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PROFESSOR: Hi. Today, we finish up the last of the basic vector operations. Today, what we're going to discuss is something called the vector product or the cross product. And it's a new type of multiplication, whereby we now multiply two vectors to obtain a vector. You see, the dot product multiplied two vectors, but it gave a scalar. This is why the dot product was called a scalar product.

Now, the idea is, again, we can go to the physical world to seek motivation. And my only problem with that is that some of the examples that motivated this are not quite as elementary as the work equals force times distance formula of last time. The cross product does come up in sort of trivial applications, in terms of moments, angular momentum, in terms of mechanics, and in more advanced situations, in terms of the Maxwell equations where, you may remember, in physics courses, going through this routine of the force and the flux and the intensity and what have you.

And at any rate, rather than waste time-- and I mean, I use the word waste here judiciously, because, for the purpose of our course, it is not nearly as crucial to motivate why we invented the cross product physically as it is to define it mathematically and to use it structurally. You see, what I want to do is simply get to the crux of the situation, discuss the concept of a cross product, and simply point out that if A and B are vectors, by A cross B-- written this way, A cross B-- we mean a vector. Now, here's the kicker again. As soon as I say vector, that means to define it, I have to tell you three ingredients, namely magnitude, direction, and sense. Right?

Now, for the magnitude of A cross B, it's simply the magnitude of A times the magnitude of B times the magnitude of the sine of the angle between A and B. And

I'm going to come back and comment on that in a few moments, about the difference between the sine here versus the cosine of the previous lecture. Again, physically, there's a motivation for this.

As far as we're concerned, all we have to do is say, this is what the magnitude is defined to be. This is the game. We define it this way.

Next thing I have to do is tell you its direction. And the direction of A cross B is, by definition, perpendicular to both A and B. All right? Another way of saying that is in the event that A and B are not parallel, then the vectors A and B determine a plane. And what we're saying is that A cross B is in a vector perpendicular to the plane determined by A and B. All right?

By the way, if A and B are parallel, they don't determine the plane, because they coincide. They have only one direction. But if A and B are parallel, the angle between them is 0 or 180, depending on whether their sense is the same or not.

Once the angle is either 0 or 180, the sine of that angle is 0. Once one of the factors here is 0, the whole product is 0. So in the event that A and B do not determine the plane, the magnitude of A cross B is 0, which means that A cross B is the zero vector. And that's a pretty trivial situation. So we just won't even worry about that.

And thirdly, I must now give you the sense of A cross B. And the sense is the usual right-hand rule, in other words, the direction in which a right-handed thread screw would turn as A is rotated into B-- and now I have to be careful because of this sine situation-- through the smaller angle. You see, look it. Notice that this sense is going to depend on what-- if you rotate A into B, you get the opposite sense than if you rotated B into A.

And not only that, but if you rotate A into B through the larger of the two angles, notice that rotating A into B through one angle-- say the smaller one-- gives you a different sense than rotating A into B through the larger of the two angles. So there's a certain ambiguity that comes up here. And the ambiguity occurs because of the fact that we're using the sine rather than the cosine.

In other words, if you reverse the sense of an angle, if you change its sine, the algebraic sine, the trigonometric sine, s-i-n changes. OK? And similarly, the sine of an angle is the negative of the sine of 360 degrees minus that angle. But that, at any rate, is the definition of the cross product.

Given two vectors, their cross product is what? A vector whose magnitude is the product of the two magnitudes times the magnitude of the sine of the angle between the two vectors. The cross product is perpendicular to each of the two vectors. And it's sense is the right-hand rule as the first is rotated into the second through the smaller of the two angles.

This is important because, you see, what happens is that as important as the cross product is from a physical point of view, arithmetically, it's a nuisance. We can take very few liberties with the cross product. It turns out that almost every rule that we would like to be true about numerical multiplication is false for vector cross products. For example, among other things, A cross B is not equal to B cross A.

In fact, A cross B is the negative of B cross A. And the reason for this is simply, as you check this thing through, A cross B and B cross A certainly have the same magnitude. They have the same direction, because each is perpendicular to both A and B.

But notice that the sense is different. Namely, A cross B tells us to rotate A into B through the smaller of the two angles. And in our particular diagram that is what, the counterclockwise direction?

On the other hand, B cross A says, rotate B into A through the smaller of the two angles. And if we do that, notice that we get the clockwise direction so that B cross A and A cross B have the same magnitude. They have the same direction. But they have the opposite sense. That is, A cross B is the negative of B cross A.

And what that means, you see, in computations is that every time you change the order of A and B in a cross product, you must remember to change the sign. In fact, what's particularly embarrassing is if you forgot to do this. If this particular term is

going to be added on to something, and you forget to change its sign, you're going to be off by more than just the error of a sign.

In other words, if something was supposed to be plus 4, and you have it as minus 4 or vice versa, the error is going to be 8 if you add it rather than subtract it, you see. So the error gets hidden in other computations. But in other words, you can change the order of the factors, but you must remember to change the sign.

Another interesting thing is that unlike in the dot product, where A dot B dot C didn't even make sense, it is true that A cross B cross C and A cross B cross C do makes sense. And by the way, what are these things? They're vectors.

B cross C is a vector, which is perpendicular to both B and C. It's perpend-- you see. And A is a vector. And the cross product of two vectors is, again, a vector, et cetera.

These are two vectors. But in general, these two vectors do not have to be equal. In fact, as I shall show you in a moment, these two vectors don't even have to be in the same plane, let alone being equal. They don't even have to be in parallel planes.

And again, without making too much of a fuss over this, let's take a look and see if we can see why. Let's look at A cross B cross C. By definition, let's just even look at the direction. Let's not even worry about magnitudes.

What is the direction of A cross (B cross C)? It must be what? It must be, on the one hand, perpendicular to A and on the other hand perpendicular to B cross C. OK? That's the definition of being a cross product.

On the other hand, what about B cross C as a vector itself? B cross C is perpendicular to both B and C. So what do you have here? You have that whatever A cross (B cross C) is, it's perpendicular to B cross C, which, in turn, is perpendicular to both B and C.

That sounds like a tongue twister. But if you decipher it, all it says is that A cross (B

cross C) is perpendicular to the perpendicular to B and C. And I call that deciphering it.

What is the perpendicular to the perpendicular? You say let's just write it this way. Here are two vectors in the plane. Here's a perpendicular. I now take a perpendicular to that. And that's going to be parallel to the original plane.

This is written up in the text. It's written up as one of our learning exercises. But just for a quick run-through now, all I want you to see is that A cross (B cross C) is parallel to the plane determined by B and C.

In fact, the easy way to memorize this is look at the vector that's in parentheses. And the cross product is what? It's parallel to the plane determined by the two vectors in parentheses.

The only time these two vectors, by the way, won't determine a plane is if they're parallel. But in that case, the cross product would have been zero anyway. In other words, to summarize what I've written on the blackboards here, A cross (B cross C) is a vector which lies in the plane determined by B and C.

After all, if the vector is parallel to the plane, you can assume it lies in the plane, because you can shift it parallel to itself. And correspondingly, (A cross B) cross C is a vector in the plane determined by A and B. In particular, there's no reason why the AB plane and the BC plane have to be the same plane.

No law says that our three vectors, A, B, and C, all lie on the same plane. So in particular, this is why the associative property doesn't hold for a cross product. Because as soon as you shift the parentheses, you actually shift the plane in which the vector exists. OK? But again, I just want to go through these highlights to make sure that you don't miss them. But these are done in more detail in the notes.

Before I start blasting the cross product too much, let me point out that certain properties that we like in numerical arithmetic are present in the cross product. For example, there is a-- I say this, there's a little structure. The distributive property holds to the cross product.

In other words, if the vector A is crossed with the sum of the two vectors, B and C-- and by the way, again notice, if B and C are vectors, B plus C is a vector. I'll put a v down here to indicate vector. A is a vector. And the cross product of two vectors is a vector.

Notice that, again, with the cross product, all of the things that you're working with happen to be vectors. But my claim is that A crossed with (B plus C) is equal to A crossed with B plus the product of A crossed with C. That's a very nice structural property. Again, as usual, Cartesian coordinates have a lot to offer us. If we switch to Cartesian coordinates, again, as usual, writing A as a1 i plus a2 j plus a3 k, and B as b1 i plus b2 j plus b3 k, and multiplying term by term, we observe a rather interesting situation.

You see, notice things like this. When you cross i with i, j with j, and k with k, you're always going to get the zero vector. And the reason for that is, we already saw that since the sine of the angle between two parallel vectors is 0, the magnitude of the cross product is 0, that means that the cross product is the zero vector. The question comes up, what happens when you cross i with j or j with i, i with k, k with j, et cetera?

I think it's easy to see that when you cross i and j and j and i, you're going to get either k or minus k. The right-handed rule is set up very, very conveniently, again, to help us with a memory device. The easiest way to remember how these things go is i, j, and k were set up to form a right-handed rule, a right-handed system.

Write i, j, and k sequentially. See, i, j, k, i, j, k, et cetera. Call the normal order the positive direction and the opposite order the negative direction.

For example, when you want i cross j, just read this list as you go along: i followed by j gives k, which is in the right order-- see, i, j, k-- so i cross j is k. On the other hand, j cross i means what? j followed by i.

j followed by i is this way. That also yields k. But you're now going in the negative direction.

To summarize these results, i cross j is the same as minus j cross i, which is k. k cross i is the negative of i cross k, which is j. i cross i, j cross j, and k cross k are all 0. And notice the use here of zero vector, because we're dealing with vectors.

And again, leaving the details to you. They're very simple to come up with. Just carry out the multiplication.

And what you find is that A cross B is given by the following. The i component of the vector is a2b3 minus a3b2. The j component is a3b1 minus a1b3. And the k component is a1b2 minus a2b1.

And if you want a convenient way of memorizing this, again, write 1, 2, 3 in their correct sequential order. And what you're saying is, each term in A cross B involves what? An a factor followed by a b factor. The subscripts involve the component, which isn't present.

See, for the first component, i, the subscripts are 2 and 3. For the second component, j, the second component, the subscripts are 3 and 1. And for the third component, k, the subscripts are 1 and 2.

And notice that the way the sines are determined is that if the subscripts occur in the correct sequential order, it's positive, otherwise negative. For example, 2 followed by 3 is the correct order. But 3 followed by 2 is the negative order. Notice, in this sequence, 1 follows 3, whereas 1 to 3 is in the negative order. See? Just the same, this works out very, very nicely.

And the reason I went through this-- it may seem like an awful lot of work to go through that to show you how logical this thing is-- is that something comes up in the textbook, which is not bad, but which I find as frightening if you have not had experience with something called determinants. You see, the author, Professor Thomas, elects to introduce a convenient device for memorizing this result. And he writes it as a 3 by 3 determinant.

Now, let me just write down the 3 by 3 determinant, and then we'll talk about it later.

I put convenient in quotation marks to emphasize to you that this is convenient only if you already know determinants. If you don't know determinants, this is not only not convenient, but it's frightening. We're always frightened of things that we don't understand too well.

And I call it a memory device, because you do not have to have determinants to remember this particular result. In fact, if one were going to list, in order, the reasons for inventing determinants, and the man was reading off of very, very long list, and you were getting bored, as soon as he came to the reason that it was convenient to list the cross product, you could heave a sigh of relief, because you would know that he's near the end of the list. In other words, to make a long story short, of all the reasons for inventing determinants, using them for a cross product representation is perhaps the least significant.

But you see, in Professor Thomas's approach, determinants are assumed to be known. And if you do know them, it's a good notation. If you don't, forget it.

The trouble is, I tell you to forget it, and you won't forget it. It always works that way. When you tell the student not to forget something, he forgets. When you say, please ignore it and forget it, he doesn't forget.

So since you may be a little bit rusty on determinants, let me give you a very, very brief review of determinants. In fact, it's not only very brief, it's a pseudo-review, because I will give you no logic behind this at all. I'm just going to tell you the recipe.

First of all, by a 2 by 2 determinant, you mean a square array of numbers consisting of two rows and two columns. You use absolute value signs to indicate the determinant. And by definition, the 2 by 2 determinant, a, b, c, d is the product of the two terms, a and d, minus the product of the two terms, b and c.

In other words, if you multiply the upper left by the lower right and subtract from that the upper right times the lower left, that is, by definition, the value of the 2 by 2 determinant, just by definition. By the way, in block seven of our course, we're going to do this in much more detail. For now I just want to give you enough of a hint, so

that if you are frightened by determinants, you can see what Professor Thomas is talking about in the text here.

Now, to expand a 3 by 3 determinant, one uses a little sign code. Starting at the upper left-hand corner, you put a little plus sign. Then you alternate plus, minus, plus, et cetera. The plus simply is a code to tell you, leave the sign of the term alone. And the minus tells you, change the sign of the term.

What that means is this. By definition, this 3 by 3 determinant is obtained as follows-- take x1 as it appears and strike out the row and column in which x1 appears and multiply x1 by the remaining 2 by 2 determinant. Then what you do is you factor out x2 with a minus sign, strike out the row and column in which x2 appears, and multiply x2 by the remaining 2 by 2 determinant that's left. Similarly, take out x3 with the sign that it appears with, strike out the row and column in which it appears, and multiply by the remaining 2 by 2 determinant. Summarized, to expand this 3 by 3 determinant, it's x1 times the determinant, y2, y3, z2, z3 minus x2 times the determinant, y1, y3, z1, z3, plus x3 times the determinant, y1, y2, z1, z2.

Now, that happens to have very practical application. As I say, for this particular situation, I just want to do it with you to have you see how it works. Let me come back to what we were doing before, and show you how, if we use this definition, we can mechanically write down A cross B in a hurry.

What did the rule say? It says, to expand the 3 by 3 determinant, take out i, as it appears, and multiply by the 2 by 2 determinant that's left, when you strike out the row and column that i appears in. OK? Then it says what? Take out j with a minus sign, strike out the row and column, multiply by the remaining 2 by 2 determinant.

Then it says, factor out k and strike out the row and column that k appears in. Multiply by the remaining 2 by 2 determinant. Well lookit, by our rule of multiplying a 2 by 2 determinant, what is this thing here? This is nothing more than a2 b3 minus a3 b2.

I'm not going to go through this, because time is short and you can carry this

operation out by yourself. What I would like you to see, however, is that this term here is precisely the same as the term we have here. In other words, if I carry out this mechanical recipe, I am going to be able to get the same answer that I got logically, only without having to worry about keeping track of things in my head. In other words, if I do know determinants, it is very convenient to write down the cross product of two vectors using Cartesian coordinates.

And just to illustrate that, let me give you a few examples. I call this example one. Find the vector perpendicular to the plane, determined by the three points, (1, 2, 3); (5, 9, 4); and (7, 6, 8).

Now, try to visualize this problem if you didn't know vectors. Try to visualize drawing this thing to scale, trying to imagine what slope the plane has and how you find a vector, or a line perpendicular to the plane. And by the way, I don't even need the word vector in here. Later I could say, find a direction. But we'll worry about that more next time.

The thing that I want to emphasize here is, I just very quickly draw myself a little convenient device. I don't draw it to scale. I assume that the points A, B, and C are not in the same straight line. And if they are, it'll become very interesting that I simply won't get-- I'll get 0 for a cross product to tell me that they don't form two non-parallel vectors.

But I mark off A, B, and C. And the first thing I do, as always, is I vectorize-- I draw the lines AB and AC, add I vectorize them. That forms for me, as usual, the vector AB and the vector AC.

Remembering how I form a vector from two points in Cartesian coordinates, again, subtracting the coordinates of the point that I'm going-- from the point that I'm going to, I subtract away the coordinates of a point that I'm leaving. In other words, 5 minus 1, 9 minus 2, 4 minus 3. The vector AB, very simply, is 4i plus 7j plus k. And similarly, the vector AC, very conveniently, is 6i plus 4j plus 5k. You see, 7 minus 1, 6 minus 2, 8 minus 3.

Now, I better stop checking these things. Because, with my luck, I've probably subtracted wrong, and you might not notice it if we just go through it rapidly. But look at how quickly I've got AB and AC now.

Now, here's the beauty. What do I want? I want a vector perpendicular to the plane determined by A, B, and C.

In particular, if I can find a vector perpendicular to both AB and AC, namely if a vector is perpendicular to two lines in a plane, it's perpendicular to the plane itself. If I can find a vector perpendicular to both AB and AC, I'm home free. That's the vector I'm looking for, or at least a vector that I'm looking for.

Do I know, conveniently, a vector perpendicular to two given vectors? And the answer had darn well better hinge somehow on the title of today's lecture, being the cross product. Yes, by definition, if I cross AB and AC, whatever vector I get is going to be perpendicular to the plane determined by A, B, and C.

And now what I'm saying is that the beauty of Cartesian coordinates is that I can now very quickly write down what AB cross AC is as a determinant. Namely, I write down i, j, k. Then I write down the components of AB. Then I write down the components of AC. See, (4, 7, 1); (6, 4, 5).

And now I do what? I strike out the row and column in which i appears. And the remaining 2 by 2 determinant is what? 7 times 5 minus 1 times 4-- in other words, 35 minus 4.

I take out minus j, strike out the row and column in which j appears. The remaining 2 by 2 determinant is 4 times 5, minus 1 times 6-- in other words, 20 minus 6. I strike out the row and column in which k appears. The remaining 2 by 2 determinant is 4 times 4 minus 7 times 6. That's 16 minus 42.

And now simplifying, what do I have? I have 31i minus 14j minus 26k. In other words, the vector 31i minus 14j minus 26k is the vector that I'm looking for. It's a vector perpendicular to the plane.

And what I would like you to see is how convenient this vector arithmetic is to studying space geometry. And in the next lesson I'm going to say some more about space geometry. But for now I thought maybe another nice example before we finish up, and let it go at that.

Let me just do one more example for you. And to simplify the computation, I might as well use the vectors I've already used. Let's take the same three points we had before, (1, 2, 3); (5, 9, 4); (7, 6, 8). And we're going to ask, what is the area of the triangle determined by these three points?

And the subtlety that I wanted to bring out here was an excuse to show you how the magnitude of the cross product of two vectors can be interpreted. Namely, look at two vectors. All right? Call them x and y. Draw them into scale from a common point. And let theta be the angle formed between these two vectors.

Consider the parallelogram formed by these two vectors. The area of a parallelogram is what? It's the base times the height.

Now, look at. The base of this parallelogram is the magnitude of y. And the height of this parallelogram-- remember, this magnitude is x. So this height is x sine theta. This is the area of the parallelogram.

But notice that this is also, by definition, the magnitude of x cross y. In other words, notice that this is, by definition, the magnitude of x cross y. x cross y is a vector whose magnitude is this.

In other words, the magnitude of x cross y is the area of the parallelogram determined by x and y, which gives you a rather nice, interesting interpretation. Imagine, for example, that x and y don't change in length. Is it clear to you that as I change the angle between x and y, the area of the parallelogram formed by x and y changes?

In other words, one extreme case is when the angle between x and y is 0, in which case the parallelogram has no area. Another extreme case is when the angle between x and y is 90 degrees. In other words, x and y are perpendicular to each

other.

But if the lengths of x and y remain fixed, as I alter the angle between them, I alter the area of the parallelogram determined by them. All right? I just thought that's a cute little device whereby you can visualize how the magnitude of the cross product of two vectors varies. Think of the two vectors at a common origin, and the magnitude is affected by the area of that parallelogram as the angle changes. OK? In other words, if you change the angle, the area of the parallelogram changes.

But at any rate, getting back to our original problem, the corollary to that is the following. We already know what the magnitude of AB cross AC is from example one. But what does that mean now, geometrically? From what we've just seen, the magnitude of AB cross AC is the area of the parallelogram, which has AB and AC as edges.

Notice that that parallelogram is precisely one-half-- I should say that--I shouldn't say that-- the triangle that we're looking for is precisely one-half the parallelogram. In other words, notice that the area of the triangle is simply 1/2 the magnitude of AB cross AC.

How do we find the magnitude of AB cross C? We square each of its components, add them, and extract the positive square root, as usual. Without going through the details, it's simply going to be what? This particular amount. And the rest is arithmetic's baby, so to speak. All right?

Now, what I'm hoping is that by this stage of a game, you now have a better feeling as to what the cross product means. Going back to the beginning of my lecture, notice that I played down the physical applications. Because one of the things that I wanted you to see is that the cross product has tremendous application, just as space geometry alone, quite apart from anything else.

Well, at any rate, what I'm going to do next time is-- now that we've talked about lines and planes and what have you-- to get down to the Cartesian representation of what lines and planes would look like. They play a very vital role in the study of

functions of several variables. We'll see more about that as the course continues to unfold. So until next time, goodbye.