Solutions

BLOCK 2:
VECTOR CALCULUS

Pretest

1. (a) $\vec{f}(t)\cdot\vec{f}'(t) = 0$, i.e., if $\vec{f}(t)$ and $\vec{f}'(t) \neq 0$ then $\vec{f}(t)\perp\vec{f}'(t)$.

   (b) The acceleration is always directed toward the center of the circle.

2. (a) $a_T = \dfrac{4t}{\sqrt{1+4t^2}}$, $a_N = \dfrac{2}{\sqrt{1+4t^2}}$

   (b) $\kappa = \dfrac{2}{(1+4t^2)^{3/2}}$

3. r=0 (the origin), $\left(\frac{5-\sqrt{17}}{4}, \pm\cos^{-1} \frac{1-\sqrt{17}}{4} \right)$ [$\approx$(0.23, $\pm$141°)],

   (-1, $\pm$90°), ($-\frac{1}{2}$, $\pm$ 60°) (Therefore, seven points in all)

4. $\frac{\pi}{8} - \frac{1}{16}$

5. $a_r = -\frac{\pi}{4}(2+\pi) \approx -4$ ft/sec^2

   $a_\theta = \frac{\pi}{2}(1-\pi) \approx -(3.4)$ ft/sec^2

Unit 1: Differentiation of Vector Functions

2.1.1(L)

The only reason this is a learning exercise is that we wanted at least one more opportunity to emphasize how the parallel structure of vector arithmetic to numerical arithmetic allows us to carry proofs over from one to the other.

While we do not include the steps used in the numerical proofs, you should be able to verify that each proof in this exercise is a "transliteration" of the corresponding numerical theorem.

The only place we must be on our guard is in any place where a scalar proof hinges on the fact that $|ab| = |a||b|$, for with respect to either the dot product or the cross product, the best we can get is

$$|\vec{a} \cdot \vec{b}| \leqslant |\vec{a}||\vec{b}|$$

$$|\vec{a} \times \vec{b}| \leqslant |\vec{a}||\vec{b}|.$$

With these remarks in mind, we continue with this exercise.

a. Here we pick $\vec{f}(x) = \vec{c}$ in our definition of $\lim_{x \to a} \vec{f}(x)$. What we must show is that for $\varepsilon > 0$ we can find $\delta > 0$ such that

$$0 < |x - a| < \delta \rightarrow |\vec{f}(x) - \vec{L}| < \varepsilon$$

In this example, $\vec{f}(x) = \vec{c}$ and we are testing $\vec{L} = \vec{c}$. Thus, we must show that for $\varepsilon > 0$ we can find $\delta > 0$ such that

$$0 < |x - a| < \delta \rightarrow |\vec{c} - \vec{c}| < \varepsilon. \qquad (1)$$

But $|\vec{c} - \vec{c}| \equiv 0$.

Therefore, $\varepsilon > 0 \rightarrow |\vec{c} - \vec{c}| < \varepsilon$.

Hence we may choose $\delta > 0$ arbitrarily to satisfy (1).

2.1.1(L) continued

b. For a given $\varepsilon > 0$ we must show that we can find $\delta > 0$ such that

$$0 < |x - a| < \delta \rightarrow |\vec{c} \cdot \vec{f}(x) - \vec{c} \cdot \vec{L}| < \varepsilon.$$

Now,

$|\vec{c} \cdot \vec{f}(x) - \vec{c} \cdot \vec{L}| =$

$|\vec{c} \cdot (\vec{f}(x) - \vec{L})| \leq$

$|\vec{c}||\vec{f}(x) - \vec{L}|$*. (2)

So given $\varepsilon > 0$ pick $\delta > 0$ such that $0 < |x - a| < \delta \rightarrow |f(x) - L| < \frac{\varepsilon}{|\vec{c}|}$.** This is possible by the definition of $\lim\limits_{x \rightarrow a} f(x) = L$.

Combining this with (2) yields

$$0 < |x - a| < \delta \rightarrow |\vec{c} \cdot \vec{f}(x) - \vec{c} \cdot \vec{L}| < |\vec{c}|\frac{\varepsilon}{|\vec{c}|}$$

therefore, $0 < |x - a| < \delta \rightarrow |\vec{c} \cdot \vec{f}(x) - \vec{c} \cdot \vec{L}| < \varepsilon$

therefore, $\lim\limits_{x \rightarrow a} \vec{c} \cdot \vec{f}(x) = \vec{c} \cdot \vec{L}$.

c. $h(x) = \vec{c} \cdot \vec{f}(x)$ (Notice that h is a scalar function.)

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*Notice that we needed nothing stronger than $|\vec{a} \cdot \vec{b}| \leq |\vec{a}||\vec{b}|$.

**This is permissible provided $|\vec{c}| \neq 0$. If, however, $|\vec{c}| = 0$, we are in no trouble since when $\vec{c} = 0$,

$$\lim_{x \rightarrow a} \vec{c} \cdot f(x) = \vec{c} \cdot \vec{L}$$

is trivially true, since both sides equal 0.

2.1.1(L) continued

Therefore, $h'(x_1) = \lim_{\Delta x \to 0} \left[\frac{h(x_1 + \Delta x) - h(x_1)}{\Delta x}\right]$

$$= \lim_{\Delta x \to 0} \left[\frac{\vec{c} \cdot \vec{f}(x_1 + \Delta x) - \vec{c} \cdot \vec{f}(x_1)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} \left[\vec{c} \cdot \left\{\frac{\vec{f}(x_1 + \Delta x) - \vec{f}(x_1)}{\Delta x}\right\}\right]$$, and by part b. above,

$$= \lim_{\Delta x \to 0} \vec{c} \cdot \lim_{\Delta x \to 0} \left[\frac{\vec{f}(x_1 + \Delta x) - \vec{f}(x_1)}{\Delta x}\right]. \qquad (3)$$

From a. $\lim_{\Delta x \to 0} \vec{c} = \vec{c}$ while, by definition,

$$\lim_{\Delta x \to 0} \left[\frac{\vec{f}(x_1 + \Delta x) - \vec{f}(x_1)}{\Delta x}\right] \vec{f}'(x).$$

Putting these results into (3) yields

$$h'(x_1) = \vec{c} \cdot \vec{f}'(x_1).$$

In other words, just as in the scalar analogue

$$[\vec{c} \cdot \vec{f}(x)]' = \vec{c} \cdot \vec{f}'(x)$$

2.1.2

a. $\vec{f}(t) = t\vec{i} + t^2\vec{j} + (2t + 1)\vec{k}$

$$\vec{g}(t) = t^3\vec{i} + 3t\vec{j} + (t^2 + 1)\vec{k}$$

Therefore, $\vec{f}(t) \times \vec{g}(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t & t^2 & 2t+1 \\ t^3 & 3t & t^2+1 \end{vmatrix}$

2.1.2 continued

$$= \vec{i}[t^2(t^2 + 1) - 3t(2t + 1)] - \vec{j}[t(t^2 + 1) - t^3(2t + 1)] + \vec{k}[t(3t) - t^3(t^2)]$$

$$= (t^4 - 5t^2 - 3t)\vec{i} + (2t^4 - t)\vec{j} + (3t^2 - t^5)\vec{k}$$

Therefore, $\frac{d}{dt}[\vec{f}(t) \times \vec{g}(t)] = (4t^3 - 10t - 3)\vec{i} + (8t^3 - 1)\vec{j} + (6t - 5t^4)\vec{k}.$ (1)

b. $\vec{f}'(t) = \vec{i} + 2t\vec{j} + 2\vec{k}$

$\vec{g}'(t) = 3t^2\vec{i} + 3\vec{j} + 2t\vec{k}$

Therefore, $[\vec{f}(t) \times \vec{g}'(t)] + [\vec{f}'(t) \times \vec{g}(t)] =$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t & t^2 & 2t+1 \\ 3t^2 & 3 & 2t \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2t & 2 \\ t^3 & 3t & t^2+1 \end{vmatrix} =$$

$\vec{i}[t^2(2t) - 3(2t + 1)] - \vec{j}[t(2t) - 3t^2(2t + 1)] + \vec{k}[3t - t^2(3t^2)] +$

$\vec{i}[2t(t^2 + 1) - 3t(2)] - \vec{j}[1(t^2 + 1) - 2t^3] + \vec{k}[1(3t) - t^3(2t)] =$

$(2t^3 - 6t - 3)\vec{i} + (6t^3 + t^2)\vec{j} + (3t - 3t^4)\vec{k} +$ (2)

$(2t^3 - 4t)\vec{i} + (2t^3 - t^2 - 1)\vec{j} + (3t - 2t^4)\vec{k} =$ (3)

$(4t^3 - 10t - 3)\vec{i} + (8t^3 - 1)\vec{j} + (6t - 5t^4)\vec{k}.$ (4)

2.1.2 continued

c. Comparing (1) and (4) yields the desired check.

d. From (2) we know that

$$\vec{f}(t) \times \vec{g}'(t) = (2t^3 - 6t - 3)\vec{i} + (6t^3 + t^2)\vec{j} + (3t - 3t^4)\vec{k} \qquad (5)$$

and since $\vec{g}(t) \times \vec{f}'(t) = -[\vec{f}'(t) \times \vec{g}(t)]$, (3) yields

$$\vec{g}(t) \times \vec{f}'(t) = (4t - 2t^3)\vec{i} + (1 + t^2 - 2t^3)\vec{j} + (2t^4 - 3t)\vec{k}. \qquad (6)$$

Adding (5) and (6) yields

$$[\vec{f}(t) \times \vec{g}'(t)] + [\vec{g}(t) \times \vec{f}'(t)] =$$

$$(-2t - 3)\vec{i} + (4t^3 + 2t^2 + 1)\vec{j} - t^4\,\vec{k}. \qquad (7)$$

e. Comparing (7) with either (4) or (1) shows that there is an error caused by changing the order of the factors in the cross product and in fact by the time we get to the final step our error is more than "just a sign change" (i.e., we cannot convert (7) to (4) just by changing a sign).

This doesn't contradict our general procedure. It merely means that we must take care with cross products not to mimic scalar proofs which use the commutativity of multiplication since the cross product is not a commutative operation.

2.1.3

a.

$$\left(\frac{d\vec{R}}{dt}\right)_{t=t_1} = \lim_{\Delta t \to 0} \frac{\Delta\vec{R}}{\Delta t}. \qquad (1)$$

$$\text{Let } \vec{k} = \frac{\Delta\vec{R}}{\Delta t} - \left(\frac{d\vec{R}}{dt}\right)_{t=t_1} \qquad (\Delta t \neq 0). \qquad (2)$$

Then, by (1),

2.1.3 continued

$$\lim_{\Delta t \to 0} \vec{k} = 0. \tag{3}$$

From (2) we have

$$\left.\begin{aligned} \Delta R &= \left(\frac{d\vec{R}}{dt}\right)_{t=t_1} \Delta t + \vec{k}\,\Delta t \\ &\text{where, from (3), } \lim_{\Delta t \to 0} \vec{k} = 0. \end{aligned}\right\} \tag{4}$$

Observe that k must be a vector rather than a scalar because it is the difference of two vectors. Notice also that such expressions as $\frac{d\vec{R}}{dt}\,\Delta t$ and $\vec{k}\,\Delta t$ are well-defined vectors since each is a scalar times a vector. (Perhaps this would be more suggestive had we written $\Delta t\,\vec{k}$ or $\Delta t\,\frac{d\vec{R}}{dt}$ which is our usual order for writing scalar multiples.)

b. We have by part a. that

$$\Delta R = \frac{d\vec{R}}{dx}\,\Delta x + \vec{k}\,\Delta x. \tag{5}$$

(Replacing t by x has no bearing on our discussion since either t or x denotes a real variable.)

From (5) we have

$$\frac{\Delta R}{\Delta u} = \frac{d\vec{R}}{dx}\,\frac{\Delta x}{\Delta u} + \vec{k}\,\frac{\Delta x}{\Delta u}. \tag{6}$$

If we now let $\Delta u \to 0$ we observe that $\lim_{\Delta u \to 0} \frac{\Delta x}{\Delta u} = \frac{dx}{du}$ by definition of the fact that x is a differentiable function of u and we also notice that $\Delta x \to 0$ as $\Delta u \to 0$ since x is a (continuous) function of u. Then, since $\vec{k} \to 0$ as $\Delta x \to 0$, equation (6) becomes

$$\lim_{\Delta u \to 0} \frac{\Delta \vec{R}}{\Delta u} = \frac{d\vec{R}}{dx}\,\frac{dx}{du} + \vec{0}\,\frac{dx}{du}$$

or

2.1.3 continued

$$\frac{d\vec{R}}{du} = \frac{d\vec{R}}{dx}\frac{dx}{du}.$$

c. $\vec{R} = (t + 1)\vec{i} + t^2\vec{j} + t^3\vec{k} \rightarrow$

$$\frac{d\vec{R}}{dt} = \vec{i} + 2t\vec{j}\ 3t^2\vec{k}. \qquad (7)$$

But $t = u^4$; hence $\frac{dt}{du} = 4u^3$. Combining this with (7), the chain rule of part b. yields

$$\frac{d\vec{R}}{du} = \frac{d\vec{R}}{dt}\frac{dt}{du} = (\vec{i} + 2t\vec{j} + 3t^2\vec{k})4u^3$$

$$= 4u^3\vec{i} + 8tu^3\vec{j} + 12t^2\ u^3\ k. \qquad (8)$$

On the other hand, letting $t = u^4$ in our original expression for $\vec{R}$, we obtain:

$$\vec{R} = (u^4 + 1)\vec{i} + u^8\vec{j} + u^{12}\ \vec{k}$$

whereupon

$$\frac{d\vec{R}}{du} = 4u^3\vec{i} + 8u^7\vec{j} + 12u^{11}\vec{k}$$

$$= 4u^3\vec{i} + 8(u^4)u^3\vec{j} + 12(u^4)^2\ u^3\vec{k}$$

$$= 4u^3\vec{i} + 8tu^3\vec{j} + 12t^2\ u^3\vec{k}$$

which checks with our result in (8).

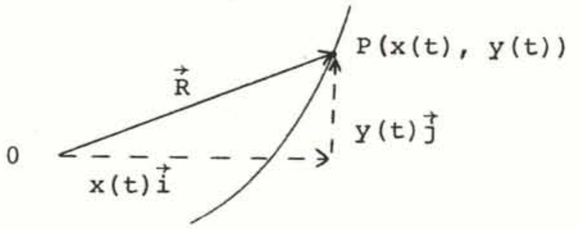
2.1.4(L)

a. We have that

$$\vec{R} = x(t)\vec{i} + y(t)\vec{j}. \qquad (1)$$

Pictorially,

2.1.4(L)



$$C: \begin{cases} x = x(t) \\ y = y(t) \end{cases} \qquad (1)$$

If we now differentiate (1) with respect to t, we obtain

$$\frac{d\vec{R}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j}. \qquad (2)$$

In terms of Cartesian coordinates, the slope of a vector is the quotient of the $\vec{j}$ component divided by the $\vec{i}$ component, and the magnitude of the vector is the square root of the sum of the squares of the components. Applying this to (2), we find that:

The slope of $\frac{d\vec{R}}{dt}$ is $(dy/dt)/(dx/dt) = dy/dx$ (3)

and the magnitude of $d\vec{R}/dt$ is

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \left|\frac{ds}{dt}\right|, \qquad (4)$$

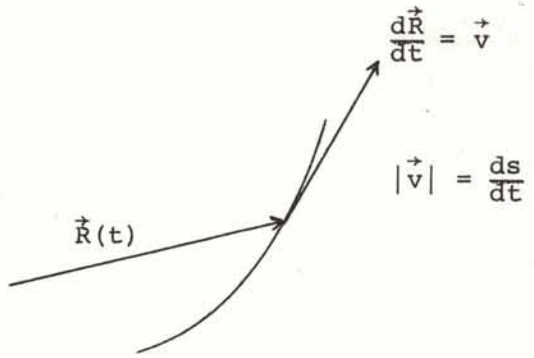
where s denotes the arclength of the curve.

If we analyze (4) we see that the magnitude of $\frac{d\vec{R}}{dt}$ is the instantaneous rate of speed of the particle along the curve (for this is precisely what ds/dt measures, namely the instantaneous rate of change of the arclength).

At the same time (3) tells us that if $\frac{d\vec{R}}{dt}$ is placed at the point P on the curve, it is tangent to the curve at that point (since the slope of the curve is dy/dx).

Thus $\frac{d\vec{R}}{dt}$ is tangent to the curve and its magnitude is the speed at which the particle moves along the curve. This is a mathematical verification of the usual physical convention that instantaneous speed is tangential to the path and its magnitude is the speed at which the arclength is being traversed. Again pictorially,

2.1.4(L) continued



NOTE: Throughout this exercise we have assumed that t denoted time. It should be noted that if $\vec{R}$ is a function of any scalar variable, say, q, then the vector $\frac{d\vec{R}}{dq}$ will still have its slope equal to $\frac{dy}{dx}$ and its magnitude will be

$$\sqrt{\frac{dx}{dq}^{2} + \frac{dy}{dq}^{2}} .$$

This follows mechanically with respect to q. I.e.,

$$\frac{d\vec{R}}{dq} = \frac{dx}{dq}\vec{i} + \frac{dy}{dq}\vec{j}.$$

From a more intuitive point of view, we realize that since $\vec{R}$ is measured from the origin to a point on the curve, $\vec{R}$ really varies only with s (arclength), but how fast it changes will depend on the rate of change of the parameter q.

In particular there will be times in our discussions when we wish to choose arclength as our parameter. For one thing arclength seems "natural" in the sense that it is independent of any particular coordinate system. Notice that if we let s = q, we find the interesting result that

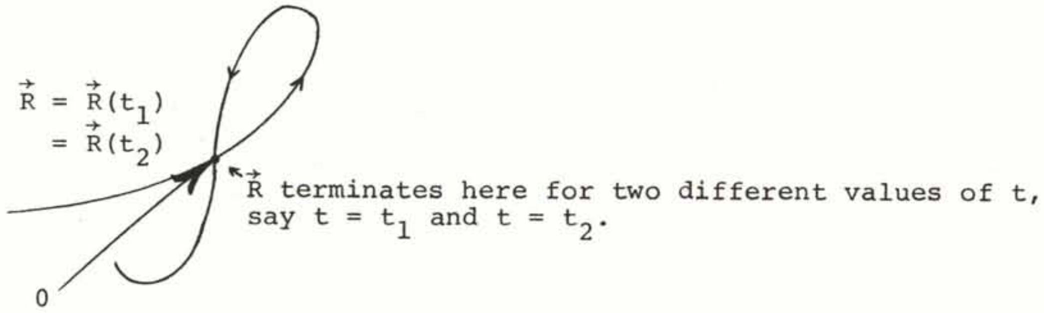
$$\left|\frac{d\vec{R}}{ds}\right| = 1,$$

since

$$dx^2 + dy^2 = ds^2.$$

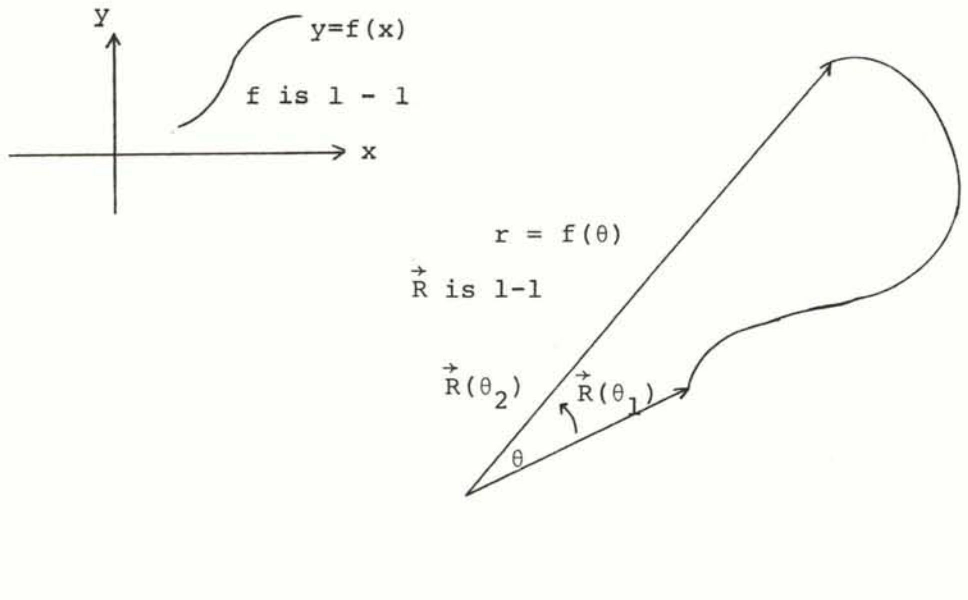
2.1.4(L) continued

b. If $\vec{R}(t_1) = \vec{R}(t_2)$ this means that the same point occurs on the path for two different values of t. Pictorially,



In other words, in terms of the vector form, the fact that $\vec{R}$ is a 1 - 1 function of t means that the path of motion never crosses itself.

Notice the difference between $\vec{R}$ being 1 - 1 and f being 1 - 1 as far as graphing is concerned. When f is 1 - 1, the usual Cartesian coordinate graph indicates that a line parallel to the x-axis meets the graph in at most one point. When $\vec{R}$ is 1 - 1, the graph can double back as far as x and y coordinates are concerned, but it cannot cross itself. Graphically,



2.1.5

a. Since $x = e^t$ and $y = e^{2t}$, the equation of motion in vector form becomes:

$$\vec{R} = e^t \vec{i} + e^{2t} \vec{j}$$

whereupon

$$\vec{v} = e^t \vec{i} + 2e^{2t} \vec{j}$$

and

$$\vec{a} = e^t \vec{i} + 4e^{2t} \vec{j}.$$

b. At $t = 0$ we have:

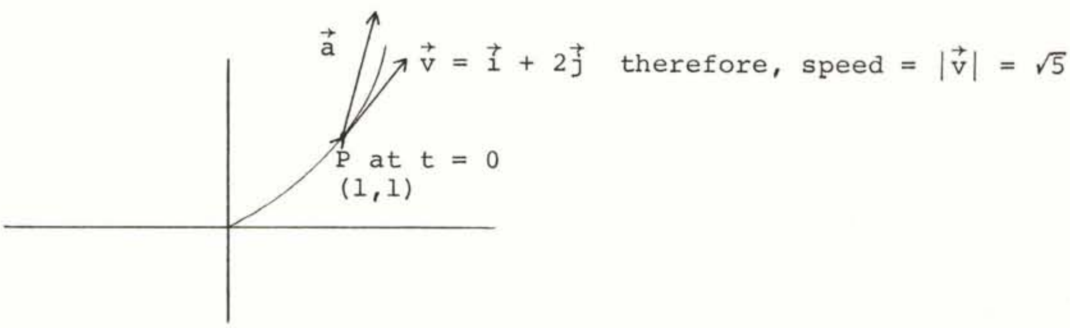
$$\vec{R} = \vec{i} + \vec{j}$$

$$\vec{v} = \vec{i} + 2\vec{j}$$

and

$$\vec{a} = \vec{i} + 4\vec{j}.$$

Hence, at $t = 0$, the particle is at (1,1) and has a speed of $v = \sqrt{(1)^2 + (2)^2} = \sqrt{5}$ ft/sec in the direction tangent to the path and an acceleration of $\sqrt{17}$ ft/sec^2 in the direction of $\vec{i} + 4\vec{j}$. Pictorially,



2.1.5 continued

c. We know that $\vec{v}$ is tangent to the curve $\vec{R} = e^t\vec{1} + e^{2t}\vec{j}$, but this is the same curve as the one whose parametric form is $x = e^t$ and $y = e^{2t}$. Thus, from part b. $\vec{1} + 2\vec{j}$ is tangent to the curve at the point corresponding to $t = 0$, and this point is $(1,1)$. The vector $\vec{1} + 2\vec{j}$ has slope 2. Hence the given curve has slope 2 at $(1,1)$.

As a check, we observe that $x = e^t$ implies that $e^{2t} = (e^t)^2 = x^2$. Thus, eliminating the parameter tells us that the equation of motion is $y = x^2$ and at $(1,1)$ the slope of this curve is 2, which checks with our answer.

(Notice that $\vec{v}$ tells us more than the direction of the curve at a point. It tells us <u>how fast</u> the particle is moving along the curve when it passes through that point.)

2.1.6(L)

a. We know from the product rule for dot products that

$$\frac{d}{dt}[\vec{f}(t) \cdot \vec{f}(t)] = \vec{f}(t) \cdot \vec{f}'(t) + \vec{f}'(t) \cdot \vec{f}(t). \qquad (1)$$

Now $\vec{f}'(t) \cdot \vec{f}(t) = \vec{f}(t) \cdot \vec{f}'(t)$ since the dot product is commutative. Thus (1) becomes

$$\frac{d}{dt}[\vec{f}(t) \cdot \vec{f}(t)] = 2[\vec{f}(t) \cdot \vec{f}'(t)]. \qquad (2)$$

Now $\vec{f}(t) \cdot \vec{f}(t) = |\vec{f}(t)|^2$.

Therefore, $\frac{d}{dt}|\vec{f}(t)|^2 = 2[\vec{f}(t) \cdot \vec{f}'(t)]$. (3)

If $|\vec{f}(t)|$ is constant* so also is $|\vec{f}(t)|^2$ and the derivative of a constant is zero. Putting this into (3) yields

$2[\vec{f}(t) \cdot \vec{f}'(t)] = 0$.

*Notice that the fact that the magnitude is constant does not require that $\vec{f}$ be constant. A vector depends on both magnitude and direction. Thus if the magnitude is constant but the direction isn't then the vector is a variable. We shall discuss this in more detail in b.

2.1.6(L) continued

Therefore, $\vec{f}(t) \cdot \vec{f}'(t) = 0$

Therefore, either $\underline{\vec{f}(t) = \vec{0}}$, $\underline{\vec{f}'(t) = \vec{0}}$, or $\underline{\vec{f}(t) \perp \vec{f}'(t)}$.

b. Let $\vec{v}$ denote the velocity of our particle. The fact that the particle has constant speed means that $|\vec{v}|$ is constant. Hence the hypothesis for part a. prevails and we have

$$\vec{v} \cdot \frac{d\vec{v}}{dt} = 0. \tag{4}$$

But, by definition, $\frac{d\vec{v}}{dt} = \vec{a}$ (= acceleration).

Hence, (4) becomes

$\vec{v} \cdot \vec{a} = 0$

from which we conclude

(1) $\vec{v} = \vec{0}$, which means the particle is stationary, a rather trivial situation

or

(2) $\vec{a} = \vec{0}$, which means that there is no acceleration, and this means that the velocity as well as the speed is constant (in other words our particle changes neither speed nor direction).

or

(3) $\vec{v} \perp \vec{a}$, which means that the acceleration is always perpendicular to the instantaneous direction of the particle (since $\vec{v}$ is in the direction of motion).

Case (3) is particularly interesting, at least in the following sense. Before we have the notion of vectors, we think of acceleration as causing a change in speed. What we have now seen is that when the speed is constant there can still be acceleration. The fact that there is acceleration means that the velocity (a vector)

2.1.6(L) continued

must be changing, but since the magnitude of the velocity (speed) is constant it must be the direction that is changing. Case (3) shows that this change in direction is always at right angles to the direction of motion.

c. In the case of a particle moving with constant speed in a circle we know from plane geometry that at any point on the circle, the tangent to the circle at that point is perpendicular to the radius drawn from the center of the circle to that point. Thus if a particle moves with non-zero constant speed in a circle, case (3) of part b. applies, and we have that the acceleration is perpendicular to the velocity. Since the velocity is tangential to the circle, it follows that the acceleration is directed along the radius.

2.1.7

a. We have

$$\vec{R} = \left[\frac{\sin^{-1} t}{2} + \frac{t\sqrt{1 - t^2}}{2}\right]\vec{i} + \tfrac{1}{2}t^2\vec{j}.$$

Hence,

$$\vec{v} = \frac{d\vec{R}}{dt} \rightarrow$$

$$\vec{v} = \left[\frac{1}{2\sqrt{1 - t^2}} + \frac{\sqrt{1 - t^2}}{2} - \frac{t^2}{2\sqrt{1 - t^2}}\right]\vec{i} + t\,\vec{j}$$

$$= \frac{1 + (1 - t^2) - t^2}{2\sqrt{1 - t^2}}\,\vec{i} + t\,\vec{j}$$

$$= \frac{1 - t^2}{\sqrt{1 - t^2}}\,\vec{i} + t\,\vec{j}$$

$$= \sqrt{1 - t^2}\,\vec{i} + t\,\vec{j}. \qquad (1)$$

2.1.7 continued

Therefore $|\vec{v}| = \sqrt{\left(\sqrt{1 - t^2}\right)^2 + t^2} = 1 = \text{constant}.$

b.

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\left(\sqrt{1 - t^2}\right)}{dt}\vec{i} + \frac{d}{dt}(t\,\vec{j})$$

$$= \frac{-t}{\sqrt{1 - t^2}}\vec{i} + \vec{j}. \qquad (2)$$

Therefore, $\vec{v} \cdot \vec{a}$ (from (1) and (2)) $= \left(\sqrt{1 - t^2}\right)\left(\frac{-t}{\sqrt{1 - t^2}}\right) + t(1)$

$$= -t + t = 0$$

c.

$$|\vec{a}|^2 = \left[\frac{-t}{\sqrt{1 - t^2}}\right]^2 + (1)^2 = \frac{t^2}{1 - t^2} + 1$$

$$= \frac{1}{1 - t^2}.$$

Therefore, the magnitude of $\vec{a}$ is variable, even though the magnitude of $\vec{v}$ is constant.

d. When $t = \frac{3}{5}$, (1) and (2) yield

$$\vec{v} = \sqrt{1 - \frac{9}{25}}\;\vec{i} + \frac{3}{5}\vec{j} = \underline{\frac{4}{5}\vec{i} + \frac{3}{5}\vec{j}}$$

$$\vec{a} = \frac{-\frac{3}{5}}{\sqrt{1 - \frac{9}{25}}}\vec{i} + \vec{j} = \frac{-\frac{3}{5}\vec{i}}{\frac{4}{5}} + \vec{j} = \underline{-\frac{3}{4}\vec{i} + \vec{j}}$$

2.1.8(L)

a. The main aim of this exercise is to show the close parallelism between finding the anti-derivative (indefinite integral) of f(t) and that of $\vec{f}(t)$.

2.1.8(L) continued

The point is that when we write $\vec{f}(t)$ in $\vec{i}$ and $\vec{j}$ components, we have

$$\vec{f}(t) = g(t)\ \vec{i} + h(t)\ \vec{j}. \tag{1}$$

To find a function $\vec{F}$ such that $\vec{F}'(t) = \vec{f}(t)$, we may argue as follows:

Since we differentiate a vector component by component, the $\vec{i}$ component of $\vec{F}(t)$ must be a function whose derivative with respect to t is g(t), and, in a similar fashion, the $\vec{j}$ component of $\vec{F}(t)$ must be a function whose derivative with respect to t is h(t).

From our knowledge of scalar calculus (and it is crucial that you recognize that g, h, G, and H are scalar functions of t, with the vector contribution coming from $\vec{i}$ and $\vec{j}$), we know that if G is any function such that $G'(t) = g(t)$, then the family of <u>all</u> functions whose derivative with respect to t is g(t) is given by $G(t) + c_1$ where c_1 is an arbitrary (scalar) constant. Similarly, if H(t) is one function such that $H'(t) = h(t)$, all other such functions are given by $H(t) + c_2$, where c_2 is another arbitrary (scalar) constant.

In other words, if $\vec{F}(t)$ is any function of the form:

$$\vec{F}(t) = [G(t) + c_1]\vec{i} + [H(t) + c_2]\vec{j}. \tag{2}$$

from (2) we see that

$$\vec{F}'(t) = g(t)\vec{i} + h(t)\vec{j},$$

and comparing this with (1), we have that

$$\vec{F}'(t) = \vec{f}(t),$$

as desired.

The fact that (2) yields all functions $\vec{F}$ such that $\vec{F}'(t) = \vec{f}(t)$ follows from the fact that if, say, the $\vec{i}$ component of $\vec{F}$ is different from the form $G(t) + c_1$, its derivative cannot be g(t).

2.1.8(L) continued

Thus, in this case $\vec{F}'(t)$ could not equal $\vec{f}(t)$ since these two vectors then have different $\vec{i}$ components. (Remember, in Cartesian coordinates in the plane, two vectors are equal if and only if the two $\vec{i}$ components are equal and the two $\vec{j}$ components are equal.)

We next note that (2) can be rewritten as

$$\vec{F}(t) = G(t)\vec{i} + H(t)\vec{j} + c_1\vec{i} + c_2\vec{j}. \qquad (3)$$

Since c_1 and c_2 are arbitrary scalars, $c_1\vec{i} + c_2\vec{j}$ is an arbitrary <u>vector</u> constant*, and (3) becomes

$$\vec{F}(t) = G(t)\vec{i} + H(t)\vec{j} + \vec{c} \qquad (4)$$

and from (4) we see that to integrate a vector function in Cartesian coordinates, we integrate component-by-component and add an arbitrary <u>vector</u> constant.

Restated, then, if $\vec{F}$ is any function such that $\vec{F}'(t) = \vec{f}(t)$, then the family of all functions whose derivative with respect to t is f(t) is given by

$\{\vec{F}(t) + \vec{c}: \vec{c}$ is an arbitrary constant <u>vector</u>.$\}$

This shows that the idea of integrating vectors, at least in Cartesian form, is analogous to our techniques in the scalar case.

b. Knowing that $\vec{a} = \frac{d\vec{v}}{dt}$, we may use part a. to conclude that if $\vec{a} = (8\cos 2t)\vec{i} + (8\sin 2t)\vec{j}$, then

$$\vec{v} = (4\sin 2t)\vec{i} + (-4\cos 2t)\vec{j} + \vec{c}_1. \qquad (1)$$

Since we are told that $\vec{v} = 0$ when $t = 0$, we may put this into (1) to obtain:

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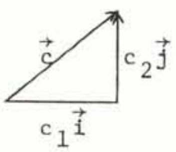
*Pictorially,



2.1.8(L) continued

$\vec{0} = 0\vec{i} - 4\vec{j} + \vec{c_1}$

whereupon we may conclude that $\vec{c}_1 = 4\vec{j}$. This, in turn means that (1) may be written as

$\vec{v} = (4\sin 2t)\vec{i} + (-4\cos 2t)\vec{j} + 4\vec{j}$

$= (4\sin 2t)\vec{i} + (4 - 4\cos 2t)\vec{j}$. (2)

Since $\vec{v} = \frac{d\vec{R}}{dt}$, we may again use part a. applied to (2) to obtain:

$\vec{R} = (-2\cos 2t)\vec{i} + (4t - 2\sin 2t)\vec{j} + \vec{c_2}$. (3)

Since $\vec{R} = \vec{0}$ when $t = 0$, (3) becomes

$\vec{0} = -2\vec{i} + \vec{c_2}$.

Therefore, $\vec{c}_2 = 2\vec{i}$.

Therefore, $\vec{R}=(-2\cos 2t)\vec{i} + (4t - 2\sin 2t)\vec{j} + 2\vec{i}$

or

$\vec{R} = (2 - 2\cos 2t)\vec{i} + (4t - 2\sin 2t)\vec{j}$.

In other words, the equation of motion is given parametrically by

$$\begin{cases} x = 2(1 - \cos 2t) \\ y = 2(2t - \sin 2t). \end{cases}$$

(Notice how the vector equation replaces a pair of scalar equations. In 3-dimensions

$\vec{R}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

2.1.8(L) continued

replaces the three scalar equations $\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$.)

c. From b. when $t = \frac{\pi}{2}$; $x = 2\left(1 - \cos 2(\frac{\pi}{2})\right) = 2(1 - \cos\pi) = 2(1 - (-1) = 4$;

while $y = 2(2[\frac{\pi}{2}] - \sin 2(\frac{\pi}{2})) = 2(\pi - 0) = 2\pi$.

Hence the particle is at $(4,2\pi)$.