{d^2\vec{R}}{ds^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^3}\right)^2\ .$

While from (6) and (9)

2.3.4 continued

$$-\kappa^2\tau = \begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}$$ where the differentiation is with respect to s.

Therefore,

$$\tau = -\frac{\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}}{(x'')^2 + (y'')^2 + (z'')^2}$$

2.3.5

a. The mimicking procedure will have us replace $\frac{d\vec{R}}{ds}$, $\frac{d^2\vec{R}}{ds^2}$, and $\frac{d^3\vec{R}}{ds^3}$ by $\frac{d\vec{R}}{dt}$, $\frac{d^2\vec{R}}{dt^2}$, and $\frac{d^3\vec{R}}{dt^3}$; or $\vec{v}$, $\vec{a}$, and $\frac{d\vec{a}}{dt}$. (In other words $\frac{d\vec{a}}{dt}$ will play a prominent role in 3-dimensions when we are interested in torsion.)

From earlier work, we have

$$\vec{v} = \frac{ds}{dt}\vec{T} \tag{1}$$

$$\vec{a} = \frac{d^2s}{dt^2}\vec{T} + \kappa\frac{ds}{dt}\vec{N} \tag{2}$$

From (2) we can compute $\frac{d\vec{a}}{dt}$ (which we do for practice, but we will only need the coefficient of $\vec{B}$ since

$$\vec{v}\cdot\left(\vec{a} \times \frac{da}{dt}\right) = \begin{vmatrix} \frac{ds}{dt} & 0 & 0 \\ \frac{d^2s}{dt^2} & \kappa\frac{ds}{dt} & 0 \\ a_1 & a_2 & a_3 \end{vmatrix}$$ where $\frac{d\vec{a}}{dt} = a_1\vec{T} + a_2\vec{N} + a_3\vec{B}$)

and expanding this determinant along the first row, we obtain

$$\frac{ds}{dt}[a_3\kappa\frac{ds}{dt} - a_2 0] = a_3\kappa\left(\frac{ds}{dt}\right)^2.$$

2.3.5 continued

In other words, neither a_1 nor a_2 appear in the value of the determinant.

At any rate, we obtain from (2) (and recalling that $\frac{d^2s}{dt^2}$, $\vec{T}$, κ, $\frac{ds}{dt}$, and $\vec{N}$ are all functions of t)

$$\frac{d\vec{a}}{dt} = \frac{d^3s}{dt^3}\vec{T} + \frac{d^2s}{dt^2}\frac{d\vec{T}}{dt} + \frac{d\kappa}{dt}\frac{ds}{dt}\vec{N} + \kappa\frac{d^2s}{dt^2}\vec{N} + \kappa\frac{ds}{dt}\frac{d\vec{N}}{dt}$$

$$= \frac{d^3s}{dt^3}\vec{T} + \frac{d^2s}{dt^2}\frac{d\vec{T}}{ds}\frac{ds}{dt} + \frac{d\kappa}{dt}\frac{ds}{dt}\vec{N} + \kappa\frac{d^2s}{dt^2}\vec{N} + \kappa\frac{ds}{dt}\frac{d\vec{N}}{ds}\frac{ds}{dt}.$$

Then since $\frac{d\vec{T}}{ds} = \kappa\vec{N}$ and $\frac{d\vec{N}}{ds} = -\kappa\vec{T} - \tau\vec{B}$, we have:

$$\frac{d\vec{a}}{dt} = \frac{d^3s}{dt^3}\vec{T} + \frac{d^2s}{dt^2}\frac{ds}{dt}\kappa\vec{N} + \frac{d\kappa}{dt}\frac{ds}{dt}\vec{N} + \kappa\frac{d^2s}{dt^2}\vec{N} + \kappa\left(\frac{ds}{dt}\right)^2(-\kappa\vec{T} - \tau\vec{B}).$$

Therefore, $$\frac{d\vec{a}}{dt} = \left[\frac{d^3s}{dt^3} - \kappa^2\left(\frac{ds}{dt}\right)^2\right]\vec{T} + \left[\kappa\frac{d^2s}{dt^2} + \frac{d\kappa}{dt}\frac{ds}{dt} + \kappa\frac{ds}{dt}\frac{d^2s}{dt^2}\right]\vec{N}$$

$$- \kappa\tau\left(\frac{ds}{dt}\right)^2\vec{B}. \qquad (3)$$

(As a check of our earlier results if we think of t as being an arbitrary parameter in (1), (2), and (3) and then let t = s, so that $\frac{ds}{dt} = 1$, $\frac{d^2s}{dt^2} = \frac{d^3s}{dt^3} = 0$, we obtain:

(1') $\frac{d\vec{R}}{ds} = \vec{T}$.

(2') $\frac{d^2\vec{R}}{ds^2} = 0\vec{T} + \kappa(1)\vec{N} = \kappa\vec{N}$.

(3') $\frac{d^3R}{ds^3} = [0 - \kappa^2(1)^2]\vec{T} + [0 + \frac{d\kappa}{ds}(1) + 0]\vec{N} - \kappa\tau(1)^2\vec{B}$

$= -\kappa^2\vec{T} + \frac{d\kappa}{ds}\vec{N} - \kappa\tau\vec{B}$

2.3.5 continued

and these check with equations (1), (2), and (5) of Exercise 2.3.4 .)

Notice also, in passing, that the derivation of equation (3) was purely mathematical and required no insight to the physics of space motion. To be sure, it might be nice to have a feeling for $\frac{d\kappa}{dt}$ as the rate of change of curvature etc., but this is completely unnecessary for the problem we are trying to solve.

Returning to the problem at hand we have:

$$\vec{v}\cdot\left(\vec{a}\times\frac{d\vec{a}}{dt}\right)=\begin{vmatrix} \frac{ds}{dt} & 0 & 0 \\ \frac{d^2s}{dt^2} & \kappa\frac{ds}{dt} & 0 \\ a_1 & a_2 & -\kappa\tau\left(\frac{ds}{dt}\right)^2 \end{vmatrix}$$

where from (3), $a_1 = \frac{d^3s}{dt^3} - \kappa^2\left(\frac{ds}{dt}\right)^2$ and

$$a_2 = \kappa\frac{d^2s}{dt^2} + \frac{d\kappa}{dt}\frac{ds}{dt} + \kappa\frac{ds}{dt}\frac{d^2s}{dt^2}\ .$$

Therefore, $\vec{v}\cdot\left(\vec{a}\times\frac{d\vec{a}}{dt}\right) = -\kappa^2\tau\left(\frac{ds}{dt}\right)^4$. (4)

(Again, as a partial check of (4), if we let t = s we obtain $\frac{d\vec{R}}{ds}\cdot\left(\frac{d^2\vec{R}}{ds^2}\times\frac{d^3\vec{R}}{ds^3}\right) = -\kappa^2\tau$, which checks with our earlier result.)

Finally, if we invoke the previously proven result that

$$\kappa = \frac{|\vec{v}\times\vec{a}|}{|\vec{v}|^3}$$

2.3.5 continued

and if we recall that $\left(\frac{ds}{dt}\right)^4$ is the same as $|v|^4$ *, then equation (4) becomes

$$\vec{v}\cdot\left(\vec{a}\times\frac{d\vec{a}}{dt}\right) = -\frac{|\vec{v}\times\vec{a}|^2}{|\vec{v}|^6}\,\tau|\vec{v}|^4 = -\frac{\tau|\vec{v}\times\vec{a}|^2}{|\vec{v}|^2}.$$

Therefore, $$\tau = \frac{-|\vec{v}|^2\left[\vec{v}\cdot\left(\vec{a}\times\frac{d\vec{a}}{dt}\right)\right]}{|\vec{v}\times\vec{a}|^2}. \qquad (5)$$

b. We had established that (Problem 2.3.5 b.)

$\vec{R} = t\,\vec{i} + 2t^2\vec{j} + t^3\vec{k}$

$\vec{v} = \vec{i} + 4t\,\vec{j} + 3t^2\vec{k}$

$\vec{a} = \quad 4\,\vec{j} + 6t\,\vec{k}$

and we now add to these equations:

$\frac{d\vec{a}}{dt} = 6\vec{k}$.

In particular at $t = 1$; $\vec{v} = \vec{i} + 4\vec{j} + 3\vec{k}$,

$\vec{a} = 4\vec{j} + 6\vec{k}$ and $\frac{d\vec{a}}{dt} = 6$; therefore, at $t = 1$ we have

$|\vec{v}| = \sqrt{1 + 16 + 9} = \sqrt{26}$

$$\vec{v}\cdot\left(\vec{a}\times\frac{d\vec{a}}{dt}\right) = \begin{vmatrix} 1 & 4 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 6 \end{vmatrix} = 24$$

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*In general it is possible that $\frac{ds}{dt}$ and $|\vec{v}|$ differ in sign, but in our case, only even powers of $\frac{ds}{dt}$ occur so everything is always non-negative.

2.3.5 continued

$$\vec{v} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 4 & 3 \\ 0 & 4 & 6 \end{vmatrix} = 12\vec{i} - 6\vec{j} + 4\vec{k}.$$

Therefore, $|\vec{v} \times \vec{a}| = \sqrt{144 + 36 + 16} = 14.$

Putting these results into (5) yields

$$\tau = -\frac{26(24)}{196} = -\frac{156}{49}$$

Therefore $|\tau| = \frac{156}{49}.$

Again, it is not our purpose to have you learn a short course in Differential Geometry or the like, but rather that you get the feeling of what curvature and torsion mean physically. The connection between 2-dimensional and 3-dimensional space curve is that the vectors $\vec{T}$ and $\vec{N}$ as they exist in the 2-dimensional case form the osculating plane at any instant in the 3-dimensional case.

In closing, it should be pointed out that the quantities $\frac{d\vec{T}}{ds}$, $\frac{d\vec{B}}{ds}$, and $\frac{d\vec{N}}{ds}$ obviously play important roles in the study of space curves. We have already shown that:

$$\frac{d\vec{T}}{ds} = \kappa\vec{N}$$

$$\frac{d\vec{B}}{ds} = \tau\vec{N}$$

$$\frac{d\vec{N}}{ds} = -\tau\vec{B} - \kappa\vec{N}.$$

These three results are known as <u>Frenet's formulas</u>.