PROFESSOR: Hi. Today we embark on a new phase of our course signified by the beginning of block two. Where in block one, after indicating what mathematical structure was in our quest to get as meaningfully as possible into the question of functions of several variables, we introduced the study of vectors. Vectors were important in their own right, so we paid some attention to that particular topic. And now we're at the next plateau, where one introduces the concept of functions.

In other words, just like in ordinary arithmetic, after we get through with the arithmetic of constants, which is what you could basically call elementary school arithmetic, you work with fixed numbers, we then move into algebra. And from algebra we move into calculus. Sooner or later, we would like to study the calculus of vector functions and the like. And why not sooner? Which is what brings us to today's lesson. We are going to talk about a rather difficult mouthful, I guess. I call today's lecture "Vector Functions of Scalar Variables." And if that sounds like a tongue twister, let me point out that what has happened now is the following.

When we did part one of this course, when we talked about functions and function machines, recall that both the input and the output of our machine were, by definition, scalars-- in other words, real numbers. Remember, we talked about functions of a single real variable. Now we have at our disposal both vectors and scalars. Consequently, the input can be either a vector or a scalar, and the output can be either a vector or a scalar. And this gives us an almost endless number of ways of combining these, whereby "almost endless," I mean four.

And the reason I say "almost endless" is this. There are only four ways. What are the four ways? Well, the four ways are what? Imagine that x is a scalar. In other words, the input is a scalar. Then the two possibilities are what? That the output is either a scalar or a vector.

On the other hand, the input could have been a vector, in which case, there would have been two additional possibilities. Namely, the output would be either a scalar or a vector. Now the reason I call these four possibilities endless or nearly endless is the fact that by the time we get through discussing all four possible cases, the course will be essentially over.

And to give you some hindsight as to what I mean, remember that we had a rather long course that was part one of this course. And notice that part one of this course was concerned only with that one case in four where both the input and the output happen to be real numbers, scalars. And it took us that long to develop the subject called what? Real functions of a single real variable-- input is scalar, output is scalar.

Now the one that we've chosen for our next block, next combination that we've chosen, is where the input will be a scalar and the output will a vector. And in terms of physical examples, this can be made very, very meaningful. For example, one very common physical situation is that when we study force on an object, force is obviously a vector-- but that the force on an object usually varies with time. But time happens to be a scalar. In other words, we might, for example, write that the vector f is a function of the scalar t.

And rather than talk abstractly, let me make up a pseudo-physical situation. By pseudo-physical, I mean I haven't the vaguest notion where this formula would come up in real life. And I think that after you saw my performance on the explanation of work equals force times distance and why objects don't rise under the influence of friction and what have you, you believe me when I tell you I have no feel for these physical things. But I say that semi-jokingly. The main reason is that it's irrelevant what the physical application is.

The important point is that each of you will find physical applications in your own way. All we really need is some generic problem that somehow signifies what all other problems look like. What I'm driving at here is, imagine that we have a force, f, which varies with time, written in Cartesian coordinates as follows. The force is e to the minus t times i plus j. Notice that t is a scalar. As t varies, the scalar e to the minus t varies. But the scalar e to the minus t varying means that the vector e to the minus t i is changing, is varying.

In other words, notice that e to the minus t i plus j is a variable vector, even though t is a scalar. For example, when t happens to be 0, what is the force? When t is 0, this would read what? The force is e to the minus 0 i plus j. That's the same as saying-- since e to the minus 0 is 1, the force-- when the time is 0-- is i plus j, which is a vector. You see, that vector changes as t changes, but t happens to be a scalar.

Now here's the interesting point or an interesting point. I look at this expression, and I'm tempted to say something like this. I say for large values of t, f of t is approximately j. Now how did I arrive at that? Well what I said was, is I said, you know, e to the minus t gets very close to 0 as t gets large. As t gets large, therefore, this component tends to 0. This component stays constantly j-- or the component is 1, the vector is j. So that as t gets large, f of t behaves like j.

And what I'm leading up to is that intuitively, I am now using the limit concept. I'm saying to myself, what? The limit of f of t as t approaches infinity et cetera is the limit of this sum. The limit of the sum is the sum of the limits. And I now take the limit term by term. And I sense that this limit is 0 and that this one is 1. Now you see, the whole point is this. What I would like to do is to study vector calculus. Meaning in this case, what? The calculus of vector functions of a scalar variable, of a scalar input.

Now here's why structure was so important in our course. What I already know how to do is study limits in the case where both my input and my output were scalars. What I would like to be able to do is structurally inherit that entire system. Because it's a beautiful system. I understand it well. I have a whole bunch of theorems in that system. If I could only incorporate that verbatim into vector calculus, not only would I have a unifying thread, but I can save weeks of work not having to re-derive theorems for vectors which automatically have to be true because of their structure.

Now that's, again, a difficult mouthful. This is written up voluminously in our notes. It's emphasized in the exercises, but I thought that I would try to show you a little bit live what this thing really means. Because I think that somehow or other, you have to hear these ideas rather than just read them to get the true feeling. What we do, for example, is we re-visit limits. And we write down the definition of limit just as it appeared in the scalar case. And let's start with an intuitive approach first, and later on, we'll get to the epsilon delta approach.

We said that the limit of f of x as x approaches a equals L means that f of x is near L provided that x is "near" a. And I've written the word "near" in quotation marks here to emphasize that we were using a geometrical phrase-- well, I guess you can't call one word a phrase, but a geometrical term-- to emphasize an arithmetic concept. Namely, to say that f of x was near L where f of x and L were numbers meant quite simply that what? The numerical value of f of x was very nearly equal to the numerical value of L. And that's the same as saying what? That the absolute value of the difference f of x minus L was very small. I just mention that, OK?

Now what we would like to do is invent a definition of limit for a vector valued function of a scalar variable. One way of doing this is of course to make up a completely brand new definition that has nothing to do with the past. But as in every field of human endeavor, one likes to plan the future and further sortes after one has modeled the successes of the past. So what we do is a very simple device and yet very, very powerful. We vectorize the definition that has already served us so well.

What I mean by that is, I go back to the definition which I've written here and put in vectors in appropriate places. For example, we're dealing with a vector function, so f has to have an arrow over it. Moreover, since f of x is a vector, anything that it approaches as a limit by definition must also be a vector. So that means the L must also be arrowized. All right? Put an arrow over the L.

On the other hand, the x and the a remain left alone because after all, they are just scalars. Remember again, x and a are inputs of our function machine. And in the particular investigation that we're making, our input happens to be a scalar. It's only the output that's a vector. So let's go back and, just as I say, arrowize or vectorize whatever has to be vectorized here.

So I now have a new definition intuitively. The limit of the vector function f of x as x approaches a is the vector L means that the vector f of x is near the vector L, provided that the scalar x is the scalar a. Now the point is that the "x is near a" doesn't need any further interpretation because a and x are still scalars. The question that comes up now is twofold. First of all, does it make sense to say that f of x is near L? Does that even make sense when you're talking about vectors? The second thing is, if it does make sense, is it the meaning that we want it to have? Does it capture our intuitive feeling? Let's take the questions in order of appearance.

First of all, what does it mean to say that f of x-- the vector f of x-- is near the vector L? In fact, let's generalize that. Given any two vectors A and B, what does it mean to say that the vector A is near the vector B? Now what my claim is, is this. Let's draw the vectors A and B as arrows in the plane here. Let's assume that they start at a common origin, which you can always assume, of course. To say that A is nearly equal to B somehow means what? That the vector A should be very nearly equal to the vector B.

Well remember, if two vectors start at a common point, the only way that they can be equal is if their heads coincide. Consequently, with A and B starting at a common point, to say that A is nearly equal to B should mean that the distance between the head of A and the head of B should be small. But how do we state the distance between the head of A and the head of B? Notice that that has a very convenient numerical form. Namely, the vector that joins A to B can be written as either A minus B or B minus A, depending on what sense you put on this. We don't care about the sense. All we care about is the magnitude. Let's just call this length the magnitude of A minus B.

And now we're in business. Namely, we say, look at it. To say that A is near B means that A is nearly equal to B, which in turn means that the magnitude of A minus B-- the magnitude of the vector-- what vector? -- the vector A minus the vector B -- is small. And keep in mind that I'm now using 'small' as I did arithmetically. Because even though A and B are vectors, and so also is A minus B, the magnitude of a vector is a number. Note this very important thing. Even when I'm dealing with vectors, as soon as I mention magnitude, I'm talking about a number. In other words then, I now define A to be near B to mean that the magnitude of the difference between A and B is small.

And again, question two-- does that caption my intuitive feeling? And the answer is, yes, it does. Is I want A to be near B to mean that they nearly coincide. And that is the same as saying that the magnitude of A minus B is small.

At any rate, having gone through this from a fairly intuitive point of view, let's now revisit limits more rigorously and rewrite our limit definition in terms of epsilons and deltas. And by the way, I hope this doesn't look like anything new to you, what I'm going to be talking about next. It's our old definition of limit that was the backbone of the entire part one of this course. I not only hope that it looks familiar to you, I hope that you read this thing almost subconsciously as second nature by now.

But the statement is what? The limit of f(x) as x approaches a equals L means, given any epsilon greater than 0, we can find a delta greater than 0 such that whenever the absolute value of x minus a is less than delta but greater than 0, the implication is that the absolute value of f(x) minus L is less than epsilon. Now if I go back to my structural properties again, it seems to me that what my first approximation, at least-- for a rigorous definition to limits of vector functions-- should be to just, as I did before, read through this definition verbatim. Don't change a thing, but just vectorize what has to be vectorized.

And because this definition that I'm going to give is so important, I elect to rewrite it. And what you're going to see next is nothing more than a carbon copy, so to speak, of this, only with appropriate arrows being put in. Namely, I am going to give, as my rigorous definition of the limit of f(x) as x approaches a equals the vector L-- it's going to mean what? Given epsilon greater than 0, I can find delta greater than 0 such that whenever the absolute value of x minus a is less than delta but greater than 0, the implication is that the magnitudes-- see I don't say absolute value now, because notice, I'm not dealing with numbers. These are vectors-- but the magnitude of f(x) minus L must be less than epsilon.

Now you see what I've done: I've obtained the second definition from the first simply by appropriate vectorization. What I have to make sure of is, what? That this thing still makes sense. And notice that it does. Notice that what this says, in plain English is, what? That if x is sufficiently close to a -- what? The difference -- the magnitude of the difference between these two vectors is as small as we wish. But we call that magnitude, what? The distance between the two vectors. This says, what? Correlating our formal language with our informal language, this simply says what? That we can make f(x) as near to L as we wish just by picking x sufficiently close to a.

So now what that tells us is that the new limit definition makes sense, just by mimicking the old definition. And what do we mean by mimicking? We mean vectorize the old definition, go through it word for word and put in vectors in appropriate places.

Now let's see what this means structurally. That's the whole key to our particular lecture today: the structural value of this. For example, what were some of the building blocks that we based calculus of a single variable on? Remember derivative was defined as a limit, and therefore all of our properties about derivatives followed from limit properties.

Well, among the limit properties that we had for part one where, what? Input and output were scalars. The limit of the sum was equal to the sum of the limits. And the limit of a product is equal to the product the limits. And remember, in all of our proofs, we use these particular structural properties. You see? The next thing we would like to know-- see after all, these structural properties, even though they're called theorems, essentially, they become the rules of the game once we've proven them.

In other words, once we've proved using epsilons and deltas that this happens to be true, notice that we never again use the fact that these are epsilons and deltas, that we have epsilons and deltas when we use this particular result. We just write it down. Because once we've proven it, we can always use it. In other words, what I'm really saying is, let me vectorize this. And I'll put a question mark now, because you see, I don't know if this is true for vectors. I don't know if the limit of a sum is the sum of the limits when we're dealing with vectors rather than with scalars.

But notice what I've done. I have structurally plagiarized from my original definition. All I did was I vectorized it. You see, what I'm going to have to do is to check from my original definition whether these properties still hold true. I'm going to talk about that in a minute. Let me just continue on for a while. See, similarly over here, we talked about the limit of a product equaling the product of the limits. So again, we'd like to use results like that. And somebody says, well, let's vectorize.

I want to show you what I mean by saying that after you vectorize, you have to be darn careful that you don't just go around saying, look at it, this is legal. I put arrows over everything. Sure it's legal, but you may not have anything that's worth keeping. Look at this expression. What are f and g in this case? The way I've written it, these happen to be vectors. See, they're both arrowized. Right? They're vectors. Have we talked about the ordinary product of two vector functions? And the answer is no. When we have two vector functions, we must either have a dot or a cross in here-- the dot product or the cross product. And if we're not going to put a dot in here, the best we can talk about is scalar multiplication.

In other words, let me-- before you tend to memorize this, let me cross that out. Because this is ambiguous. It doesn't make sense. And what I am saying is that unfortunately-- not unfortunately, but if I want to be rigorous about this-- there happens to be three interpretations here. Namely, I can do what? I can think of f(x) as being a scalar and g(x) as being a vector. Or I can think of f(x) and g(x) as both being vector functions, but in one case having the dot product and in the other case, the cross product. In other words, to vectorize this particular equation, I guess what I have to do is write down these three possibilities.

Namely, notice in this case, this is, what? This is a scalar multiple of a vector function. Granted that the scalar multiple is a variable in this case, this still makes sense. Namely, for a fixed x-- for a fixed x, f(x) is a constant, and g(x) is a vector. A constant times a vector is a vector.

For a fixed x, f and g here are fixed vectors. And I can talk about the dot product of two vectors. And for a fixed x over here, f(x) and g(x) are again fixed vectors. And consequently, I can talk about their cross-product. And the question is, will these properties still be true when we're dealing with vector functions? The answer happens to be yes. But it's not yes automatically.

In other words, what we're going to do in the notes is to mimic-- to prove these results for vectors, what we are going to do is to mimic every single proof that we gave in the scalar case, only replacing scalars by appropriate vectors whenever this is supposed to happen. The trouble is that certain results which were true for scalars may not automatically be true for vectors.

Let me give you an example. Somehow or other to prove that the limit of a sum was equal to the sum of the limits, we used the property of absolute values that said that the absolute value of a sum was less than or equal to the sum of the absolute values. And we proved that for numbers. Suppose I now vectorize this. See? This now means magnitudes, right? And if a and b are any vectors in the plane, a and b no longer have to be in the same direction-- how do we know that vector addition has the same magnitude property that scalar addition has? See, can we be sure that this recipe is true?

Well let's go and see what this recipe means. By the way, this does happen to be true in vector arithmetic. And it happens to be known as the triangle inequality. And the reason for that is, is if I draw a triangle calling two of the sides a and b, respectively, and the third side c, we do know from elementary geometry that the sum of the lengths of two sides of a triangle is greater than or equal to the length of the third side of the triangle.

What I'm driving at now is, let me just vectorize this and show you something that I think is very cute here. If I put arrows here, this becomes the vector a, now; this becomes the vector b; this becomes the vector c. OK? But in terms of our definition of vector addition, noticed that c goes from the tail of a to the head of b. a and b are lined up properly for addition. So this is the vector a plus b.

Now state the triangle inequality in terms of this picture. The triangle inequality says what? The third side of the triangle-- well what length is this? If the vector is a plus b, the length is the magnitude of a plus b-- must be less than or equal to the sum of the lengths of the other two sides. But those are just this. And that verifies the recipe that we want.

In other words, it's going to turn out that every vector property that we need-- what do we mean by need-- that happened to be true for scalar calculus-- is going to be sufficient in vector calculus as well. For example, when we proved the product rule for scalars-- not the product rule, but the limit of a product was the product of the limits-- we used such things as the absolute value of a product was equal to the product of the absolute values. Notice, for example, in terms of case one over here, if I vectorize b and leave a as a scalar, the magnitude of a times the vector b is equal to the magnitude of a times the magnitude of b, just by the definition of what we meant by a scalar multiple.

After all, a times b meant what? You just kept the direction of b constant, but multiplied b by the magnitude of a. Meaning what? By a, if was a positive; minus a, if a was negative, this is certainly true. By the way, there is a very important caution that I'd like to warn you about. And this is done in great detail in the notes. It's very important to point out that when I vectorize this in terms of a dot and a cross product, it is not true that the magnitude of a dot b is equal to the magnitude of a times the magnitude of b.

You see, what I'm really saying here is, how was a dot b defined? a dot b, by definition, was what? It was the magnitude of a times the magnitude of b, times the cosine of the angle between a and b. Notice that if I take magnitudes here-- I want to keep this separate, so you can see what I'm talking about here--

Look at it. This factor here can be no bigger than 1. But in general, it's going to be less than 1. In other words, notice that the magnitude of a dot b is actually less than or equal to the magnitude of a times the magnitude of b. In fact, its equality holds only if this is a 0 degree angle or 180 degree angle. Because only then is the magnitude of the cosine equal to 1. Similar result holds for the cross-product. The interesting point is, and I'm not going to take the time to do it here, because I want this lecture to be basically an overview-- but the important point is that you don't need equality to prove our limit theorems.

Remember, every one of our limit theorems was a string of inequalities. And actually-- and as I say, I'm going to do this in more detail in the notes-- the only result that we needed to prove theorems even in the part one section of our course was the fact that the less than or equal part be valid. The fact that it was equal when we were dealing with numbers was like frosting on the cake. We didn't need that strong a condition.

The key overview that I want you to get from this present discussion is that, when we vectorize our definition of limit, that that new definition, having the same structure as the old-- meaning that all the properties of magnitudes of vectors that we need are carried over from absolute values of numbers, all previous limit theorems-- what do I mean by "all previous limit theorems"? I mean all the limit theorems of part one of this course are still valid. And with that in mind, I can now proceed to differential calculus.

Namely what? I'm going to do the same thing again. I write down the definition of derivative as it was in part one of our course. And now what I do is I just vectorize everything in sight, provided it's supposed to be vectorized. Remember x and delta x are scalars here. f is the function. I just vectorize our definition. The first thing I have to do is to check to see whether the new expression makes sense. Notice that the numerator of my bracketed expression is now a vector. The denominator is a scalar. And a vector divided by a scalar makes perfectly good sense.

In other words, this can be read as what? The scalar multiple (1/delta x) times the vector [f(x + delta x) minus f(x)]. So this definition makes very good sense. It captures the meaning of average rate of change because you see, it's what? It's the total change in f over the change in x equal to delta x. So it's an average rate of change. Again, that's discussed in more detail in the text and in our notes, and in the exercises.

But the point is that since every derivative property that we had in part one of our course followed from our limit properties, the fact that the limit properties are the same for vectors as they are for scalars now means that all derivative formulas are still valid. In other words, the product rule is still going to hold. The derivative of a sum is still going to be the sum of the derivatives. The derivative of a constant is still going to be 0.

And keep this in mind. It's very important. I could re-derive every single one of these results from scratch as if scalar calculus had never been invented, just by using my basic epsilon delta definition. The beauty of structure is, is that since the structures are alike-- you see, since they are played by the same rules of the game-- the theorems of vector calculus are going to be precisely those of scalar calculus. And I don't have to take the time to re-derive them all.

As I say in the notes, I re-derive a few just so that you can get an idea of how this transliteration takes place. At any rate, let me close today's lesson with an example. Let's take motion in a plane. Let's suppose we have a particle moving along a curve C. And that the motion of the particle is given in parametric form, meaning we know both the x- and the y-coordinates of the particle at any time, t, as functions of t. Notice, by the way, that what requires two equations in scalar form can be written as a single vector equation, namely at time t-- let's say at time t, the particle was over here.

Notice that if we now let r be the vector from the origin to the position of the particle that r is a vector, right? Because to specify r, I not only have to give its length, I have to give its direction. r is a vector which varies with our scalar t. All right? So r is a vector function of the scalar t. Now what is the i component of r? Well, the i component is x. And the j component is y. Notice that the pair of simultaneous parametric scalar equations-- x = x(t), y = y(t), can be written as the single vector equation r(t) = x(t) i + y(t) j, where x(t) and y(t) are scalar functions of the scalar variable t. OK?

Now the question comes up, wouldn't it be nice to just take dr dt here? Well, I mean, that's about as motivated as you can get. Meaning what? Let's see what happens. I know how to differentiate now. I'm going to use the fact that the derivative of a product of a sum is a sum of the derivatives, that i and j are constants. And a constant times a function, to differentiate that, you skip over the constant and differentiate the function. x(t) and y(t) are scalar functions, and I already know how to differentiate scalar functions.

So all I'm going to assume now is that x(t) and y(t) are differentiable scalar functions, which means now that instead of just talking about motion in a plane, I'm assuming that the motion is smooth. At any rate, I now differentiate term by term. And look what I get. This is what? It's (dx/dt times i) plus (dy/dt times j). In other words, the dr/dt is this particular vector.

And this particular vector is fascinating. Why is it fascinating? Well, for one thing, let's compute its magnitude. The magnitude of any vector in i and j components is, what? The square root of the sum of the squares of the components. In this case, that's the square root of dx/dt squared plus dy/dt squared. But remember from part one of our course, this is exactly the magnitude of ds/dt where s is arc length. Remember I put the absolute value sign in here because we always take the positive square root. Notice we're assuming that arc length is traverse in a given direction at a particular time, t.

But what is ds/dt? It's the change in arc length with respect to time. And that's precisely what we mean by speed along the curve. On the other hand, if we look at the slope of dr/dt, it's what? It's the slope-- it's the term by what? You take the j component, which is dy/dt, divided by the i component, which is dx/dt. By the chain rule, we know that that's dy/dx. Therefore, the vector dr/dt has its magnitude equal to the speed along the curve. And its direction is tangential to the curve. And what better motivation than that to define a velocity vector?

After all, we've already done that in elementary physics. In elementary physics, what was the velocity vector defined to be? At a given point, it was the vector whose direction to the curve-- whose direction was tangential to the path at that point and whose magnitude was numerically equal to the speed along the curve that the particle had at that point. And that's precisely what dr/dt has. In other words, what we have done is given a self-contained mathematical derivation as to why the derivative of the position vector r, with respect to time, should physically be called the velocity vector.

And then again, analogously to ordinary physics, if v is the velocity vector, we define the acceleration vector to be the derivative of the velocity vector with respect to time. Now I was originally going to close over here, but my feeling is that this seems a little bit abstract to you, so maybe we should take a couple more minutes and do a specific problem.

Let's take the particle that moves along the path whose parametric equations are y = t^3 + 1, and x = t^2. In other words, notice that for a given value of t-- t is a scalar-- for a given value of t, I can figure out where the particle is at any time along the curve. For example, when the time is 2, when t is 2, x is 4, y is 9. So at t = 2, the particle is at the point (4, 9), et cetera. Notice that to understand this problem, one does not need vectors. But if one knows vectors, one introduces the radius vector, r. What is the radius vector? Its i component is the value of x, and its j component is the value of y. So the radius vector r = (t^2) i + (t^3 + 1) j.

Now, by these results, I can very quickly compute both the velocity and the acceleration of my particle at any time. Namely, to find the velocity, I just differentiate this with respect to t. This is going to give me what? The derivative of t^2 is 2t. The derivative of t^3 + 1 is 3t^2. So my velocity vector is just (2t) i + (3t^2)

j. Now to get the acceleration vector, I simply have to differentiate the velocity vector with respect to time. And I get what? 2 i + (6t) j.

And I now have the acceleration, the velocity, and, by the way, the position of my particle at any time t. In particular, since I've already computed that the particle is at the point (4, 9) comma when t is 2, let's carry through the rest of our investigation when t is 2. When t is 2, notice that in vector form, r is equal to 4i + 9j; v is equal to 4i + 12j; and a is equal to 2i + 12j. Pictorially-- and by the way, to show you how to get this, all I'm saying here is that if we come back here recalling that t^3 is t^2 to the 3/2 power-- this simply says that y is equal to x^(3/2) + 1. So I drew the path in just so that you can get an idea of what's going on over here.

All I'm saying is when somebody says, where is the particle at time t equals 2, it's at the point (4, 9). And notice the correlation again. The point (4, 9) is the point at which the vector 4i plus 9j terminates. Because you see, that vector originates at (0, 0).

I can compute the velocity vector-- namely what? The velocity vector has its slope equal to 3-- 12 over 4. And its magnitude is the square root of (4^2 + 12^2). That's the square root of 160, which is roughly 12.5. I can now draw that velocity vector to scale. In a similar way, the slope of the acceleration vector at that point is 12 over 2, which is 6. So the slope is 6. That's a pretty steep line here. And the magnitude is what? The square root of (4 + 144)-- the square root of 148, which I just call approximately 12, to give you a rough idea. I can draw this into scale as well.

In other words, in just one short overview lesson, notice that we have introduced what we mean by vector functions, what we mean by limits. We've inherited the entire structure of part one. I can now introduce a position vector, differentiate it with respect to time to find velocity and acceleration, and I'm in business, being able to solve vector problems in the plane-- kinematics problems, if you will.

We're going to continue on in this vein for the remainder of this block. We have other coordinate systems to talk about. Next time, I'm going to talk about tangential and normal components of vectors and the like. But all the time, the theory that we're talking about is the same. Namely, we are given motion in a plane. And we can now, in terms of vector calculus, study that motion very, very effectively. At any rate, until next time, goodbye.

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