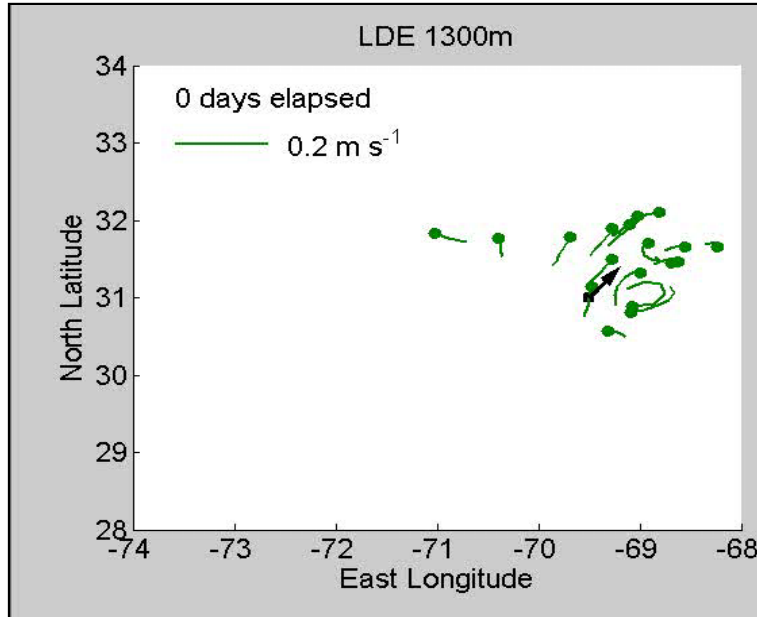


Lagrangian and Eulerian Representations of Fluid Flow: Kinematics and the Equations of Motion



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Summary: This essay introduces the two methods that are widely used to observe and analyze fluid flows, either by observing the trajectories of specific fluid parcels, which yields what is commonly termed a Lagrangian representation, or by observing the fluid velocity at fixed positions, which yields an Eulerian representation. Lagrangian methods are often the most efficient way to sample a fluid flow and the physical conservation laws are inherently Lagrangian since they apply to moving fluid volumes rather than to the fluid that happens to be present at some fixed point in space. Nevertheless, the Lagrangian equations of motion applied to a three-dimensional continuum are quite difficult in most applications, and thus almost all of the theory (forward calculation) in fluid mechanics is developed within the Eulerian system. Lagrangian and Eulerian concepts and methods are thus used side-by-side in many investigations, and the premise of this essay is that an understanding of both systems and the relationships between them can help form the framework for a study of fluid mechanics.

The transformation of the conservation laws from a Lagrangian to an Eulerian system can be envisaged in three steps. (1) The first is dubbed the Fundamental Principle of Kinematics; the fluid velocity at a given time and fixed position (the Eulerian velocity) is equal to the velocity of the fluid parcel (the Lagrangian velocity) that is present at that position at that instant. Thus while we often speak of Lagrangian velocity or Eulerian velocity, it is important to keep in mind that these are merely (but significantly) different ways to represent a given fluid flow. (2) A similar relation holds for time derivatives of fluid properties: the time rate of change observed on a specific fluid parcel, $D()/Dt = \partial()/\partial t$ in the Lagrangian system, has a counterpart in the Eulerian system, $D()/Dt = \partial()/\partial t + \mathbf{V} \cdot \nabla()$, called the material derivative. The material derivative at a given position is equal to the Lagrangian time rate of change of the parcel present at that position. (3) The physical conservation laws apply to extensive quantities, i.e., the mass or the momentum of a specific fluid volume. The time derivative of an integral over a moving fluid volume (a Lagrangian quantity) can be transformed into the equivalent Eulerian conservation law for the corresponding intensive quantity, i.e., mass density or momentum density, by means of the Reynolds Transport Theorem (Section 3.3).

Once an Eulerian velocity field has been observed or calculated, it is then more or less straightforward to compute parcel trajectories, a Lagrangian property, which are often of great practical interest. An interesting complication arises when time-averaging of the Eulerian velocity is either required or results from the observation method. In that event, the FPK does not apply. If the high frequency motion that is filtered out is wavelike, then the difference between the Lagrangian and Eulerian velocities may be understood as Stokes drift, a correlation between parcel displacement and the spatial gradient of the Eulerian velocity.

In an Eulerian system the local effect of transport by the fluid flow is represented by the advective rate of change, $\mathbf{V} \cdot \nabla()$, the product of an unknown velocity and the first partial derivative of an unknown field variable. This nonlinearity leads to much of the interesting and most of the challenging phenomena of fluid flows. We can begin to put some useful bounds upon what advection alone can do. For variables that can be written in conservation form, e.g., mass and momentum, advection alone can not be a net source or sink when integrated over a closed or infinite domain. Advection represents the transport of fluid properties at a definite rate and direction, that of the fluid velocity, so that parcel trajectories are the characteristics of the advection equation. Advection by a nonuniform velocity may cause linear and shear deformation (rate) of a fluid parcel, and it may also cause a fluid parcel to rotate. This fluid rotation rate, often called vorticity follows a particularly simple and useful conservation law.

Cover page graphic: SOFAR float trajectories (green worms) and horizontal velocity measured by a current meter (black vector) during the Local Dynamics Experiment conducted in the Sargasso Sea. Click on the figure to start an animation. The float trajectories are five-day segments, and the current vector is scaled similarly. The northeast to southwest oscillation seen here appears to be a (short) barotropic Rossby wave; see Price, J. F. and H. T. Rossby, 'Observations of a barotropic planetary wave in the western North Atlantic', *J. Marine Res.*, **40**, 543-558, 1982. An analysis of the potential vorticity balance of this motion is in Section 7. These data and much more are available online from <http://ortelius.who.edu/> and other animations of float data North Atlantic are at <http://www.argo.ucsd.edu/Acpictures.html>

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1 The challenge of fluid mechanics is mainly the kinematics of fluid flow.

This essay introduces a few of the concepts and mathematical tools that make up the foundation of fluid mechanics. Fluid mechanics is a vast subject, encompassing widely diverse materials and phenomena. This essay emphasizes aspects of fluid mechanics that are relevant to the flow of what one might term ordinary fluids, air and water, that make up the Earth’s fluid environment.^{1,2} The physics that govern the geophysical flow of these fluids is codified by the conservation laws of classical mechanics: conservation of mass, and conservation of (linear) momentum, angular momentum and energy. The theme of this essay follows from the question — How can we apply these conservation laws to the analysis of a fluid flow?

¹Footnotes provide references, extensions or qualifications of material discussed in the main text, along with a few homework assignments. They may be skipped on first reading.

²An excellent web page that surveys the wide range of fluid mechanics is <http://physics.about.com/cs/fluidynamics/>

In principle the answer is straightforward; first we erect a coordinate system that is suitable for describing a fluid flow, and then we derive the mathematical form of the conservation laws that correspond to that system. The definition of a coordinate system is a matter of choice, and the issues to be considered are more in the realm of kinematics — the description of fluid flow and its consequences — than of dynamics or physical properties. However, the physical properties of a fluid have everything to do with the response to a given force, and so to appreciate how or why a fluid is different from a solid, the most relevant physical properties of fluids and solids are reviewed briefly in Section 1.1.

Kinematics of fluid flow are considered beginning in Section 1.2. As we will see in a table-top experiment, even the smallest and simplest fluid flow is likely to be fully three-dimensional and time-dependent. It is this complex kinematics, more than the physics *per se*, that makes classical and geophysical fluid mechanics challenging. This kinematics also leads to the first requirement for a coordinate system, that it be able to represent the motion and properties of a fluid at every point in a domain, as if the fluid material was a smoothly varying continuum. Then comes a choice, discussed beginning in Section 1.3 and throughout this essay, whether to observe and model the motion of moving fluid parcels, the Lagrangian approach that is closest in spirit to solid particle dynamics, or to model the fluid velocity as observed fixed points in space, the Eulerian approach. These each have characteristic advantages and both are systems are widely used, often side-by-side. The transformation of conservation laws and of data from one system to the other is thus a very important part of many investigations and is the object at several stages of this essay.

1.1 Physical properties of materials; what distinguishes fluids from solids?

Classical fluid mechanics, like classical thermodynamics, is concerned with macroscopic phenomena (bulk properties) rather than microscopic (molecular-scale) phenomena. In fact, the molecular makeup of a fluid will be studiously ignored in all that follows, and the crucially important physical properties of a fluid, e.g., its mass density, ρ , heat capacity, C_p , among others, must be provided from outside of this theory, Table (1). It will be assumed that these physical properties, along with flow properties, e.g., the pressure, P , velocity, V , temperature, T , etc., are in principle definable at every point in space, as if the fluid was a smoothly varying continuum, rather than a swarm of very fine, discrete particles (molecules).³

The space occupied by the material will be called the domain. Solids are materials that have a more or less intrinsic configuration or shape and do not conform to their domain under nominal conditions. Fluids do not have an intrinsic shape; gases are fluids that will completely fill their domain (or container) and liquids are fluids that form a free surface in the presence of gravity.

An important property of any material is its response to an applied force, Fig. (1). If the force on the face of a cube, say, is proportional to the area of the face, as will often be the case, then it is appropriate to consider the force per unit area, called the stress, and represented by the symbol S ; \mathbf{S} is a three component stress vector and \mathbb{S} is a nine component stress tensor that we will introduce briefly here and in much more detail in Section 3.4. The SI units of stress are Newtons per meter squared, which is commonly represented by a derived unit, the Pascal, or Pa. Why there is a stress and how the stress is related to the physical

³Readers are presumed to have a college-level background in physics and multivariable calculus and to be familiar with basic physical concepts such as pressure and velocity, Newton's laws of mechanics and the ideal gas laws. We will review the definitions when we require an especially sharp or distinct meaning.

Some physical properties of air, sea and land (granite)

	density $\rho, \text{kg m}^{-3}$	heat capacity $C_p, \text{J kg}^{-1} \text{C}^{-1}$	bulk modulus B, Pa	sound speed c, m s^{-1}	shear modulus K, Pa	viscosity $\nu, \text{Pa s}$
air	1.2	1000	1.3×10^5	330	na	18×10^{-6}
sea water	1025	4000	2.2×10^9	1500	na	1×10^{-3}
granite	2800	2800	4×10^{10}	5950	2×10^{10}	$\geq 10^{22}$

Table 1: Approximate, nominal values of some thermodynamic variables that are required to characterize materials to be described by a continuum theory. These important data must be derived from laboratory studies. For air, the values are at standard temperature, 0 C, and nominal atmospheric pressure, 10^5 Pa. The bulk modulus shown here is for adiabatic compression; under an isothermal compression the value for air is about 30% smaller; the values are nearly identical for liquids and solids. na is not applicable. The viscosity of granite is temperature-dependent; granite is brittle at low temperatures, but appears to flow as a highly viscous material at temperatures above a few hundred C.

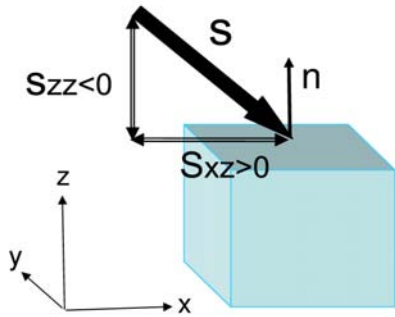


Figure 1: An orthogonal triad of Cartesian unit vectors and a small cube of material. The surrounding material is presumed to exert a stress, \mathbf{S} , upon the face of the cube that is normal to the z axis. The outward-directed unit normal of this face is $\mathbf{n} = \mathbf{e}_z$. To manipulate the stress vector it will usually be necessary to resolve it into components: S_{zz} is the projection of \mathbf{S} onto the \mathbf{e}_z unit vector and is negative, and S_{xz} is the projection of \mathbf{S} onto the \mathbf{e}_x unit vector and is positive. Thus the first subscript on S indicates the direction of the stress component and the second subscript indicates the orientation of the face upon which it acts. This ordering of the subscripts is a convention, and it is not uncommon to see this reversed.

properties and the motion of the material are questions of first importance that we will begin to consider in this section. To start we can take the stress as given.

The component of stress that is normal to the upper surface of the material in Fig. (1) is denoted S_{zz} . A normal stress can be either a compression, if $S_{zz} \leq 0$, as implied in Fig. (2), or a tension, if $S_{zz} \geq 0$. The most important compressive normal stress is almost always due to pressure rather than to viscous effects, and when the discussion is limited to compressive normal stress only we will identify S_{zz} with the pressure.

1.1.1 The response to pressure — in linear deformation liquids are not very different from solids

Every material will undergo some volume change as the ambient pressure is increased or decreased, though the amount varies quite widely from gases to liquids and solids. To make a quantitative measure of the volume change, let P_0 be the nominal pressure and h_0 the initial thickness of the fluid sample; denote the pressure change by δP and the resulting thickness change by δh . The normalized change in thickness, $\delta h / h_0$,

is called the linear deformation (linear in this case meaning that the displacement is in line with the stress). The linear deformation is of special significance in this one-dimensional configuration because the volume change is equal to the linear deformation, $\delta V = V_0 \delta h / h_0$ (in a two- or three-dimensional fluid this need not be the case, Section 6.3). The mass of material, $M = \rho V$, is not affected by pressure changes and hence the mass density, $\rho = M / V$, will change inversely with the linear deformation;

$$\frac{\delta \rho}{\rho_0} = -\frac{\delta V}{V_0} = -\frac{\delta h}{h_0}, \quad (1)$$

where δ is a small change, $\delta \ll 1$. Assuming that the dependence of thickness change upon pressure can be observed in the laboratory, then $\delta h = \delta h(P_0, \delta P)$ together with Eq. (1) are the rudiments of an equation of state, the functional relationship between density, pressure and temperature, $\rho = \rho(P, T)$ or equivalently, $P = P(\rho, T)$, with T the absolute temperature in Kelvin.

The archetype of an equation of state is that of an ideal gas, $PV = nRT$ where n is the number of moles of the gas and $R = 8.31 \text{ Joule moles}^{-1} \text{ K}^{-1}$ is the universal gas constant. An equivalent form that shows pressure and density explicitly is

$$P = \rho RT / M, \quad (2)$$

where $\rho = nM / V$ is the mass density and M is the molecular weight (kg/mole). If the composition of the material changes, then the appropriate equation of state will involve more than three variables, for example the concentration of salt if sea water, or water vapor if air.

An important class of phenomenon may be described by a reduced equation of state having state variables density and pressure alone,

$$\rho = \rho(P), \text{ or equivalently, } P = P(\rho). \quad (3)$$

It can be presumed that ρ is a monotonic function of P and hence that $P(\rho)$ should be a well-defined function of the density. A fluid described by Eq. (3) is said to be ‘barotropic’ in that the gradient of density will be everywhere parallel to the gradient of pressure, $\nabla \rho = (\partial \rho / \partial P) \nabla P$, and hence surfaces of constant density will be parallel to surfaces of constant pressure. The temperature of the fluid will change as pressure work is done on or by the fluid, and yet temperature need not appear as a separate, independent state variable provided conditions approximate one of two limiting cases: If the fluid is a fixed mass of ideal gas, say, that can readily exchange heat with a heat reservoir having a constant temperature, then the gas may remain isothermal under pressure changes and so

$$\rho = \rho_0 \frac{P}{P_0}, \text{ or, } P = P_0 \frac{\rho}{\rho_0}. \quad (4)$$

The other limit, which is more likely to be relevant, is that heat exchange with the surroundings is negligible because the time scale for significant conduction is very long compared to the time scale (lifetime or period) of the phenomenon. In that event the system is said to be adiabatic and in the case of an ideal gas the density and pressure are related by the well-known adiabatic law,⁴

$$\rho = \rho_0 \left(\frac{P}{P_0} \right)^{\frac{1}{\gamma}}, \text{ or, } P = P_0 \left(\frac{\rho}{\rho_0} \right)^{\gamma}. \quad (5)$$

⁴An excellent online source for many physics topics including this one is *Hyperphysics*; <http://hyperphysics.phy-astr.gsu.edu/hbase/thermo/adiab.html#c1>

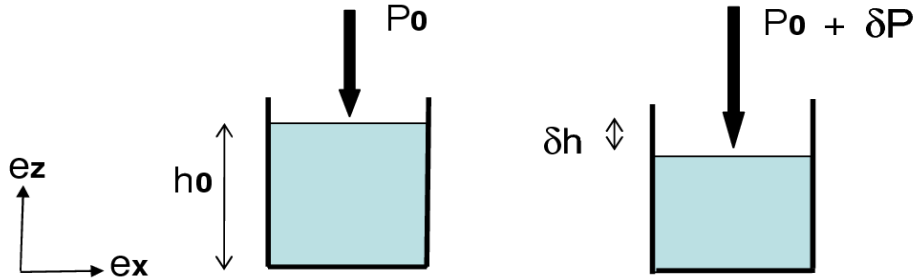


Figure 2: A solid or fluid sample confined within a piston has a thickness h_0 at the ambient pressure P_0 . If the pressure is increased by an amount δP , the material will be compressed by the amount δh and the volume decreased in proportion. The work done during this compression will raise the temperature of the sample, perhaps quite a lot if the material is a gas, and we have to specify whether the sidewalls allow heat flux into the surroundings (isothermal compression) or not (adiabatic compression); the B in Table 1 is the latter.

The parameter $\gamma = C_p/C_v$ is the ratio of specific heat at constant pressure to the specific heat at constant volume; $\gamma \approx 1.4$ for air and nearly independent of pressure or density. In an adiabatic process, the gas temperature will increase with compression (work done on the gas) and hence the gas will appear to be less compressible, or stiffer, than in an otherwise similar isothermal process, Eq. (2).

A convenient measure of the stiffness or inverse compressibility of the material is

$$B = \frac{S_{zz}}{\delta h/h} = -V_0 \frac{\delta P}{\delta V} = \rho_0 \frac{\delta P}{\delta \rho}, \quad (6)$$

called the bulk modulus. Notice that B has the units of stress or pressure, Pa, and is much like a normalized spring constant; B times the normalized linear strain (or volume change or density change) gives the resulting pressure change. The numerical value of B is the pressure increase required to compress the volume by 100% of V_0 . Of course, a complete compression of that sort does not happen outside of black holes, and the bulk modulus should be regarded as the first derivative of the state equation, accurate for small changes around the ambient pressure, P_0 . Gases are readily compressed; a pressure increase $\delta P = 10^4$ Pa, which is 10% above nominal atmospheric pressure, will cause an air sample to compress by about $B^{-1} 10^4 Pa = \delta V/V_0 = 7\%$ under adiabatic conditions. Most liquids are quite resistant to compressive stress, e.g., for water, $B = 2.2 \times 10^9$ Pa, which is less than but comparable to the bulk modulus of a very stiff solid, granite (Table 1). Thus the otherwise crushing pressure in the abyssal ocean, up to about 1000 times atmospheric pressure in the deepest trench, has a rather small effect upon sea water, compressing it and raising the density by only about five percent above sea level values. Water is stiff enough and pressure changes associated with geophysical flows small enough that for many purposes water may be idealized as an incompressible fluid, as if B was infinite. Surprisingly, the same is often true for air.

The first several physical properties listed in Table 1 suggest that water has more in common with granite than with air, our other fluid. The character of fluids becomes evident in their response to anything besides a compressive normal stress. Fluids are qualitatively different from solids in their response to a tensile normal stress, i.e., $S_{zz} \geq 0$, is resisted by many solid materials, especially metals, with almost the same strength that they exhibit to compression. In contrast, gases do not resist tensile stress at all, while liquids do so only very,

very weakly when compared with their resistance to compression. Thus if a fluid volume is compressed along one dimension but is free to expand in a second, orthogonal, direction (which the one-dimensional fluid confined in a piston, Fig. (2), can not, of course) then the volume may remain nearly constant though the fluid may undergo significant linear deformation, compression and a compensating expansion, in orthogonal directions.

1.1.2 The response to shear stress — solids deform and fluids flow

A stress that is parallel to (in the plane of) the surface that receives the stress is called a ‘shear’ stress.⁵ A shear stress that is in the x direction and applied to the upward face of the cube in Fig. (1) would be labeled S_{xz} and a shear stress in the y -direction, S_{yz} . A measure of a material’s response to a steady shear stress is the shear deformation, r/h , where r is the steady (equilibrium) sideways displacement of the face that receives the shear stress and h is the column thickness (Fig. 3, and note that the cube of material is presumed to be stuck to the lower surface). The corresponding stiffness for shear stress, or shear modulus, is then defined as

$$K = \frac{S_{xz}}{r/h}, \quad (7)$$

which has units of pressure. The magnitude of K is the shear stress required to achieve a shear deformation of $r/h = 1$, which is past the breaking point of most solid materials. For many solids the shear modulus is comparable to the bulk modulus (Table 1).⁶

Fluids are qualitatively different from solids in their response to a shear stress. Ordinary fluids such as air and water have no intrinsic configuration, and hence fluids do not develop a restoring force that can provide a static balance to a shear stress.⁷ There is no volume change associated with a pure shear deformation and thus no coupling to the bulk modulus. Hence, there is no meaningful shear modulus for a fluid since r/h will not be steady. Rather, the distinguishing physical property of a fluid is that it will move or ‘flow’ in response to a shear stress, and a fluid will continue to flow so long as a shear stress is present.

When the shear stress is held steady, and assuming that the geometry does not interfere, the shear deformation rate, $h^{-1}(dr/dt)$, may also be steady or have a meaningful time-average. In analogy with the shear modulus, we can define a generalized viscosity, γ , to be the ratio of the measured shear stress to the overall (for the column as a whole), and perhaps time-averaged shear deformation rate,

$$\gamma = \frac{S_{xz}}{h^{-1}dr/dt}. \quad (8)$$

⁵The word *shear* has an origin in the Middle English *scheren*, which means to cut with a pair of sliding blades (as in ‘Why are you scheren those sheep in the kitchen? If I’ve told you once I’ve told you a hundred times .. blah, blah, blah...’) A velocity shear is a spatial variation of the velocity in a direction that is perpendicular to the velocity vector.

⁶The distinction between solid and fluid seems clear enough when considering ordinary times and forces. But materials that may appear unequivocally solid when observed for a few minutes may be observed to flow, albeit slowly, when observed over many days or millenia. Glaciers are an important example, and see the pitch drop experiment of footnote 2.

⁷There is no volume change associated with a pure shear deformation and thus no coupling to the bulk modulus. There does occur a significant linear deformation, compression and expansion, in certain directions that we will examine in a later section, 6.3.

While fluids have no *intrinsic* restoring forces or equilibrium configuration, nevertheless, there are important restoring forces set up within fluids in the presence of an acceleration field. Most notably, gravity will tend to restore a displaced free surface back towards level. Earth’s rotation also endows the atmosphere and oceans with something closely akin to angular momentum that provides a restoring tendency for horizontal displacements; the oscillatory wave motion seen in the cover graphic is an example.

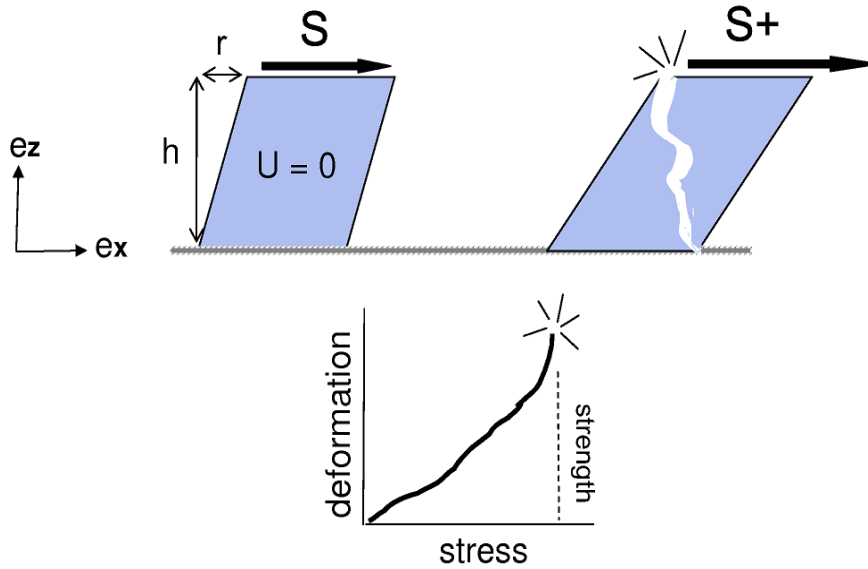


Figure 3: A vector stress, \mathbf{S} , is imposed upon the upper face of a cube of solid material that is attached to a lower surface. Given the orientation of this face with respect to the unit vectors, this stress can also be represented by a single component, S_{xz} , of the stress tensor (Section 2.2.1). For small values of the stress, a solid will come to a static equilibrium in which an elastic restoring force balances the shear stress. The shear deformation (also called the shear ‘strain’) may be measured as r/h for small angles. It is fairly common that homogeneous materials exhibit a roughly linear stress/deformation relationship for small deformations. But if the stress exceeds the strength of the material, a solid may break, an irreversible transition. Just before that stage is reached the stress/deformation ratio is likely to decrease.

This ratio of shear stress to shear deformation rate will depend upon the kind of fluid material and also upon the flow itself, i.e., the speed, $U = dr/dt$ of the upper moving surface and the column thickness, h . This generalized viscosity times a unit, overall velocity shear $U(z = h)/h = h^{-1}(dr/dt) = 1 \text{ s}^{-1}$ is the shear stress required to produce the unit velocity shear.

Laminar flow at small Reynolds number: If the flow depicted in Fig. 4 is set up carefully, it may happen that the fluid velocity U will be steady, with velocity vectors lying smoothly, one on top of another, in layers or ‘laminar’ flow (the upper left of Fig. 4). The ratio

$$\boxed{\nu = \frac{S_{xz}}{\partial U / \partial z}} \tag{9}$$

is then a property of the fluid alone, called just viscosity, or sometimes dynamic viscosity.⁸

Newtonian fluids, air and water: Fluids for which the viscosity in laminar flow is a thermodynamic property of the fluid alone and not dependent upon the shear stress magnitude are dubbed ‘Newtonian’ fluids,

⁸There are about twenty boxed equations in this essay, beginning with Eq. (9), that you will encounter over and over again in a study of fluid mechanics. These boxed equations are sufficiently important that they should be memorized, and you should be able to explain in detail what each term and each symbol means.

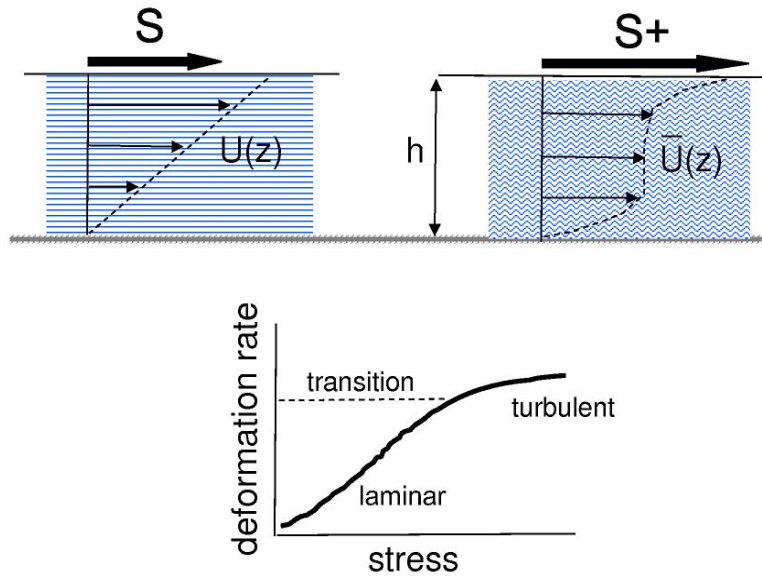


Figure 4: A vector stress, S , is imposed upon the upper face of a cube of fluid material that is sitting on a no-slip lower surface. Since we are considering only the z -dependence of the flow, it is implicit that the fluid and the stress are uniform in the horizontal. The response of a fluid to a shear stress is quite different from that of a solid in as much as a fluid has no intrinsic shape and so develops no elastic restoring force in response to a deformation. Instead, an ordinary fluid will move or flow so long as a shear stress is imposed and so the relevant kinematic variable is the shear deformation rate. For small values of the stress and assuming a Newtonian fluid, the fluid velocity, $U(z)$, may come into a laminar and steady state with a uniform vertical shear, $\partial U/\partial z = U(h)/h = \text{constant} = S_{xz}/\nu$, that can be readily observed and used to infer the fluid viscosity, ν , given the measured stress. For larger values of stress (right side) the flow may undergo a reversible transition to a turbulent state in which the fluid velocity $\bar{U}(z)$ is likely to be well-defined provided the external conditions are held constant. In this turbulent flow state, the time-averaged shear $\partial \bar{U}/\partial z$ will vary with z , being larger near the boundaries. The shear stress and the time-averaged overall deformation rate, $\bar{U}(h)/h$, are not related by a constant viscosity as obtains in the laminar flow regime, and across the turbulent transition the stress/deformation rate ratio will increase.

in recognition of Isaac Newton's insightful analysis of frictional effects in fluid flow. Air and water are found to be Newtonian fluids to an excellent approximation.⁹

If the fluid is Newtonian, then it is found empirically that the conditions for laminar flow include that a nondimensional parameter called the Reynolds number, Re , must satisfy the inequality

$$Re = \frac{\rho U h}{\nu} \leq 400, \quad (10)$$

where U is the speed of the upper (moving) surface relative to the lower, fixed, no-slip surface. In practice this means that the speed must be very low or the column thickness very small. The laminar flow velocity $U(z)$ of a Newtonian fluid will vary linearly with z and the velocity shear at each point in z will then be equal to the overall shear deformation rate, $\partial U/\partial z = h^{-1}(dr/dt)$, the particular laminar flow sketched in Fig. 4 upper left.

Assuming that we know the fluid viscosity and its dependence upon temperature, density, etc., then the relationship Eq. (9) between viscosity, stress and velocity shear may just as well be turned around and used to estimate the viscous shear stress from a given velocity shear. This is the way that viscous shear stress will be incorporated into the momentum balance of a fluid parcel (Section 3.4.3). It is important to remember, though, that Eq. (9) is not an identity, but rather a contingent experimental law that applies only for laminar, steady flow. If instead the fluid velocity is unsteady and two- or three dimensional, i.e., turbulent, then for a given upper surface speed $U(h)$, the shear stress will be larger, and sometimes quite a lot larger, than the laminar value predicted by Eq. (9) (Figure 4).¹⁰ Evidently then, Eq. (9) has to be accompanied by Eq. (10) along with a description of the geometry of the flow, i.e., that h is the distance between parallel planes (and not the distance from one plate or the diameter of a pipe, for example). In most geophysical flows the equivalent Reynolds number is enormously larger than the upper limit for laminar flow indicated by Eq. (10) and consequently geophysical flows are seldom laminar and steady, but are much more likely to be turbulent and unsteady. Thus it frequently happens that properties of the flow, rather than physical properties of the fluid alone, determine the stress for a given velocity shear in the ocean or atmosphere.

⁹To verify that air and water are Newtonian requires rather precise laboratory measurements that may not be readily accessible. But to understand what a Newtonian fluid *is*, it is very helpful to understand what a Newtonian fluid *is not*, and there is a wide variety of non-Newtonian fluids that we encounter routinely. Many high molecular weight polymers such as paint and mayonnaise are said to be 'shear-thinning'. Under a small stress these materials may behave like very weak solids, i.e., they will deform but not quite flow until subjected to a shear stress that exceeds some threshold that is often an important characteristic of the material. 'Shear-thickening' fluids are less common, and can seem quite bizarre. Here's one you can make at home: a solution of about three parts cold water and two parts of corn starch powder will make a fluid that flows under a gentle stress. When the corn starch solution is pushed too vigorously it will quickly seize up, forming what seems to be a solid material. Try adding a drop of food coloring to the cold water, and observe how or whether the dyed material can be stirred and mixed into the remainder. Sketch the qualitative stress/deformation (or rate of deformation) relationship for these non-Newtonian fluids, as in Figs. (3) and (4). How does water appear to a very small, swimming bug? What would our life be like if water was significantly, observably non-Newtonian for the phenomenon of our everyday existence?

¹⁰Viscosity and turbulence can in some limited respects mimic one another; a given stress and velocity shear can be consistent with either a large viscosity in laminar flow, or, a smaller viscosity (and thus higher Reynolds number) in turbulent flow. The pioneering investigators of liquid helium assumed that the flow in the very small laboratory apparatus used to estimate viscosity must be laminar, when in fact it was turbulent. This delayed the recognition that superfluid helium has a nearly vanishing viscosity (A. Griffin, Superfluidity: a new state of matter. In *A Century of Nature*. Ed. by L. Garwin and T. Lincoln. The Univ. of Chicago Press, 2003.) An excellent introduction to modern experimental research on turbulence including some Lagrangian aspects is by R. Ecke, The turbulence problem, available online at <http://library.lanl.gov/cgi-bin/getfile?01057083.pdf>

1.2 A first look at the kinematics of fluid flow

Up to now we have confined the fluid sample within a piston or have assumed that the lower face was stuck to a no-slip surface and confined between infinite parallel plates. These special geometries are appropriate for analyzing the physical properties of a fluid in a laboratory but not much else. Suppose now that the fluid parcel¹¹ is free to move in any of three dimensions in response to an applied force. We presume that an applied force will cause a fluid parcel to accelerate exactly as expected from Newton's laws of mechanics. In this most fundamental respect, a fluid parcel is not different from a solid particle.

But before we decide that fluids are indeed just like solids, let's try the simplest fluid flow experiment. Some day your fluid domain will be grand and important, the Earth's atmosphere or perhaps an ocean basin, but for now you can make useful qualitative observations in a domain that is small and accessible; even a teacup will suffice because the fundamentals of kinematics are the same for flows big and small. To initiate flow in a tea cup we need only apply an impulse, a gentle, linear push on the fluid with a spoon, say, and then observe the result. The motion of the fluid bears little resemblance to this simple forcing. The fluid that is directly pushed by the spoon can not simply plow straight ahead, both because water is effectively incompressible for such gentle motion and because the inertia of the fluid that would have to be displaced is appreciable. Instead, the fluid flows mainly around the spoon from front to back, forming swirling coherent features called vortices that are clearly two-dimensional, despite that the forcing was a one-dimensional push. This vortex pair then moves slowly through the fluid, and careful observation will reveal that most of the linear (one-dimensional) momentum imparted by the push is contained within their translational motion. Momentum is conserved, but the fluid flow that results would be hard to anticipate if one's intuition derived solely from solid mechanics. If the initial push is made a little more vigorous, then the resulting fluid motion will spontaneously become three-dimensional and irregular, or turbulent (as in the high Reynolds number flow between parallel plates, Fig. 4).

After a short time, less than a few tens of seconds, the smallest spatial scales of the motion will be damped by viscosity leaving larger and larger scales of motion, often vortices, with increasing time. This damping process is in the realm of physics since it depends very much upon a physical property of the fluid, the viscosity, and also upon the physical scale (i.e., the size) of the flow features. Thus even though our intent in this essay is to emphasize kinematics, we can not go far without acknowledging physical phenomena, in this case damping of the motion due to fluid viscosity. The last surviving flow feature in a tea cup forced by an impulse is likely to be a vortex that fills the entire tea cup.

These details of fluid flow are all important, but for now we want to draw only the broadest inferences regarding the form that a theory or description of a fluid flow must take. These observations shows us that every parcel that participates in fluid flow is literally pushed and pulled by all of the surrounding fluid parcels via shear stress and normal stress. A consequence is that we can not predict the motion of a given parcel in isolation from its surroundings, rather we have to predict the motion of the surrounding fluid parcels as well. How extensive are these so-called surroundings? It depends upon how far backward or forward in time we may care to go, and also upon how rapidly signals including waves are propagated within the fluid. If we

¹¹A fluid 'particle' is equivalent to a solid particle in that it denotes a specific small piece of the material that has a vanishing extent. If our interest is position only, then a fluid particle would suffice. A fluid 'parcel' is a particle with a small but finite area and volume and hence can be pushed around by normal and shear stresses. When we use 'point' as a noun we will always mean a point in space, i.e., a position, rather than a fluid particle or parcel.

follow a parcel long enough, or if we need to know the history in detail, then every parcel will have a dependence upon the entire domain occupied by the fluid. In other words, even if our goal was limited to calculating the motion of just one parcel or the flow at just one place, we would nevertheless have to solve for the fluid motion over the entire domain at all times of interest. As we have remarked already and you have observed (if you have studied your teacup) fluid flows may spontaneously develop motion on all accessible spatial scales, from the scale of the domain down to a scale set by viscous or diffusive properties of the fluid, typically a fraction of a millimeter in water. Thus what we intended to be the smallest and simplest (but unconstrained) dynamics experiment turns out to be a remarkably complex, three-dimensional phenomenon that fills the entire, available domain and that has spatial scales much smaller than that imposed by the forcing.¹² The tea cup and its fluid flow are well within the domain of classical physics and so we can be confident that everything we have observed is consistent with the classical conservation laws for mass, momentum, angular momentum and energy.

It is the complex kinematics of fluid flow that most distinguishes fluid flows from the motion of otherwise comparable solid materials. The physical origin of this complex kinematics is the ease with which fluids undergo shear deformation. The practical consequence of this complex kinematics is that an appropriate description and theory of fluid flow must be able to define motion and acceleration on arbitrarily small spatial scales, i.e., that the coordinates of a fluid theory or model must vary continuously. This is the phenomenological motivation for the continuum model of fluid flow noted in the introduction to Section 1.1 (there are interesting, specialized alternatives to the continuum model noted in a later footnote 32).

1.3 Two ways to observe fluid flow and the Fundamental Principle of Kinematics

Let's suppose that our task is to observe the fluid flow within some three-dimensional domain that we will denote by R^3 . There are two quite different ways to accomplish this, either by tracking specific, identifiable fluid material volumes that are carried about with the flow, the Lagrangian method, or by observing the fluid velocity at locations that are fixed in space, the Eulerian method (Fig. 5). Both methods are commonly used in the analysis of the atmosphere and oceans, and in fluid mechanics generally. Lagrangian methods are natural for many observational techniques and for the statement of the fundamental conservation theorems. On the other hand, almost all of the theory in fluid mechanics has been developed in the Eulerian system. It is for this reason that we will consider both coordinate systems, at first on a more or less equal footing, and will emphasize the transformation of conservation laws and data from one system to the other.

¹²How many observation points do you estimate would be required to define completely the fluid flow in a teacup? In particular, what is the smallest spatial scale on which there is a significant variation of the fluid velocity? Does the number depend upon the state of the flow, i.e., whether it is weakly or strongly stirred? Does it depend upon time in any way? Which do you see more of, linear or shear deformation rate? The viscosity of water varies by a factor of about four as the temperature varies from 100 to 0 C. Can you infer the sense of this viscosity variation from your observations? To achieve a much larger range of viscosity, consider a mixture of water and honey. What fundamental physical principles, e.g., conservation of momentum, second law of thermodynamics, can you infer from purely qualitative observations and experiments?

The fluid motion may also include waves: capillary waves have short wavelengths, only a few centimeters, while gravity waves can have any larger wavelength, and may appear mainly as a sloshing back and forth of the entire tea cup. Waves can propagate momentum and energy much more rapidly than can the vortices. Capillary and gravity waves owe their entire existence to the free surface, and may not appear at all if the speed at which the spoon is pushed through the fluid does not exceed a certain threshold. Can you estimate roughly what that speed is? It may be helpful to investigate this within a somewhat larger container.

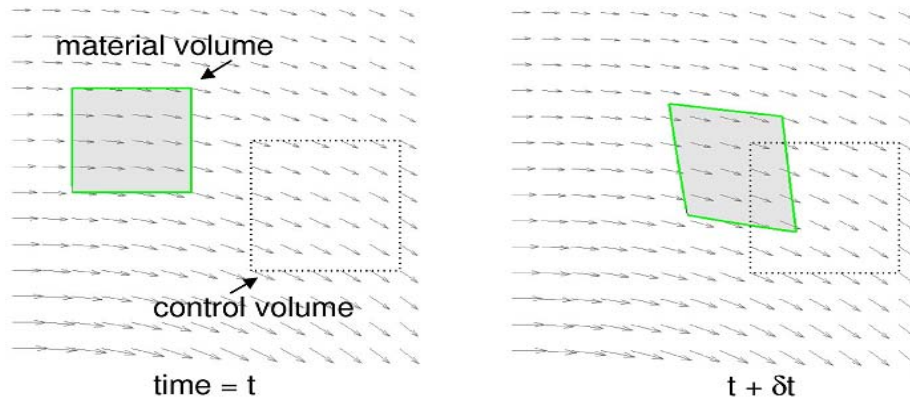


Figure 5: A velocity field, represented by a regular array of velocity vectors, and within which there is a material fluid volume (green boundary and shaded) and a control volume (dotted boundary). The (Lagrangian) material volume is made up of specific fluid parcels that are carried along with the flow. The (Eulerian) control volume is fixed in space, and the sides are imaginary and completely invisible so far as the flow is concerned. The fluid material inside a control volume is continually changing, assuming that there is some fluid flow. The essence of a Lagrangian representation is that we observe and seek to describe the position, pressure, and other properties of material volumes; the essence of an Eulerian representation is that we observe and seek to describe the fluid properties inside control volumes. The continuum model assumes that either a material volume or a control volume may be made as small as is necessary to resolve the phenomenon of fluid flow.

The most natural way to observe a fluid flow is to observe the trajectories of discrete material volumes or parcels, which is almost certainly your (Lagrangian) observation method in the tea cup experiment. To make this quantitative we will use the Greek uppercase ξ to denote the position vector of a parcel whose Cartesian components are the lowercase (ξ, ψ, ω) , i.e., ξ is the x -coordinate of a parcel, ψ is the y -coordinate of the parcel and ω is the z -coordinate. If we knew the density, ρ , as a function of the position, i.e., $\rho(\xi, \psi, \omega)$ we could just as well write this as $\rho(x, y, z)$ and we will have occasion to do this in later sections. An important question is how to identify specific parcels? For the purpose of a continuum theory we will need a scheme that can serve to tag and identify parcels throughout a domain and at arbitrarily fine spatial resolution. One possibility is to use the position of the parcels at some specified time, say the initial time, $t = 0$; denote the initial position by the Greek uppercase alpha, A , with Cartesian components, (α, β, γ) . We somewhat blithely assume that we can determine the position of parcels at all later times, t , to form the parcel trajectory, also called the pathline,

$$\xi = \xi(A, t) \tag{11}$$

The trajectory ξ of specific fluids parcels is a dependent variable in a Lagrangian description (along with pressure and density) and the initial position A and time, t , are the independent variables.¹³

The velocity of a parcel, often termed the ‘Lagrangian’ velocity, V_L , is just the time rate change of the

¹³We are not going to impose a time limit on parcel identity. But in practice, how long can you follow a parcel (a small patch of dye) around in a tea cup before it effectively disappears by diffusion into its surroundings?

parcel position holding A fixed, where this time derivative will be denoted by

$$\frac{D}{Dt} = \frac{d}{dt} |_{A=\text{constant}} \quad (12)$$

When this derivative is applied to a Lagrangian variable that depends upon A and t , say the parcel position, it is simply a partial derivative with respect to time,

$$\boxed{V_L(A, t) = \frac{D\xi(A, t)}{Dt} = \frac{\partial \xi(A, t)}{\partial t}} \quad (13)$$

where V_L is the Lagrangian velocity. If instead of a fluid continuum we were dealing with a finite collection of solid particles or floats, we could represent the particle identity by a subscript appended to ξ and the time derivative would then be an ordinary time derivative since there would be no independent variable A . Aside from this, the Lagrangian velocity of a fluid parcel is exactly the same thing as the velocity of a (solid) particle familiar from classical dynamics.

If tracking fluid parcels is impractical, perhaps because the fluid is opaque, then we might choose to observe the fluid velocity by means of current meters that we could implant at fixed positions, x . The essential component of every current meter is a transducer that converts fluid motion into a readily measured signal - e.g., the rotary motion of a propeller or the Doppler shift of a sound pulse. But regardless of the mechanical details, the velocity sampled in this way, termed the 'Eulerian' velocity, V_E , is intended to be the velocity of the fluid parcel that is present, instantaneously, within the fixed, control volume sampled by the transducer. Thus the Eulerian velocity is defined by what is here dubbed the Fundamental Principle of Kinematics, or FPK,

$$\boxed{V_E(x, t) |_{x=\xi(A, t)} = V_L(A, t)} \quad (14)$$

where x is fixed and the A on the left and right sides are the same initial position. In other words, the fluid velocity at a fixed position, the x on the left side, is the velocity of the fluid parcel that happens to be at that position at that instant in time. The velocity V_E is a dependent variable in an Eulerian description, along with pressure and density, and the position, x , and time, t , are the independent variables; compare this with the corresponding Lagrangian description noted just above.

One way to appreciate the difference between the Lagrangian velocity V_L and the Eulerian velocity V_E is to note that ξ in the Lagrangian velocity of Eq. (13) is the position of a moving parcel, while x in Eq. (14) is the arbitrary and fixed position of a current meter. Parcel position is a result of the fluid flow rather than our choice, aside from the initial position. As time runs, the position of any specific parcel will change, barring that the flow is static, while the velocity observed at the current meter position will be the velocity of the sequence of parcels (each having a different A) that move through that position as time runs. It bears emphasis that the FPK is valid instantaneously and does not, in general, survive time-averaging, as we will see in a later Section 5.2.

The float and current meter data of the cover graphic afford an opportunity to check the FPK in practice: when the flow is smoothly varying on the horizontal scale of the float cluster, and when the floats surround the current meter mooring, the Lagrangian velocity (the green worms) and the Eulerian velocity (the single black

vector) appear to be very similar. But at other times, and especially when the velocity is changing direction rapidly in time or in space, the equality expected from the FPK is not clearly present.¹⁴

Our usage Lagrangian and Eulerian is standard; if no such label is appended, then Eulerian is almost always understood as the default.¹⁵ The Lagrangian/Eulerian usage should not be interpreted to mean that there are two physical fluid velocities. For a given fluid flow there is a unique fluid velocity that can be sampled in two quite different ways, by tracking specific parcels (Lagrangian) or by observing the motion of fluid parcels that flow through a fixed site (Eulerian). The formal statement of this, Eq. (14), is not very impressive, and hence we have given it an imposing title. Much of what we have to say in this essay follows from variants or extensions of the FPK combined with the familiar conservation laws of classical physics.

1.4 The goal and the plan of this essay; Lagrangian to Eulerian and back again

Now that we have learned (or imagined) how to observe a fluid flow, we can begin to think about surveying the entire domain in order to construct a representation of the complete fluid flow. This will require an important decision regarding the sampling strategy; should we make these observations by tracking a large number of fluid parcels as they wander throughout the domain, or, should we deploy additional current meters and observe the fluid velocity at many additional sites? In principle, either approach could suffice to define the flow if done in sufficient, exhaustive detail (an example being the ocean circulation model of Fig. 6).¹⁶ Nevertheless, the observations themselves and the analysis needed to understand these observations would be quite different, as we will see in examples below. And of course, in practice, our choice of a sampling method will be decided as much by purely practical matters - the availability of floats or current meters - as by any Lagrangian or Eulerian preference we might hold. Thus it commonly happens that we may make observations in one system, and then apply theory or diagnostic analysis in the other. A similar kind of duality arises in the development of models and theories. The (Lagrangian) parcels of a fluid flow follow conservation laws that are identical with those followed by the particles of classical dynamics; nevertheless the theory commonly applied to a continuum model of fluid flow is almost always Eulerian.

The goal of this essay is to develop an understanding of both systems, and especially to appreciate how Lagrangian and Eulerian concepts and models are woven together to implement the observation and analysis of fluid flows. The plan is to describe further the Lagrangian and Eulerian systems in Section 2 and 3, respectively. As we will see in Section 2.3, the three-dimensional Lagrangian equations of motion are quite

¹⁴If a model seems to be consistent with relevant observations, then there may not be much more to say. Much more interesting is the case of an outright failure. What would we do here if the float and current meter velocities did *not* appear to be similar? We would not lay the blame on Eq. (14), which is, in effect, an identity, i.e., it defines what we mean by the Eulerian velocity. Instead, we would start to question, in roughly this order, 1) if $\xi = x$ as required by the FPK, since this would imply a collision between float and current meter (none was reported), 2) if some time-averaging had been applied (it was, inevitably, and time-averaging can have a surprising effect as noted above), 3) whether the float tracking accuracy was sufficient, and then perhaps 4) whether the current meter had been improperly calibrated or had malfunctioned.

¹⁵This usage is evidently inaccurate as historical attribution; Lamb, *Hydrodynamics*, 6th ed., (Cambridge Univ. Press, 1937) credits Leonard Euler with developing both representations, and it is not the least bit descriptive of the systems in the way that ‘material’ and ‘field’ are, somewhat. This essay nevertheless propagates the Lagrangian and Eulerian usage because to try to change it would cause almost certain confusion with little chance of significant benefit.

¹⁶An application of Lagrangian and Eulerian observational methods to a natural system (San Francisco Bay) is discussed by http://sfbay.wr.usgs.gov/watershed/drifter_studies/eul_lagr.html A recent review of Lagrangian methods is by Yeung, P. K., Lagrangian investigations of turbulence, *Ann. Rev. of Fluid Mech.*, **34**, 115-142, 2002.

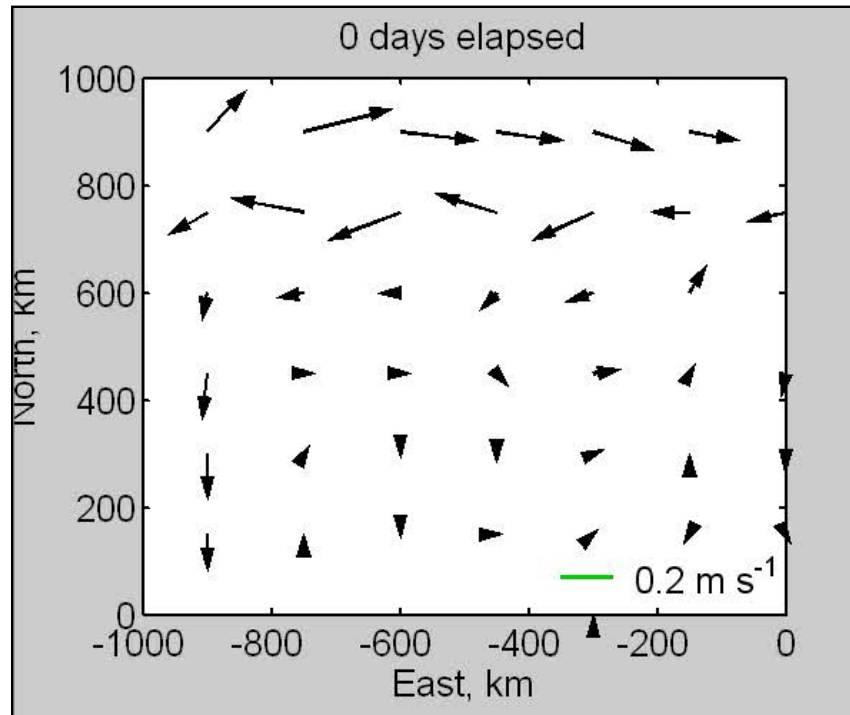


Figure 6: An ocean circulation model solved in the usual Eulerian system, and then sampled for the Eulerian velocity (the regularly spaced black vectors) and analyzed for a comparable number of parcel trajectories (the green worms). If you are viewing this with Acrobat Reader, click on the figure to begin an animation. The domain is a square basin 2000 km by 2000 km driven by a basin-scale wind having negative curl, as if a subtropical gyre. Only the northwestern quadrant of the model domain and only the upper most layer of the model are shown here. The main circulation feature is a rather thin western and northern boundary current that flows clockwise. There is also a well-developed westward recirculation just to the south of the northern boundary current. This westward flow is (baroclinically) unstable and oscillates with a period of about 60 days, comparable to the period of the north-south oscillation of the float cluster seen in the cover graphic. This model solution, like many, suffers from poor horizontal resolution, the grid interval being one fourth the interval between velocity vectors plotted here. As one consequence, the simulated fluid must be assigned an unrealistically large, generalized viscosity, Eq. (8), that is more like very cold honey than water (footnote 10). The Reynolds number of the computed flow is thus lower than is realistic and there is less variance in small scale features than is realistic, but as much as the grid can resolve. How would you characterize the Eulerian and Lagrangian representations of this circulation? In particular, do you notice any systematic differences? This ocean model is available from the author's web page.

difficult when pressure gradients are included, which is nearly always necessary, and the object of Section 3 is therefore to derive the Eulerian equations of motion, which are used almost universally for problems of continuum mechanics. As we remarked above, it often happens that Eulerian solutions for the velocity field need to be transformed into Lagrangian properties, e.g., trajectories as in Fig. (6), a problem considered in Sections 4 and 5. In an Eulerian system the process of transport by the fluid flow is represented by advection, the nonlinear and inherently difficult part of most fluid models and that is considered in Section 6. Section 7 applies many of the concepts and tools considered here in an analysis of the Lagrangian, oceanic data seen in the cover graphic. And finally, Section 8 is a brief summary.

This essay is pedagogical in aim and in style. It has been written for students who have some background in fluid mechanics, and who are beginning to wonder how to organize and consolidate the many topics that make up fluid mechanics. This essay starts from an elementary level and is intended to be nearly self-contained. Nevertheless, it is best viewed as a supplement rather than as a substitute for a comprehensive fluids textbook,¹⁷ even where topics overlap.¹⁸ There are two reasons. First, the plan is to begin with a Lagrangian perspective and then to transform step by step to the Eulerian system that we almost always use for theory. This is not the shortest or easiest route to useful results, which is instead a purely Eulerian treatment that is favored rightly by introductory textbooks. Second, many of the concepts or tools that are introduced here — the velocity gradient tensor, Reynolds Transport Theorem, the method of characteristics — are reviewed only briefly compared to the depth of treatment that most students will need if they were seeing these things for the very first time.

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2 The Lagrangian (or material) coordinate system.

One helpful way to think of a fluid flow is that it carries or maps parcels from one position to the next, e.g., from a starting position, A , into the positions ξ at some later time. Given a starting position A and a time, we presume that there is a unique ξ . Each trajectory that we observe or construct must be tagged with a unique A and thus for a given trajectory A is a constant. In effect, the starting position is carried along with the parcel, and thus serves to identify the parcel. A small patch of a scalar tracer, e.g., dye concentration, can be used in the exactly the same way to tag one or a few specific parcels, but our coordinate system has to do much more; our coordinate system must be able to describe a continuum defined over some domain, and hence A must

¹⁷Modern examples include excellent texts by P. K. Kundu and I. C. Cohen, *Fluid Mechanics* (Academic Press, 2001), by B. R. Munson, D. F. Young, and T. H. Okiishi, *Fundamentals of Fluid Mechanics, 3rd ed.* (John Wiley and Sons, NY, 1998), by D. C. Wilcox, *Basic Fluid Mechanics* (DCW Industries, La Canada, CA, 2000) and by D. J. Acheson, *Elementary Fluid Dynamics* (Clarendon Press, Oxford, 1990). A superb text that emphasizes experiment and fluid phenomenon is by D. J. Tritton, *Physical Fluid Dynamics* (Oxford Science Pub., 1988). Two other classic references, comparable to Lamb but more modern are by Landau, L. D. and E. M. Lifshitz, 'Fluid Mechanics', (Pergamon Press, 1959) and G. K. Batchelor, 'An Introduction to Fluid Dynamics', (Cambridge U. Press, 1967). An especially good discussion of the physical properties of fluids is Ch. 1 of Batchelor's text.

¹⁸A rather advanced source for fluid kinematics is Chapter 4 of Aris, R., *Vectors, Tensors and the Basic Equations of Fluid Mechanics*, (Dover Pub., New York, 1962). A particularly good discussion of the Reynolds Transport Theorem (discussed here in Section 3.2) is by C. C. Lin and L. A. Segel, *Mathematics Applied to Deterministic Problems in the Natural Sciences* (MacMillan Pub., 1974). A new and quite advanced monograph that goes well beyond the present essay is by A. Bennett, *Lagrangian Fluid Dynamics*, Cambridge Univ. Press, 2006.

vary continuously over the entire domain of the fluid. The variable A is thus the independent, spatial coordinate in a Lagrangian coordinate system. This kind of coordinate system in which parcel position is the fundamental dependent spatial variable is sometimes and appropriately called a ‘material’ coordinate system.

We will assume that the mapping from A to ξ is continuous and unique in that adjacent parcels will never be split apart, and neither will one parcel be forced to occupy the same position as another parcel.¹⁹ This requires that the fluid must be a smooth continuum down to arbitrarily small spatial scales. With these conventional assumptions in place, the mapping of parcels from initial to subsequent positions, Eq. (11), can be inverted so that a Lagrangian representation, which we described just above, can be inverted to yield an Eulerian representation,

$$\begin{array}{ccc} \xi = \xi(A, t) & \iff & A = A(\xi, t) \\ \text{Lagrangian representation} & & \text{Eulerian representation} \end{array} \quad (15)$$

at least in principle. In the Lagrangian representation we presume to know the starting position, A , the independent variable, and treat the subsequent position ξ as the dependent variable — in the Eulerian representation we take the fixed position, $X = \xi$ as the independent variable (the usual spatial coordinate) and ask what was the initial position of the parcel now present at this position, i.e., A is treated as the dependent variable. In the study of fluid mechanics it seldom makes sense to think of parcel initial position as an observable in an Eulerian system (in the way that it does make sense in the study of elasticity of solid continuum dynamics). Hence, we will not make use of the right hand side of Eq. (15) except in one crucial way, we will assume that trajectories are invertible when we transform from the A coordinates to the ξ coordinates, a Lagrangian to Eulerian transformation later in this section, and will consider the reverse transformation, Eulerian to Lagrangian in Section 3.1. As we will see, in practice these transformations are not as symmetric as these relations imply, if, as we already suggested, initial position is not an observable in an Eulerian representation.

An example of a flow represented in the Lagrangian system will be helpful. For the present purpose it is appropriate to consider a one-dimensional domain denoted by \mathbb{R}^1 . Compared with a three-dimensional domain, \mathbb{R}^3 , this minimizes algebra and so helps to clarify the salient features of a Lagrangian description. However, there are aspects of a three-dimensional flow that are not contained in one space dimension, and so we will have to generalize this before we are done. But for now let’s assume that we have been given the trajectories of all the parcels in a one-dimensional domain with spatial coordinate x by way of the explicit formula²⁰

$$\xi(\alpha, t) = \alpha(1 + 2t)^{1/2}. \quad (16)$$

Once we identify a parcel by specifying the starting position, $\alpha = \xi(t = 0)$, this handy little formula tells us the x position of that specific parcel at any later time. It is most unusual to have so much information presented in such a convenient way, and in fact, this particular flow has been concocted to have just enough complexity to be interesting for our purpose here, but has no physical significance. There are no parameters in

¹⁹The mapping from A to ξ can be viewed as a coordinate transformation. A coordinate transformation can be inverted provided that the Jacobian of the transformation does not vanish. The physical interpretation is that the fluid density does not vanish or become infinite. See Lin and Segel (footnote 18) for more on the Jacobian and coordinate transformations in this context.

²⁰When a list of parameters and variables is separated by commas as $\xi(\alpha, t)$ on the left hand side of Eq (16), we mean to emphasize that ξ is a function of α , a parameter since it is held constant on a trajectory, and t , an independent variable. When variables are separated by operators, as $\alpha(1 + 2t)$ on the right hand side, we mean that the variable α is to be multiplied by the sum $(1 + 2t)$.

Eq. (16) that give any sense of a physical length scale or time scale, i.e., whether this is meant to describe a flow on the scale of a millimeter or an ocean basin. In the same vein, the variable t , called ‘time’ must be nondimensional, $t = \text{time divided by some time scale}$ if this equation is to satisfy dimensional homogeneity. We need not define these space or time scales so long as the discussion is about kinematics, which is scale-independent.

The velocity of a parcel is readily calculated as the time derivative holding α constant,

$$V_L(\alpha, t) = \frac{\partial \xi}{\partial t} = \alpha(1 + 2t)^{-1/2} \quad (17)$$

and the acceleration is just

$$\frac{\partial^2 \xi}{\partial t^2} = -\alpha(1 + 2t)^{-3/2}. \quad (18)$$

Given the initial positions of four parcels, let’s say $\alpha = (0.1, 0.3, 0.5, 0.7)$ we can readily compute the trajectories and velocities from Eqs. (16) and (17) and plot the results in Figs. 7a and 7b. Note that the velocity depends upon the initial position, α . If V_L did not depend upon α , then the flow would necessarily be spatially uniform, i.e., all the fluid parcels in the domain would have exactly the same velocity. The flow shown here has the following form: all parcels shown (and we could say all of the fluid in $\alpha > 0$) are moving in the direction of positive x ; parcels that are at larger α move faster (Eq. 17); all of the parcels having $\alpha > 0$ are also decelerating and the magnitude of this deceleration increases with α (Eq. 18). If the density remained nearly constant, which it does in most geophysical flows but does not in the one-dimensional flow defined by Eq. 16, then it would be appropriate to infer a force directed in the negative x direction (more on this below).

2.1 The joy of Lagrangian measurement

Consider the information that the Lagrangian representation Eq. (16) provides; in the most straightforward way possible it shows where fluid parcels released into a flow at the initial time and position $x = \alpha$ will be found at some later time. If our goal was to observe how a fluid flow carried a pollutant from a source (the initial position) into the rest of the domain, then this Lagrangian representation would be ideal. We could simply release or tag parcels over and over again at the source position, and then observe where the parcels were carried by the flow. By releasing a cluster of parcels we could observe how the flow deformed or rotated the fluid, e.g., the float cluster shown on the cover page and taken up in detail in Section 7.

In a real, physical experiment the spatial distribution of sampling by Lagrangian methods is inherently uncontrolled, and we can not be assured that any specific portion of the domain will be sampled unless we launch a parcel there. Even then, the parcels may spend most of their time in regions we are not particularly interested in sampling, a hazard of Lagrangian experimentation. Whether this is important is a practical, logistical matter. It often happens that the major cost of a Lagrangian measurement scheme lies in the tracking apparatus, with additional floats or trackable parcels being relatively cheap; Particle Imaging Velocimetry noted in the next section being a prime example. In that circumstance there may be almost no limit to the number of Lagrangian measurements that can be made.

If our goal was to measure the force applied to the fluid, then by tracking parcels in time it is straightforward to estimate the acceleration. Given that we have defined and can compute the acceleration of a fluid parcel, we go on to assert that Newton’s laws of classical dynamics apply to a fluid parcel in exactly the

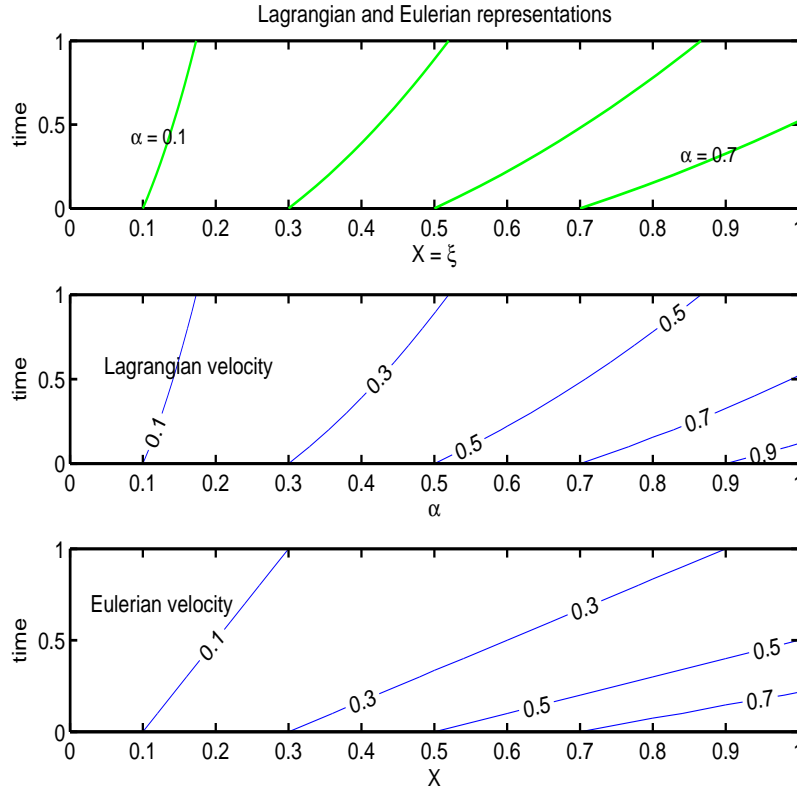


Figure 7: Lagrangian and Eulerian representations of the one-dimensional, time-dependent flow defined by Eq. (16). (a) The solid lines are the trajectories $\xi(\alpha, t)$ of four parcels whose initial positions were $\alpha = 0.1, 0.3, 0.5$ and 0.7 . (b) The Lagrangian velocity, $V_L(\alpha, t) = \partial\xi/\partial t$, as a function of initial position, α , and time. The lines plotted here are contours of constant velocity, not trajectories, and although this plot looks exactly like the trajectory data plotted just above, it is a completely different thing. (c) The corresponding Eulerian velocity field $V_E(y, t)$, and again the lines are contours of constant velocity.

form used in classical (solid particle) dynamics, i.e.,

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{\mathbf{F}}{\rho}, \quad (19)$$

where \mathbf{F} is the net force per unit volume imposed upon that parcel by the environment, and ρ is the mass per unit volume of the fluid. In virtually all geophysical and most engineering flows, the density remains nearly constant at $\rho = \rho_0$, and so if we observe that a fluid parcel undergoes an acceleration, we can readily infer that there must have been a force applied to that parcel. It is on this kind of diagnostic problem that the Lagrangian coordinate system is most useful, generally. These are important and common uses of the Lagrangian coordinate system but note that they are all related in one way or another to the *observation* of fluid flow rather than to the calculation of fluid flow that we will consider in Section 2.4. There is more to say about Lagrangian observation, and we will return to this discussion as we develop the Lagrangian equations of motion later in this section.

2.2 Transforming a Lagrangian velocity into an Eulerian velocity

You may feel that we have only just begun to know this Lagrangian velocity, Eqs. (16) and (17), but let's go ahead and transform it into the equivalent Eulerian velocity field, the transformation process being important in and of itself. We have indicated that a Lagrangian velocity is some function of A and t ,

$$V_L(A, t) = \frac{\partial \xi(A, t)}{\partial t} = \frac{D\xi}{Dt}.$$

Given that parcel trajectories can be inverted to yield $A(\xi, t)$, Eq. (15), we can write the left hand side as a composite function (Section 9.1), $V_L(A(\xi, t), t)$, whose dependent variables are the arguments of the inner function, i.e., ξ and t . If we want to write this as a function of the inner arguments alone, then we should give this function a new name, V_E for Eulerian velocity is appropriate since this will be velocity as a function of the spatial coordinate $x = \xi$, and t . Thus,

$$V_E(x, t) = V_L(A(\xi, t), t), \quad (20)$$

which is another way to state the FPK.²¹

In the example of a Lagrangian flow considered here we have the complete (and unrealistic) knowledge of all the parcel trajectories via Eq. (16) and so we can make the transformation from the Lagrangian velocity Eq. (17) to the Eulerian velocity explicitly. Formally, the task is to eliminate all reference in Eq. (17) to the parcel initial position, α , in favor of the position $x = \xi$. This is readily accomplished since we can invert the trajectory Eq. (16) to find

$$\alpha = \xi(1 + 2t)^{-1/2}, \quad (21)$$

which is the left side of Eq. (15). In other words, given a position, $x = \xi$, and the time, t , we can calculate the initial position, α , from Eq. (21). Substitution of this $\alpha(\xi, t)$ into Eq. (17), substituting x for ξ , and a little rearrangement gives the velocity field

$$V_E(x, t) = u(x, t) = x(1 + 2t)^{-1} \quad (22)$$

which is plotted in Fig. 7c. Notice that this transformation from the Lagrangian to Eulerian system required algebra only; the information about velocity at a given position was already present in the Lagrangian description and hence all that we had to do was rearrange and relabel. To go from the Eulerian velocity back to trajectories will require an integration (Section 3.1).

Admittedly, this is not an especially interesting velocity field, but rather a simple one, and partly as a consequence the (Eulerian) velocity field looks a lot like the Lagrangian velocity of moving parcels, cf., Fig. 7b and Fig. 7c. However, the independent spatial coordinates in these figures are qualitatively different - the Lagrangian data of (b) is plotted as a function of α , the initial x -coordinate of parcels, while the Eulerian data of (c) is plotted as a function of the usual field coordinate, the fixed position, x . To compare the Eulerian and

²¹It would be sensible to insist that the most Fundamental Principle of fluid kinematics is that trajectories may be inverted, Eq. (15), combined with the properties of composite functions noted in Section 9. What we call the FPK, Eq. (14), is an application of this more general principle to fluid velocity. However, Eq. (14) has the advantage that it starts with a focus on fluid flow, rather than the somewhat abstract concept of inverting trajectories.

the Lagrangian velocities as plotted in Fig. 7 is thus a bit like comparing apples and oranges; they are not the same kind of thing despite that they have the same dimensions and in this case they describe the same flow.

Though different generally, nevertheless there are times and places where the Lagrangian and Eulerian velocities are equal, as evinced by the Fundamental Principle of Kinematics or FPK, Eq. (14). By tracking a particular parcel in this flow, in Fig. 8 we have arbitrarily chosen the parcel tagged by $\alpha = 0.5$, and by observing velocity at a fixed site, arbitrarily, $x = 0.7$, we can verify that the corresponding Lagrangian and Eulerian velocities are equal at $t = 0.48$ when the parcel arrives at that fixed site, i.e., when $x = 0.5 = \xi(\alpha = 0.7, t = 0.48)$, consistent with the FPK (Fig. 8b). Indeed, there is an *exact* equality since there has been no need for approximation in this transformation Lagrangian \rightarrow Eulerian.²² In Section 3.1 we will transform this Eulerian velocity field into the equivalent Lagrangian velocity.

2.3 The Lagrangian equations of motion in one dimension

If our goal is to carry out a forward calculation in the Lagrangian system, i.e., to predict rather than to observe fluid flow, then we would have to specify the net force, the \mathbf{F} of Eq. (19), acting on parcels. This is something we began to consider in Section 1.1 and will continue here; to minimize algebra we will retain the one-dimensional geometry. Often the extension of one-dimensional results to three-dimensions is straightforward. But that is unfortunately not the case for the Lagrangian equations of motion, as we will note in Section 2.4. Also, in what follows below we are going to consider the effects of fluid velocity and pressure only, while omitting the effects of diffusion, which, as we noted in Section 1, is likely to be important in many real fluid flows. The (molecular) diffusion of heat or momentum that occurs in a fluid is however, not fundamentally different from the diffusion of heat in a solid, for example, and for our present purpose can be omitted.

2.3.1 Mass conservation; mass is neither lost or created by fluid flow

Consider a one-dimensional flow, so that the velocity is entirely in the x-direction, and all variations of the pressure, fluid density, and fluid velocity are in the x-direction only (Fig. 9). Suppose that in the initial state there is a material volume of fluid that occupies the interval $\alpha_1 < x < \alpha_2$. The cross-sectional area of this material volume will be denoted by A (not to be confused with the initial position vector A that is not needed here). At some later time, this volume will be displaced to a new position where its endpoints will be at $x = \xi_1$ and $x = \xi_2$.

The mass of the volume in its initial state is just

$$M = A\bar{\rho}_0(\alpha)(\alpha_2 - \alpha_1), \quad (23)$$

where the overbar indicates mean value. After the material volume is displaced, the end points will be at $\xi_1(\alpha_1, t)$, etc., and the mass in the displaced position will be

$$M = A\bar{\rho}(\alpha, t)(\xi_2 - \xi_1), \quad (24)$$

²²Here's one for you: assume Lagrangian trajectories $\xi = a(e^t + 1)$ with a a constant. Compute and interpret the Lagrangian velocity $V_L(\alpha, t)$ and the Eulerian velocity field $V_E(x, t)$. Suppose that two parcels have initial positions $\alpha = 2a$ and $2a(1 + \delta)$ with $\delta \ll 1$; how will the distance between these parcels change with time? How is the rate of change of this distance related to V_E ? (Hint: consider the divergence of the velocity field, $\partial V_E / \partial x$.) Suppose the trajectories are instead $\xi = a(e^t - 1)$.

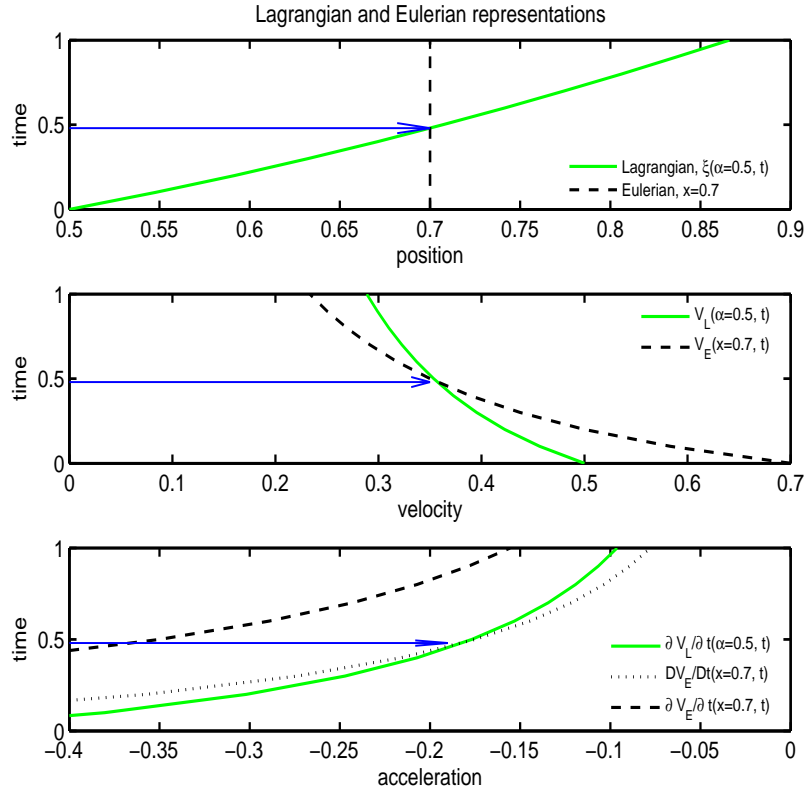


Figure 8: Lagrangian and Eulerian representations of the one-dimensional, time-dependent flow defined by Eq. (16). (a) Positions; the position or trajectory (green, solid line) of a parcel, ξ , having $\alpha = 0.5$. A fixed observation site, $y = 0.7$ is also shown (dashed line) and is a constant in this diagram. Note that this particular trajectory crosses $y = 0.7$ at time $t = 0.48$, computed from Eq. (21) and marked with an arrow in each panel. (b) The Lagrangian velocity of the parcel defined by $\alpha = 0.5$ and the Eulerian velocity at the fixed position, $y = 0.7$. Note that at $t = 0.48$ the Lagrangian velocity of this parcel and the Eulerian velocity at the noted position are exactly equal, but not otherwise. That this equality holds is at once trivial - a non-equality could only mean an error in the calculation - but also consistent with and illustrative of the FPK, Eq. (3). (c) Accelerations; the Lagrangian acceleration of the parcel (green, solid line) and the Eulerian acceleration evaluated at the fixed position $x = 0.7$. There are two ways to compute a time rate change of velocity in the Eulerian system; the partial time derivative is shown as a dashed line, and the material time derivative, DV_E/Dt , is shown as a dotted line. The latter is the counterpart of the Lagrangian acceleration in the sense that at the time the parcel crosses the Eulerian observation site, $DV_E/Dt = \partial V_L/\partial t$, discussed in Section 3.2.

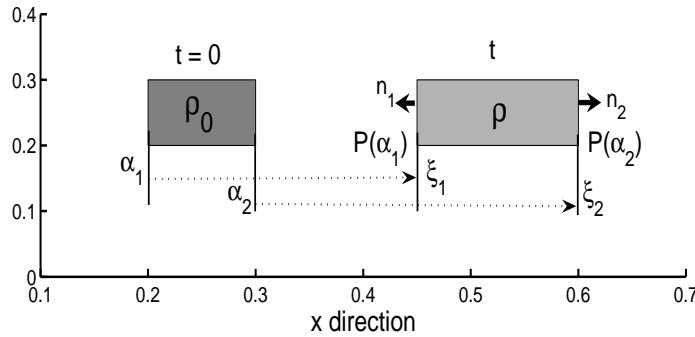


Figure 9: A schematic of a moving fluid parcel used to derive the Lagrangian conservation equations for mass (density) and momentum. This volume is presumed to have an area normal to the x -direction of A (not shown) and motion is presumed to be in the x -direction only. In the Lagrangian system the independent coordinates are the initial x -position of a parcel, α , and the time, t . The dependent variables are the position of the parcel, $\xi(\alpha, t)$, the density of the parcel, $\rho(\alpha, t)$ and the pressure, $P(\alpha, t)$.

and exactly equal to the initial mass. How can we be so sure? Because the fluid parcels that make up the volume can not move through one another or through the boundary, which is itself a specific parcel. Thus the material in this volume remains the same under fluid flow and hence the name ‘material volume’; a two-dimensional example is sketched in Fig. (12). (The situation is quite different in a ‘control volume’, an imaginary volume that is fixed in space, Fig. (5), and hence is continually swept out by fluid flow, as discussed in Section 3.) Equating the masses in the initial and subsequent states,

$$M = A\bar{\rho}_0(\alpha)(\alpha_2 - \alpha_1) = A\bar{\rho}(\alpha, t)(\xi_2 - \xi_1),$$

and thus the density of the parcel at later times is related to the initial density by

$$\bar{\rho}(\alpha, t) = \bar{\rho}_0(\alpha) \frac{\alpha_2 - \alpha_1}{\xi_2 - \xi_1}.$$

If we let the interval of Eqs. (23) and (24) be small, in which case we will call the material volume a parcel, and assuming that ρ is smoothly varying, then the ratio of the lengths becomes the partial derivative, and

$$\boxed{\rho(\alpha, t) = \rho_0(\alpha) \left(\frac{\partial \xi}{\partial \alpha} \right)^{-1}} \quad (25)$$

which is exact (since no terms involving products of small changes have been dropped). The term $\partial \xi / \partial \alpha$ is called the linear deformation, and is the normalized volume change of the parcel. In the case sketched in Fig.(9), the displacement increases in the direction of increasing α , and hence $\partial \xi / \partial \alpha > 1$ and the fluid flow is accompanied by an increase in the volume of a parcel, compared with the initial state. (Notice that with the present definition of ξ as the position relative to the coordinate axis (and not to the initial position) then $\partial \xi / \partial \alpha = 1$ corresponds to zero change in volume.) In Section 1.1 we considered a measure of linear deformation, $\delta h / h$, that applied to a fluid column as a whole; this is the differential, or pointwise, version of the same thing.

This one-dimensional Lagrangian statement of mass conservation shows that density changes are inversely related to the linear deformation. Thus when a material volume is stretched (expanded) compared with the initial state, the case shown schematically in Fig. (9), the density of the fluid within that volume will necessarily be decreased compared with ρ_0 . Indeed, in this one-dimensional model that excludes diffusion,

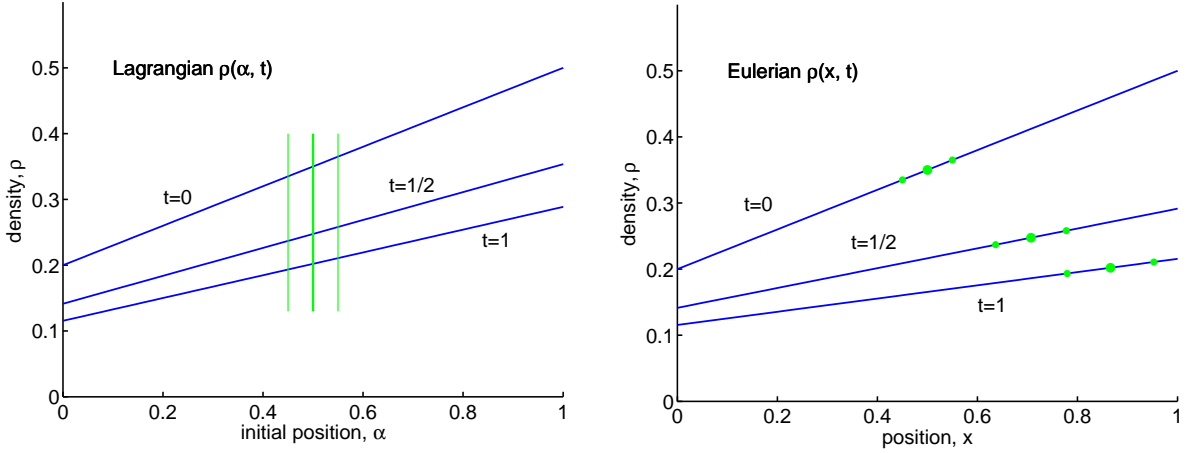


Figure 10: The Lagrangian and Eulerian representations (left and right) of the density of Eqs. (27) and (69), the latter is in Section 3.4. The density is evaluated at $t = 0, 1/2, 1$. The green dots in the Eulerian figure are parcel position and density for three parcels, $\alpha = 0.5$ (the bigger, central dot) and $\alpha = 0.45$ and 0.55 . Note that the distance between these parcels increases with time, i.e, the material volume of which they are the endpoints is stretched (see the next figure) and thus the Lagrangian density shown at left decreases with increasing time; so does the Eulerian density shown at right.

the only way that the density of a *material volume* can change is by linear deformation (stretching or compression) regardless of how fast or slow the fluid may move and regardless of the initial profile. On the other hand, if we were to observe density at a fixed site, the Eulerian perspective that will be developed in Section 3.4, this process of density change by stretching or compression will also occur, but in addition, density at a fixed site will also change merely because fluid of a different density may be transported or advected to the site by the flow (Fig. 10). Very often this advection process will be much larger in amplitude than is the stretching process, and if one's interest was to observe density changes of the fluid as opposed to density changes at a fixed site, then a Lagrangian measurement approach might offer a significant advantage.

As an example of density represented in a Lagrangian system we will assume an initial density

$$\rho_0(\alpha) = \rho_c + \Gamma\alpha \quad (26)$$

that is embedded in the Lagrangian flow, Eq. (16), $\xi = \alpha(1 + 2t)^{1/2}$. It is easy to compute the linear deformation, $\partial\xi/\partial\alpha = (1 + 2t)^{1/2}$, and by Eqs. (25) and (26) the Lagrangian density evolves according to

$$\rho(\alpha, t) = \frac{\rho_c + \Gamma\alpha}{(1 + 2t)^{1/2}}. \quad (27)$$

This density is evaluated for $\rho_c = 0.2$ and $\Gamma = 0.3$ and at several α s and times in Fig. (10, left). The density of a moving parcel, i.e., the density at constant α , thus decreases with time, rather quickly from $0 < t < 1/2$ and more slowly from $1/2 < t < 1$.²³

²³Suppose that there was no linear deformation in this one-dimensional example, or more generally, that there was no volume change of three-dimensional flow. How would the density vary in that case? In this instance the Lagrangian solution for density is very simple to calculate, which is somewhat unusual. Anticipating the Eulerian solutions of Section 3 and the method of characteristics in Section 6.2, how could you use this Lagrangian solution to find the Eulerian solution, i.e., what else do you need to know?

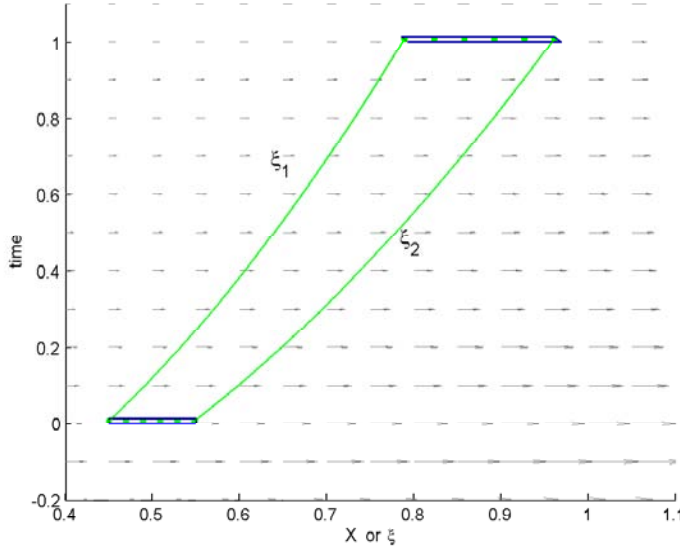


Figure 11: The one-dimensional velocity field $u(x, t)$ of Eq. (22) is plotted as the array of velocity vectors. The grey bar represents a material line whose end points are the parcels at ξ_1 and ξ_2 and whose trajectories $\xi(\alpha, t)$ are the green lines. This velocity field is divergent and hence the length of the material line increases in time; the mass density thus decreases with time (shown in the previous figure). Notice that the parcels that make up this one-dimensional material volume always maintain their relative order, though the distance between them may increase or decrease.

It is helpful to define a normalized measure of the deformation or nondimensional density anomaly by

$$C(\alpha, t) = \left(\frac{\partial \xi}{\partial \alpha} \right)^{-1} - 1 = \frac{\rho - \rho_0}{\rho_0}, \quad (28)$$

which is called the condensation. If $\frac{\partial \xi}{\partial \alpha} > 1$ so that the parcel has been stretched out, then $C < 0$ and the fractional density change will be negative. For small amplitude motions the condensation hovers around zero, and thus C makes a useful measure of the amplitude of the motion. The condensation that accompanies our simple Lagrangian flow Eq. (16) is

$$C(\alpha, t) = \frac{1}{(1 + 2t)^{1/2}} - 1$$

and is not small for $t = O(1)$. This is an artifact of using a one-dimensional flow: if a one-dimensional flow has some spatial variation then it will have a comparably large condensation.

2.3.2 Momentum conservation; $F = Ma$ in a one dimensional fluid flow

As we noted in Section 1.1, a fluid parcel is presumed to follow Newton's laws of classical mechanics and for the present purpose we are going to consider that pressure is the only force acting upon a material volume. Pressure is important in almost every flow; small scale flows may also be effected directly by viscous effects that we can omit for now. In the Lagrangian system the pressure force on the right side will be $-P(\alpha_2)A$ and the pressure force on the left side $P(\alpha_1)A$ (the sign changes with the unit normal). The net pressure force on the volume is then the sum, $F = -(P(\alpha_2) - P(\alpha_1))A$. The mass of the material volume is just $M = \rho_0 A(\alpha_2 - \alpha_1)$, and the acceleration is just $a = \frac{\partial^2 \xi}{\partial t^2}$. Newton's second law applied to this material volume is then

$$\begin{aligned} F &= Ma \\ -(P(\alpha_2) - P(\alpha_1))A &= \rho_0 A(\alpha_2 - \alpha_1) \frac{\partial^2 \xi}{\partial t^2}. \end{aligned} \quad (29)$$

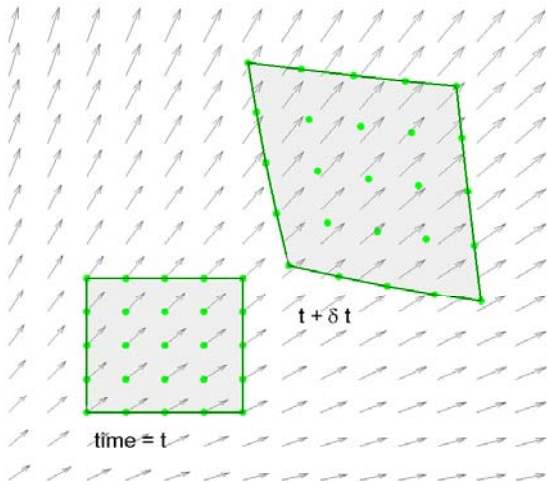


Figure 12: A fluid material ‘volume’ in R^2 , V , that includes N parcels shown as green dots. The material volume and the parcels are advected in a steady, divergent velocity field, and after elapsed time δt the volume has expanded as has the distance between parcels. The number of parcels N within the marked volume remains exactly the same since parcels never cross through the boundary; N is thus said to be conserved under fluid flow. The volume increases and hence the number density, N/V , decreases with time.

Dividing by the α interval and repeating the limit procedure noted above gives the one-dimensional Lagrangian equation of motion,

$$\boxed{\frac{\partial^2 \xi(\alpha, t)}{\partial t^2} = -\frac{1}{\rho_0} \frac{\partial P(\alpha, t)}{\partial \alpha}} \quad (30)$$

which is remarkably simple. A couple of things to take note of: (1) The density that appears in this momentum equation is the given, initial density, ρ_0 , which could depend upon α , and not the variable density, ρ . The latter follows because in the Lagrangian system we are observing a material volume of fluid that conserves mass in the way that we defined in the previous section. (2) This is a partial differential equation, since there are derivatives with respect to both time and α . *Why* there is a pressure variation is a non-trivial question; pressure is as much an unknown in this equation as is the parcel position itself, and so all we can say is that pressure variations are almost always required to represent the interactions of a parcel with its surroundings, and for example to enforce that two parcels can not occupy the same position. If this was the momentum balance of a solid particle then we probably would not have this latter dependence, or at least we wouldn’t have to account for this dependence throughout a continuum, as we do with a fluid.

The derivative of pressure in Eq. (30) is with respect to α , the initial position, or material coordinate, and so this is the genuine Lagrangian form of momentum conservation. It is not the force per unit volume, which is the pressure *gradient*, $\partial P/\partial x$, that you may already know and that will arise in the Eulerian equation of motion considered in Section 3. We can calculate the pressure gradient from this by invoking the invertibility of the trajectory, Eq. (15), i.e., that $\alpha(\xi)$ exists and so we can differentiate $P(\alpha(\xi))$. Now apply the chain rule for differentiation of a composite function (Section 9.2) to calculate the derivative of pressure with respect to ξ ,

$$\frac{\partial P(\alpha(\xi))}{\partial \xi} = \frac{\partial P(\alpha(\xi))}{\partial \alpha} \frac{\partial \alpha}{\partial \xi}$$

and thus

$$\frac{\partial P(\alpha)}{\partial \alpha} = \frac{\partial P(\alpha(\xi))}{\partial \xi} \left(\frac{\partial \alpha}{\partial \xi}\right)^{-1} = \frac{\partial P}{\partial x} \frac{\partial x}{\partial \alpha}.$$

Note that the derivatives $\partial/\partial\alpha$ and $\partial/\partial x$ are related by the linear deformation. Substitution into Eq. (30) and the use of Eq. (25) gives

$$\frac{\partial^2 \xi(\alpha, t)}{\partial t^2} = -\frac{1}{\rho} \frac{\partial P}{\partial x}, \quad (31)$$

a form that is especially useful for diagnosing forces from observations of the acceleration. When the acceleration on the left hand side is evaluated from observations, the right side will be the net force per unit volume, \mathbf{F} , divided by the actual, variable density. Without some additional information we would not know the cause of the force, i.e., whether due to a pressure gradient, as presumed here, or a frictional stress.

2.3.3 The one-dimensional Lagrangian equations reduce to an exact wave equation

The equation of state, $P = P(\rho)$, Eq. (3), provides a third and crucial piece of information by linking pressure with density, and we already know that the density is related inversely to the strain. Thus the system of equations, Eqs. (3), (25) and (30) are in principle a complete set for the three unknown variables ξ , ρ and P . It is often useful to reduce such a system of equations to a single equation in one variable, if possible, and in this one-dimensional system it is easy. The pressure term may be eliminated from the momentum equation (30) by substituting the derivative of pressure from the equation of state,

$$\frac{\partial P(\rho)}{\partial \alpha} = \frac{dP(\rho)}{d\rho} \frac{\partial \rho}{\partial \alpha}.$$

The derivative of density may in turn be eliminated by using the mass conservation relation, Eq. (25),

$$\frac{\partial \rho}{\partial \alpha} = -\rho_0 \frac{\partial^2 \xi}{\partial \alpha^2} / \left(\frac{\partial \xi}{\partial \alpha} \right)^2.$$

Substitution into the momentum equation then yields an exact, nonlinear wave equation in ξ alone:

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{\frac{dP}{d\rho}(\rho_0 \frac{\partial \xi}{\partial \alpha})}{\left(\frac{\partial \xi}{\partial \alpha} \right)^2} \frac{\partial^2 \xi}{\partial \alpha^2}. \quad (32)$$

It is nonlinear because the term $(\frac{\partial \xi}{\partial \alpha})^2$ is nonlinear, and it is also nondimensional; the term $\frac{dP}{d\rho}(\rho_0 \frac{\partial \xi}{\partial \alpha})$ is also likely to be nonlinear, depending upon the appropriate equation of state, and it has the dimensions $P/\rho \doteq \text{length}^2 \text{time}^{-2}$ of a velocity². For some purposes, e.g., modeling the very intense acoustic waves that are produced by an explosion, this nonlinear equation is necessary. But for many phenomena the nonlinear equation is not warranted and is an unnecessary encumbrance. For example, the condensation accompanying a mid-audio range sound having a frequency of 1000 Hz, and an amplitude that would be considered appreciable to the ear is very small, C is $O(10^{-6})$.²⁴ Thus the derivative $\partial \xi / \partial \alpha$ is very, very close to 1 and the density that appears in $dP/d\rho$ can be taken to be the nominal density, ρ_0 , with very little error. These approximations of small amplitude motion are often called the acoustic approximations, and with these in place the resulting governing equation is the linear, elementary wave equation,

$$\frac{\partial^2 \xi}{\partial t^2} = c^2 \frac{\partial^2 \xi}{\partial \alpha^2}. \quad (33)$$

²⁴D. H. Towne, *Wave Phenomenon*, Dover Pub., 1967, is highly recommended for more on acoustic phenomena. An interesting web page is <http://www.du.edu/jcalvert/waves/wavhom.htm>.

where the constant coefficient $c_0^2 = \frac{dP}{d\rho}(\rho_0)$ is the nondispersive wave speed squared. We will return to this after doing a more or less comparable derivation in the Eulerian system.²⁵

2.4 The agony of the three-dimensional Lagrangian equations

The one-dimensional form of the Lagrangian equations (25) and (30) is quite useful and in some respects even more so than the Eulerian counterpart that we will develop in Section 3. If fluid dynamics was truly one-dimensional then Eq. (32) would be our touchstone; we might never need anything more. However, one-dimensional phenomenon are, as we pointed out in Section 1.2, the exception, and these one-dimensional equations do not admit, e.g., shear deformation, which two and three-dimensional fluids do with aplomb. Thus, in a one-dimensional flow the parcels maintain their relative order, while in a three-dimensional flow the trajectories and parcel positions may become completely entangled, with no apparent order (this is evident in the flow in a teacup but in none of the figures plotted here). Far more often we are faced with fully three-dimensional and time-dependent phenomenon whose analysis requires three-dimensional equations of motion. As we are going to find here, the three-dimensional Lagrangian equations are very awkward for most purposes, and consequently they are very seldom used. Hence there is little practical motive to write them out. However if you are curious to see in what way the three-dimensional Lagrangian equations are awkward, and so to understand why we almost always use the Eulerian equations instead, then read on. Otherwise you can skip ahead to Section 3 with no real loss.

To see how or why the three-dimensional complication arises we are going to transform the pressure gradient

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{-1}{\rho} \nabla P - g \mathbf{e}_z, \quad (34)$$

where ∇ is the usual gradient operator, into the Lagrangian (material) coordinates. Better for this purpose is to use the component equations,

$$\frac{\partial^2 \xi}{\partial t^2} = -\frac{1}{\rho} \frac{\partial P}{\partial x}, \quad \frac{\partial^2 \psi}{\partial t^2} = -\frac{1}{\rho} \frac{\partial P}{\partial y} \quad \text{and} \quad \frac{\partial^2 \omega}{\partial t^2} = -\frac{1}{\rho} \frac{\partial P}{\partial z} - g. \quad (35)$$

On first sight these three-dimensional Lagrangian equations look promising, but on second sight there is a serious problem with the pressure gradient, and if we had included viscous stress terms, even worse there. The problem is that the gradient of pressure is taken with respect to the usual spatial (field) coordinates (x, y, z) , which in this Lagrangian system are *dependent* variables, i.e., they are equivalent to the (ξ, ψ, ω) , so that we have a kind of mixed notation in Eq. (34). When rewritten in the notation used for Lagrangian variables the equations of motion are then

$$\frac{\partial^2 \xi}{\partial t^2} = -\frac{1}{\rho} \frac{\partial P}{\partial \xi}, \quad \frac{\partial^2 \psi}{\partial t^2} = -\frac{1}{\rho} \frac{\partial P}{\partial \psi} \quad \text{and} \quad \frac{\partial^2 \omega}{\partial t^2} = -\frac{1}{\rho} \frac{\partial P}{\partial \omega} - g \quad (36)$$

which make it clear that we have a problem: the pressure gradient involves the derivative of one unknown with respect to another unknown, which is extraordinarily difficult. The spatial derivative we would much

²⁵Can you show that the linear wave speed in the case of an adiabatic ideal gas is just $c_0^2 = \gamma P_0 / \rho_0$, and for the exact nonlinear wave equation, $c^2 = c_0^2 \frac{\partial \xi}{\partial \alpha}^{-(\gamma+1)}$. Thus the wave speed is greater in regions of condensation, $C > 1$, i.e., where the density is increased compared to ρ_0 .

prefer is with respect to the independent variable, which in the Lagrangian system is the initial position, i.e., the (α, β, γ) . In general, we have to allow that the present position of a parcel will be dependent upon all three components of the initial position of that parcel. By the chain rule, and given that $\alpha = \alpha(\xi, \psi, \omega)$:

$$\frac{\partial P}{\partial \alpha} = \frac{\partial P}{\partial \xi} \frac{\partial \xi}{\partial \alpha} + \frac{\partial P}{\partial \psi} \frac{\partial \psi}{\partial \alpha} + \frac{\partial P}{\partial \omega} \frac{\partial \omega}{\partial \alpha}.$$

Solving this for the pressure gradient $\partial P/\partial \xi$ and substitution into the ξ -component of the Lagrangian momentum equation gives

$$\frac{\partial^2 \xi}{\partial t^2} = - \left(\frac{\partial P}{\partial \alpha} - \frac{\partial P}{\partial \psi} \frac{\partial \psi}{\partial \alpha} - \frac{\partial P}{\partial \omega} \frac{\partial \omega}{\partial \alpha} \right) / \rho \frac{\partial \xi}{\partial \alpha},$$

which does not look as promising. To put this and the other component equations into a form in which only a single pressure term appears in each component equation we can multiply the x-component equation of Eq. (36) by $\partial \xi/\partial \alpha$, the y-component equation by $\partial \psi/\partial \alpha$ and the z-component by $\partial \omega/\partial \alpha$ and add the resulting three equations; the procedure is repeated for the other components with the result (Lamb,¹⁵ Article 1.13):

$$\begin{aligned} \frac{\partial^2 \xi}{\partial t^2} \frac{\partial \xi}{\partial \alpha} + \frac{\partial^2 \psi}{\partial t^2} \frac{\partial \psi}{\partial \alpha} + \left(\frac{\partial^2 \omega}{\partial t^2} + g \right) \frac{\partial \omega}{\partial \alpha} &= - \frac{1}{\rho} \frac{\partial P}{\partial \alpha}, \\ \frac{\partial^2 \xi}{\partial t^2} \frac{\partial \xi}{\partial \beta} + \frac{\partial^2 \psi}{\partial t^2} \frac{\partial \psi}{\partial \beta} + \left(\frac{\partial^2 \omega}{\partial t^2} + g \right) \frac{\partial \omega}{\partial \beta} &= - \frac{1}{\rho} \frac{\partial P}{\partial \beta}, \\ \frac{\partial^2 \xi}{\partial t^2} \frac{\partial \xi}{\partial \gamma} + \frac{\partial^2 \psi}{\partial t^2} \frac{\partial \psi}{\partial \gamma} + \left(\frac{\partial^2 \omega}{\partial t^2} + g \right) \frac{\partial \omega}{\partial \gamma} &= - \frac{1}{\rho} \frac{\partial P}{\partial \gamma}. \end{aligned} \quad (37)$$

The three-dimensional mass conservation equation is listed for completeness,

$$\rho = \rho_0 J \left(\frac{\partial(\xi, \psi, \omega)}{\partial(\alpha, \beta, \gamma)} \right)^{-1} = \rho_0 \begin{vmatrix} \frac{\partial \xi}{\partial \alpha} & \frac{\partial \xi}{\partial \beta} & \frac{\partial \xi}{\partial \gamma} \\ \frac{\partial \psi}{\partial \alpha} & \frac{\partial \psi}{\partial \beta} & \frac{\partial \psi}{\partial \gamma} \\ \frac{\partial \omega}{\partial \alpha} & \frac{\partial \omega}{\partial \beta} & \frac{\partial \omega}{\partial \gamma} \end{vmatrix}^{-1} \quad (38)$$

where J is the determinant of the Jacobian matrix of the transformation from A to ξ (footnote 19).

At the outset of this essay we asked the question: How can we apply the conservation principles of classical physics to a fluid flow? Here in Eqs. (37) and (38) is one possible answer. However, the three-dimensional Lagrangian momentum equations written for a continuum (and hence they contain a pressure gradient) are, not to put too fine a point on it, frightful, and as we have already noted they are not widely useful. With no intent to make this situation seem hopeless, there are two significant hurdles to their use: (1) Some form of nonlinearity is inevitable in a theory of fluid mechanics, but the Lagrangian form of it, in which each acceleration term is multiplied by a deformation term, e.g., $\partial \xi/\partial \alpha$, $\partial \xi/\partial \beta$, is awkward for most purposes and certainly difficult to interpret.²⁶ In some applications it would be appropriate to linearize these equations, but the result is the same linear equation that arises also from the Eulerian equations (and which are easier to work with, in general). (2) The Lagrangian equations involve a second derivative with respect to

²⁶Can you spot the one-dimensional momentum equation (30) in the thicket of terms shown in Eqs. (37)? And, for bonus points, where is our friendly one-dimensional density equation (25) in the three-dimensional Jacobian?

time, where even one integration can be challenging. In effect, the solution of the Lagrangian equations requires solving for parcel trajectories in one fell swoop. In the Eulerian system we can solve for the velocity (one integration in time), and can then integrate the velocity solution to find parcel trajectories (examples of this are sprinkled throughout the next three sections), if they are required. In many problems we may have no interest in knowing the trajectories of specific parcels and if so then a Lagrangian solution may tell us much more than we want to know. That would not be grounds for complaint, except that Lagrangian solutions are seldom possible in the first place. To summarize this discussion — it is a fair generalization²⁷ that the three-dimensional Lagrangian momentum equations, Eq. (37), are not as suitable for most theoretical purposes as are the three-dimensional Eulerian equations that we will take up in the next section.

3 The Eulerian (or field) coordinate system.

We have been keen on extolling the virtues of Lagrangian observation, but we should admit to some inherent problems, as well. The spatial sampling of Lagrangian data is more or less uncontrolled since the parcels will go wherever the flow takes them, and that may not be where our interest lies. For example, if our goal was to observe the time-averaged flow through a channel, then it would probably be preferable to moor a current meter directly in the channel rather than chase floats or drifters in and out. Similarly, if high temporal resolution was desirable then again it would be preferable to install a rapidly sampling current meter rather than to attempt to resample with Lagrangian methods. Indeed, most of the applications of engineering or geophysical fluid dynamics require fluid properties at fixed sites, e.g., the pressure along the surface of a wing or the temperature and wind at a city center, and not the fluid conditions of moving, material volumes. To say it a little differently, most applications of fluid mechanics are site-specific and not fluid-specific.

A simple example of an Eulerian velocity is the one-dimensional flow from Section 2.3 and repeated here,

$$V_E(x, t) = u(x, t) = x(1 + 2t)^{-1}. \quad (39)$$

If we needed to know the velocity at some specific position, x , we have it here. For example, the numerical model of Fig. (6) produces a regular array of Eulerian velocity in three space dimensions, though with finite horizontal and vertical resolution. Most of our effort from here on will be devoted to the generation or the analysis of such Eulerian data.

²⁷To which there are many interesting exceptions, each of which helps to clarify why it is true generally: In this course we will examine a (Lagrangian) model of a finite number of interacting free vortices. To the approximation of potential flow theory these vortices interact only by advecting one another about. Some important microscale aspects of a dilute gas, e.g., diffusivity or viscosity or reaction dynamics, can be modeled using kinetic theory that treats individual gas molecules as hard spheres that interact via two-particle collisions. An accessible and highly recommended introduction to gas theory is Ch. 11 of A. J. Garcia, ‘Numerical Methods for Physics’ (Prentice Hall, New Jersey, 2000). A collection of stars may be modeled as a dilute gas with interactions via gravitational mass attraction. Notice that in each of these three cases the ‘fluid’ is treated as an aggregate of particles, rather than as a continuum, and the pressure gradient is not required. But in addition there are also continuum problems wherein Lagrangian theory is tractable and even advantageous: in surface gravity wave problems, where the free surface is a constant in Lagrangian coordinates, and nonlinear acoustic waves where the full nonlinear equations may be reduced to a single governing equation, at least in one space dimension, Eq. (32). Finally, the method of characteristics considered in Section 6.2 takes a parcel-tracking perspective on a continuum fluid model, but is usually applicable only in circumstances that do not lead to complex trajectories. For further examples see the monograph by A. Bennett noted in footnote 18.

An important and rapidly developing observational technique involves the generation of the Eulerian velocity field from Lagrangian data by the analysis procedure of interpolating or mapping irregularly sampled Lagrangian data $V_L(\alpha, t)$ on to a spatial grid. To know where to assign the velocity we will also have to know the position, $x = \xi(\alpha, t)$. This kind of procedure, a direct application of the FPK, may be used to make maps of entire flow fields. One such method is known as Particle Imaging Velocimetry, or PIV. On the laboratory scale, the PIV technique uses successive images of what can be very small and inexpensive neutrally buoyant particles that can be seeded into a laboratory flow in huge numbers and then illuminated by a pulsed laser source and imaged. Provided that the particles (or the pattern that they form) can be recognized from one image to the next, then it is fairly simple to differentiate parcel position with respect to time and so form a map, sometimes in three-dimensions and in great detail, of the velocity throughout the domain. If the trajectories are fully resolved in time, then the resulting snapshot of Lagrangian velocity can be assigned to the midpoint of the trajectory segment. The same basic technique can be applied to observe winds on a planetary scale by tracking either naturally occurring features, clouds, or balloons. Satellite-tracked drifters on the ocean surface make it possible to measure directly quite detailed maps of the time-mean ocean surface velocity, and acoustically-tracked floats may be used to observe the mid-depth velocity (e.g., the cover graphic).²⁸ As we will see in Section 5.2, time-averaging can introduce surprising consequences into this Lagrangian/Eulerian transformation.

It was noted at the end of Section 2.2 that observations of (well-resolved) trajectories make a rather direct means for diagnosing the net force acting on the associated fluid parcel. The Eulerian equivalent measurement, velocity at a fixed point, is not so readily utilized for that purpose, but to understand why we will first have to transform velocity from the Eulerian to the Lagrangian system, the next subsection, and then develop the equations for mass and momentum conservation.

3.1 Transforming an Eulerian velocity field to Lagrangian trajectories

Given the (complete) one-dimensional Eulerian velocity of Eq. (39), we should be able to compute the corresponding Lagrangian velocity. If we had the trajectories, $\xi(\alpha, t)$, then we could simply substitute for ξ directly into Eq. (39) to find $V_L(\alpha, t)$, the sort of algebraic procedure that sufficed for the Lagrangian to Eulerian transformation of Section 2.2. However, since we are presuming to start with an Eulerian representation it is unrealistic to suppose that we already have the trajectories. Rather, we will have to integrate the velocity in time while holding α constant to find the trajectories for this flow and then we can easily differentiate the trajectories to get the Lagrangian velocity, if that is really what we want. This is the first of several occasions when we will carry out this Eulerian to Lagrangian transformation; here the aim is to show the purely formal steps in a very simple example.

The mathematical task then, is to calculate trajectories by integrating the FPK, Eq. (14),

$$\frac{D\xi}{Dt} = \xi(1 + 2t)^{-1}, \quad (40)$$

²⁸Links to PIV web pages are found at <http://www.efluids.com/efluids/pages/products/PIV-PREV.HTM> A nice illustration of the use of PIV to diagnose a Taylor-Couette flow can be found at <https://engineering.purdue.edu/~wereley/pivresearch.html> The tracers used in this technique may be naturally occurring features as well, see <http://www.irisa.fr/vista/Themes/Demos/MouvementFluide/infra.html> or deliberately seeded floats or drifters, see <http://iprc.soest.hawaii.edu/research/slides/theme2/04-spring-t2-1.pdf> and <http://www.whoi.edu/scientist/abower/> A study of the North Atlantic surface circulation derived from satellite-tracked surface drifters is at <http://asl.whoi.edu/pdf/papers/natldriftdrift.pdf>

where here we have written the moving parcel position as ξ , which is the dependant variable here. In effect we have turned around the FPK by assuming that we know the Eulerian velocity field: and thus the Lagrangian velocity of a moving parcel is the (Eulerian) velocity at the position of that parcel, and, we should probably add, at that instant.

The time derivative in Eq. (40) is an ordinary time derivative, since time is the only one independent variable and ξ is the dependent variable; the spatial independent variable α that defines which parcel and which trajectory enters with the initial condition. To integrate this particular velocity we can easily separate variables, $d\xi/\xi = dt/(1 + 2t)$, and then integrate over appropriate limits; the position goes from α to ξ while time runs from 0 to t and thus

$$\int_{\alpha}^{\xi} \frac{d\xi}{\xi} = \int_0^t \frac{dt}{1 + 2t}.$$

These integrals are easily evaluated,

$$\ln(\xi) - \ln(\alpha) = \ln\left(\frac{1}{2}(1 + 2t)\right),$$

and after taking exponentials and a little rearrangement we have the trajectories of this flow,

$$\xi(\alpha, t) = \alpha(1 + 2t)^{1/2}.$$

We have seen this before as Eq. (16). If needed, the Lagrangian velocity may then be easily computed by taking the partial time derivative of the trajectories,

$$V_L(\alpha, t) = \alpha(1 + 2t)^{-1/2}.$$

There is little more to say about this solution, but a couple remarks about the general procedure are apropos. (1) It should be clear from this and the previous transformation of Section 2.2 that either a complete Lagrangian representation — trajectories from all possible starting points in the domain for all time — or a complete Eulerian velocity field — the velocity at every point in the domain at all times — is sufficient information to define the complete flow and including to transform from one representation to the other. Of course, having complete information of this sort is a bit unusual, and the practical question is how this plays out when we have only partial information, something we will consider beginning in Section 5. (2) There is considerably more information contained within the basic Lagrangian description, i.e., the trajectories, than we have with an Eulerian description, i.e., velocity only, and thus to get the trajectories requires an integration of the Eulerian velocity. This integration will usually have to be carried out numerically even if one has a solution for V_E (the case above is unrealistically simple since we could separate variables), and the integration certainly has to be done numerically if V_E is specified only on a numerical grid, as in the numerical solution of Fig. 6.

3.2 Transforming time derivatives from Lagrangian to Eulerian systems; the material derivative

We have seen in several examples that transforming the velocity back and forth from Lagrangian to Eulerian systems is straightforward in principle, provided that we understand the content of the FPK. Transforming the acceleration (Fig. 8c), or time derivatives generally, is just slightly more involved. In the Eulerian system a

partial derivative with respect to time represents the rate of change observed at a fixed point in space; this is not equal to the Lagrangian time rate change in which α is held constant (i.e., parcel identity is held constant) except in the degenerate case that there is no spatial variation of the flow. To learn how to transform time derivatives we will write the time rate of change following a parcel, i.e., the Lagrangian time rate of change, in terms of Eulerian (or field) variables. This leads to the material time derivative operator, and is the second of three steps required to derive equations suitable for modelling most fluid flows.

To accomplish the transformation of a time derivative we will follow the spirit of the procedure of Section 2.2; assume a fluid variable, say a scalar variable b in the Lagrangian system, $b_L(\alpha, t)$. The time rate of change holding α fixed is just

$$\frac{Db_L}{Dt} = \frac{\partial b_L}{\partial t}, \quad (41)$$

the partial time derivative, as we have noted before. In the case that b is the fluid velocity given by Eq. (17), then

$$\frac{DV_L}{Dt} = -\alpha(1 + 2t)^{-3/2}, \quad (42)$$

which we have also seen before as Eq. (18).

Now presume that the trajectory can in principle be inverted so that we can write $\alpha = \alpha(\xi, t)$ and hence $b_L(\alpha(\xi, t), t)$. The function of the inner variables alone is then $b_E(\xi, t)$, which is the Eulerian representation of b , or,

$$b_E(\xi, t) = b_L(\alpha(\xi, t), t). \quad (43)$$

Now take the time derivative holding α constant of this b_E . By application of the chain rule for composite functions (Section 9.2),

$$\frac{Db_E(\xi, t)}{Dt} = \frac{\partial b_E}{\partial \xi} \frac{\partial \xi}{\partial t} + \frac{\partial b_E}{\partial t}. \quad (44)$$

The time derivative $\partial \xi / \partial t$ holding α constant is just the velocity of a fluid parcel, u , and writing ξ as x , then with minor rearrangement,

$$\frac{Db_E}{Dt} = \frac{\partial b_E}{\partial t} + u \frac{\partial b_E}{\partial x}. \quad (45)$$

The Eulerian representation of the D/Dt derivative is then, in one dimension,

$$\boxed{\frac{D(\)}{Dt} = \frac{\partial(\)}{\partial t} + u \frac{\partial(\)}{\partial x}} \quad (46)$$

We can check the result Eq. (45) by transforming the material derivative of the Eulerian velocity field $u(x, t)$ of Eq. (22) into the corresponding Lagrangian form to compare with the Lagrangian acceleration given by Eq. 42. The material time derivative of this velocity field is just

$$\frac{DV_E}{Dt} = \left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \right) x(1 + 2t)^{-1} = -2x(1 + 2t)^{-2} + x(1 + 2t)^{-2} = -x(1 + 2t)^{-2}.$$

To find the corresponding Lagrangian acceleration we can eliminate this x in favor of α using the known trajectory for this flow, Eq. (16) (here it does seem fair to use the trajectory since we are merely checking a result), and we find

$$\frac{DV_E}{Dt}(x(\alpha, t), t) = -\alpha(1 + 2t)^{-3/2},$$

which is indeed Eq. (42). Thus, the DV_E/Dt given by Eq. (45) is indeed the Eulerian representation of DV_L/Dt , as we had intended. Another way to check and interpret this very important result is noted below.

The operator D/Dt goes by a profusion of different names — the convective derivative, the substantive (or substantial) derivative, the Stokes derivative and the material derivative (our choice) — giving a clue to its great importance. D/Dt is often said to be the time derivative ‘following the flow’. This is appropriate insofar as $\partial\xi/\partial t$ in Eq. (44) is with α held constant and so is the fluid velocity. However, this name could be misleading if it was interpreted to mean parcel tracking in the Lagrangian sense, i.e., for an extended time. Rather, the operator D/Dt gives, entirely in field coordinates (just x in the one-dimensional case), the time rate of change that would be observed by a parcel moving through the point x at the time t ; thus D/Dt follows the flow only in an instantaneous sense. An example is in Fig. 8c; the Lagrangian acceleration and the material time derivative of velocity (evaluated in their respective systems) are equal when they are evaluated at a common position and time, $\alpha = 0.5$, $x = 0.7$ and $t = 0.48$, in the same way that the corresponding Lagrangian and Eulerian velocities are equal at a common position and time, Figs. 8a and b. The material derivative is then the time rate of change equivalent of the FPK and it is also the second key step in the transformation of dynamics from the Lagrangian into the Eulerian coordinate system.²⁹

In the example above we assumed a one-dimensional domain R^1 to simplify as much as possible. The same relations hold in a three-dimensional flow having Cartesian coordinates (x, y, z) and velocity (u, v, w) , the material time derivative represented in an Eulerian system is then

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}, \quad (47)$$

or using more compact vector notation,

$$\boxed{\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla} \quad (48)$$

Notice that the subscript E has been dropped from the velocity vector, since Eulerian should be clear. The gradient operator expanded in Cartesian coordinates is

$$\nabla = \mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z},$$

with \mathbf{e} the unit vector. The advective derivative term $\mathbf{V} \cdot \nabla$ of (48) is, in effect, a scalar that is multiplied onto the variable being differentiated it is sometimes written $(\mathbf{V} \cdot \nabla)$ to emphasize this property.³⁰

²⁹A couple of questions for you. (1) Can you show that the material derivative of parcel position is just the velocity? In this respect, the material derivative could also be said to be more fundamental than is the FPK. (2) Suppose a one-dimensional domain as above, and a fluid variable that has an initial, Lagrangian distribution $b = c\alpha$, with trajectories $\xi = \alpha + Vt$, and V a constant. Given that b is conserved in every material volume, find the equivalent material derivative in the Eulerian system. Interpret the local rate of change of b . (3) Suppose instead that $Db/Dt = a$, with a a constant; how does $b(x, t)$ evolve?

³⁰In the Cartesian coordinate system we find that the D/Dt produces four terms for each component of a vector being differentiated, one term being the partial with respect to time, and three terms arising from the advective rate of change. However, when D/Dt is expanded in cylindrical polar coordinates we get five terms, the additional advective term arising from the θ dependence of the \mathbf{r} and $\boldsymbol{\theta}$ unit vectors (in the Cartesian system the unit vectors are constant). In spherical polar coordinates the D/Dt operator expands to six terms. It is sometimes helpful to represent the advective derivative of velocity by the following vector identity $(\mathbf{V} \cdot \nabla)\mathbf{V} = \frac{1}{2}\nabla(\mathbf{V}^2) + (\nabla \times \mathbf{V}) \times \mathbf{V}$, which is less likely to be misinterpreted. This form is especially useful applied to an ideal fluid, wherein frictional effects vanish and so too may the curl of the velocity, also called the vorticity (Section 6.4).

3.3 Transforming integrals and their time derivatives; the Reynolds Transport Theorem

We have emphasized that the conservation laws of classical physics apply to specific fluid material volumes, Fig. 5, and not to points in space or control volumes that are fixed in space. But we have also indicated (in Section 2.2) that most of the theory of fluid flow is more useful (simpler) when developed in an Eulerian coordinate system rather than the Lagrangian coordinate system in which the conservation laws are first given. Thus there is a need to know the Eulerian equivalent of an integral taken over a moving, material fluid volume. This leads to the third key piece of the transformation of dynamics from Lagrangian to Eulerian form, the Reynolds Transport Theorem, or RTT.

The essential content of the RTT can be understood by considering the integral in R^1 of an intensive fluid property, let's say, b , over a moving (linear) material volume of fluid (Fig. 11), the results being easily extended to R^3 ,

$$B(t) = \int_{\xi_1}^{\xi_2} b(\xi, t) d\xi. \quad (49)$$

$B(t)$ is thus an extensive property.³¹ An extensive property of the sort defined here may follow a conservation law, $\frac{DB}{Dt} = 0$, or if there is a source then, $\frac{DB}{Dt} = -\nabla Q$, (examples will follow). To make the best use of this we need to write the integral Eq. (49) in terms of the field variables, just x in one dimension. The integral itself is easy since at a given time $x_1 = \xi_1$, and $d\xi = dx$, etc., and so

$$B(t) = \int_{x_1}^{x_2} b(x, t) dx. \quad (50)$$

The next step is to write the time derivative (holding α constant) of the integral, DB/Dt , in field coordinates. This is not quite so easy because the limits are the positions of moving parcels, e.g., $x_1 = \xi_1(\alpha_1, t)$, and hence we can not simply move the time derivative inside the integral as it is written here.

A Heuristic Approach: To proceed we will represent the integral by a sum over some large number K of infinitesimal segments, $\delta\xi_k$,

$$\begin{aligned} \xi_2 - \xi_1 &= \sum_K \delta\xi_k, \\ B(t) &= \int_{\xi_1}^{\xi_2} b(\xi, t) d\xi \approx \sum_K b(\xi_k, t) \delta\xi_k. \end{aligned} \quad (51)$$

The number K is fixed while the length of the total interval, $\xi_2 - \xi_1$, may change with time and thus we have to allow that the material segments $\delta\xi_k$ will vary with time. With that understood, the time derivative can be

³¹It is sometimes helpful to make clear whether a given fluid property is 'intensive' or 'extensive'. (1) An intensive property of a fluid is measurable at a point, e.g., density, while an extensive property of a fluid is an integral over a finite volume, e.g., the volume itself or the mass of the volume of fluid. An extensive property is thus probably not directly measurable. (2) Imagine two volumes of fluid, V_1 and V_2 , having densities ρ_1 and ρ_2 . Now suppose that the volumes are added together to make a new volume, V_3 . Aside from from nonlinear effects in the equation of state, the new volume will be $V_3 = V_1 + V_2$, while the new density will be $\rho_3 = (\rho_1 V_1 + \rho_2 V_2)/V_3$. Thus extensive property, the masses, add up when volumes are combined, while the corresponding intensive property, the densities, are a volume-weighted average (again, aside from nonlinearity of the equation of state). (3) Extensive properties such as mass, and internal energy, may be subject to a conservation law, while the corresponding intensive properties, density and temperature, will generally not be. Nevertheless, as we will see in this section, conservation laws for extensive properties will lead to useful differential balance equations for the intensive properties.

taken through the summation operator,

$$\frac{DB}{Dt} \approx \sum_K \left(\frac{Db(\xi_k)}{Dt} \delta\xi_k + b(\xi_k) \frac{D\delta\xi_k}{Dt} \right). \quad (52)$$

The first term on the right hand side is expected, and the second term accounts for a possible change in the material length, $D\delta\xi_k/Dt$. The time derivative of this length is just the velocity difference at the location of the two parcels that mark its endpoints,

$$\frac{D\delta\xi_k}{Dt} = \frac{D\delta x_k}{Dt} = \frac{D}{Dt}(x_{k+1} - x_k) = u_{k+1} - u_k,$$

where $u_k = \frac{Dx_k}{Dt}$ is the fluid velocity at $x_k = \xi_k$, etc. When the length δx is infinitesimal the velocity difference may be written

$$(u_{k+1} - u_k) = \frac{\partial u}{\partial x} \delta x$$

and hence

$$\frac{D\delta x}{Dt} = \frac{\partial u}{\partial x} \delta x.$$

The normalized rate of change of length, $\frac{1}{\delta\xi} \frac{D\delta\xi}{Dt} = \frac{\partial u}{\partial x}$ is the linear deformation rate, something we have seen before (and recall the linear deformation of Sections 1.1 and 2.2) and that will see again in Section 7.4.

Collecting these results into Eq. (52) gives

$$\frac{DB}{Dt} = \frac{D}{Dt} \int_{\xi_1}^{\xi_2} b(\xi, t) d\xi \approx \sum_k \left(\frac{Db(x_k)}{Dt} + b(x_k) \frac{\partial u}{\partial x} \right) \delta x_k \approx \int_{x_1}^{x_2} \left(\frac{Db}{Dt} + b \frac{\partial u}{\partial x} \right) dx \quad (53)$$

We can now state the one-dimensional Reynolds Transport Theorem (or RTT) which relates the time derivative over a material volume to the equivalent field quantities,

$$\boxed{\frac{DB}{Dt} = \frac{D}{Dt} \int_{\xi_1}^{\xi_2} b(\xi, t) d\xi = \int_{x_1}^{x_2} \left(\frac{Db}{Dt} + b \frac{\partial u}{\partial x} \right) dx} \quad (54)$$

where $x_1 = \xi_1$, etc. at the time the transformation is made. Thus the RTT shows us that to transform a time derivative of an integral over a material volume into field coordinates we have to account for the possible time rate of change of the material volume.

A Formal Approach: A second, more formal approach to the RTT may also be helpful (this follows Aris¹⁸ Sec 4.22). The integral Eq. (50) is over moving (time-dependent positions) and hence we can not take the derivative inside the integral as written. However, if the integral is rewritten in terms of the initial positions of the parcels, i.e., α_1 and α_2 , then upon changing variables (Section 9.3) from ξ to $\xi(\alpha, t)$,

$$B(t) = \int_{\xi_1}^{\xi_2} b(\xi, t) d\xi = \int_{\alpha_1}^{\alpha_2} b(\xi(\alpha, t), t) \frac{\partial \xi}{\partial \alpha} d\alpha. \quad (55)$$

Note that we also must transform the integration limits, $\alpha_1 = \xi_1$, etc. The time derivative can be taken inside the second integral since the material coordinates α are held constant during the derivative ($\alpha = \text{const}$ is what is meant by ‘material volume’) and thus

$$\frac{DB(t)}{Dt} = \int_{\alpha_1}^{\alpha_2} \left(\frac{Db(\xi(\alpha, t), t)}{Dt} \frac{\partial \xi}{\partial \alpha} + b(\xi(\alpha, t), t) \frac{D}{Dt} \frac{\partial \xi}{\partial \alpha} \right) d\alpha. \quad (56)$$

The interesting term here is $\frac{D}{Dt} \frac{\partial \xi}{\partial \alpha}$; by interchanging the order of differentiation, and then by using that $u(\alpha) = u(\xi(\alpha, t), t)$,

$$\frac{D}{Dt} \frac{\partial \xi}{\partial \alpha} = \frac{\partial}{\partial \alpha} \frac{D\xi}{Dt} = \frac{\partial u}{\partial \alpha} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial \alpha}. \quad (57)$$

Collecting terms in the integrand yields

$$\frac{DB(t)}{Dt} = \int_{\alpha_1}^{\alpha_2} \left(\frac{Db(\xi(\alpha, t), t)}{Dt} + b(\xi(\alpha, t), t) \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial \alpha} \right) d\alpha. \quad (58)$$

This can be transformed to the field coordinates by going from right to left in Eq. (55),

$$\frac{DB(t)}{Dt} = \int_{x_1}^{x_2} \left(\frac{Db(x, t)}{Dt} + b(x, t) \frac{\partial u}{\partial x} \right) dx, \quad (59)$$

which is the same as Eq. (54).

An application of the RTT: To exercise and verify the RRT in this one-dimensional form we will show that

$$\frac{DB(t)}{Dt} = \frac{D}{Dt} \int_{\alpha_1}^{\alpha_2} b(\xi(\alpha, t), t) \frac{\partial \xi}{\partial \alpha} d\alpha = \int_{x_1}^{x_2} \left(\frac{Db(x, t)}{Dt} + b(x, t) \frac{\partial u}{\partial x} \right) dx, \quad (60)$$

where for b we will use the density of the one-dimensional flow introduced in Sections 2 and 3. The middle, Lagrangian term requires $\rho(\alpha, t) = (\rho_c + \Gamma\alpha)(1 + 2t)^{-1/2}$ and $\xi(\alpha, t) = \alpha(1 + 2t)^{1/2}$, and from this we easily find that $\frac{\partial \xi}{\partial \alpha} = (1 + 2t)^{1/2}$ and hence the integrand

$$\rho(\alpha, t) \frac{\partial \xi}{\partial \alpha} = \rho_0(\alpha),$$

is a function of α alone. The time derivative of the integral is thus zero on any interval since α is held constant in this time derivative. Thus the Lagrangian term vanishes on all intervals, consistent with mass conservation of a material volume.

To calculate the right hand, Eulerian integral we need the Eulerian density and velocity, $\rho(x, t) = \rho_c(1 + 2t)^{1/2} + \Gamma x(1 + 2t)^{-1}$, and, $u(x, t) = x(1 + 2t)^{-1}$. From these the material derivative of density and the divergence of the velocity are just $\frac{D\rho(x, t)}{Dt} = -\rho_c(1 + 2t)^{-3/2} - \Gamma x(1 + 2t)^{-2}$, and, $\frac{\partial u}{\partial x} = (1 + 2t)^{-1}$. Substitution into the integrand of the Eulerian term then shows that

$$\frac{D\rho(x, t)}{Dt} + \rho(x, t) \frac{\partial u}{\partial x} = 0, \quad (61)$$

and hence the integral vanishes on every interval, consistent with the Lagrangian representation of this integral noted just above. This last result has an important interpretation that we will discuss beginning in the next section.³²

Generalization of the RTT to three-dimensions: If instead of a one-dimensional ‘volume’ used in the examples above we transform a true three-dimensional volume, $dVol = dx dy dz$ in Cartesian coordinates, then it is easy to show that the time derivative of the differential volume is

$$\boxed{\frac{D}{Dt} dVol = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) dVol = \nabla \cdot \mathbf{V} dVol} \quad (62)$$

³²We have noted that Lagrangian and Eulerian functions can not, in general, be said to be equal, so how can Eq. 60 possibly be correct?

$\nabla \cdot \mathbf{V}$ is the divergence of the fluid velocity in three dimensions and is the volumetric (or normalized) time rate change of the differential volume, i.e., $\nabla \cdot \mathbf{V} = dVol^{-1}(d/dt)(dVol)$. The RTT applied to a scalar b in three dimensions is then a straightforward generalization of our one-dimensional form, Eq. (54),

$$\boxed{\frac{D}{Dt} \iiint_{material} b dVol = \iiint_{field} \left(\frac{Db}{Dt} + b \nabla \cdot \mathbf{V} \right) dVol} \quad (63)$$

The RTT is an exact, kinematic relationship that holds for any intensive fluid property, e.g., b could be mass density, ρ , momentum density, $\rho \mathbf{V}$, or any other. The RTT serves one crucial purpose — to show how fluid flow effects the time rate change of the corresponding extensive quantity, i.e., the B in our notation, and moreover, to show the result in field coordinates. The RTT is thus the last piece needed to give a practical (widely useful) answer to the question we posed in the opening paragraph of Section 1, viz, how can we apply the conservation principles of classical physics to a fluid flow? We will find the consequences for conservation of mass and momentum below. Physics, as opposed to kinematics, enters the development when we specify how or whether the corresponding extensive quantity, e.g., mass or momentum, is conserved under fluid flow and the form of external sources, if any.

3.4 The Eulerian equations of motion

3.4.1 Mass conservation represented in field coordinates

An important application of the RTT is to the mass of a moving, three-dimensional volume of fluid,

$$M = \iiint_{material} \rho dVol, \quad (64)$$

which is thus an extensive property defined on a specific material volume. There is no source or sink for mass in the classical physics that we presume holds, and thus the physics of mass conservation is that the mass M of a specific material volume must remain exactly constant,

$$\frac{DM}{Dt} = 0, \quad (65)$$

for all flow conditions. This is the three-dimensional version of mass conservation for a material volume we considered in Section 2.2.1 and sketched in Figs. (12) and (11). It is important to appreciate that we could make no such general assertion for the intensive property, mass (volume) density, ρ , nor could we assert this conservation property for a control volume that was fixed in space.

The mass conservation Eq. (65) has a clear and precise physical meaning, but it is not in and of itself directly useful as a means to predict mass or density in most models of fluid flow. We can write it in a form that will be by application of the Reynolds Transport Theorem, Eq. (63) and use of Eq. (64),

$$\frac{DM}{Dt} = \frac{D}{Dt} \iiint_{material} \rho dVol = \iiint_{field} \left(\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{V} \right) dVol = 0. \quad (66)$$

If this integral relation holds at all times and for all positions within a domain, and if the integrand is smooth (no discontinuities), then the integrand must vanish at all times and positions in that domain³³ yielding the differential form of the mass conservation relation,

$$\boxed{\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{V} = 0} \quad (67)$$

The meaning of Eq. (67) is also clear: say the material derivative of density is negative, $\frac{D\rho}{Dt} < 0$ at a point in space, then we can conclude that there must also be a divergence of the fluid velocity, $\nabla \cdot \mathbf{V} > 0$, at that point. Thus the fluid parcel that is instantaneously at that point will have decreasing density and an increasing volume. This is a kinematic relationship that holds regardless of what we might say was the cause of the density change, i.e., whether due to a pressure variation or heat exchange with the surroundings, or even a phase change of the fluid material. Thus Eq. (67) is not by itself sufficient to predict or understand *why* density might change, which comes instead from a thermodynamic state equation for the fluid, as we will discuss further below. However, it is worth noting that Eq. (67) does *not* hold if the fluid volume exchanges material with its surroundings, e.g., salt in the case of sea water or water vapor if air, in which case it would not be a constant material volume.

The mass conservation relation may be written in two other forms that emphasize the local time rate of change of density. By expanding the material derivative we can see the separate effects of advection and of velocity divergence,

$$\boxed{\frac{\partial \rho}{\partial t} + \mathbf{V} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{V} = 0} \quad (68)$$

or in Cartesian components,

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} + \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0.$$

The meaning of Eq. (68) is that if the density changes at a fixed point, $\partial \rho / \partial t < 0$, say, then the sum of the advective term $\mathbf{V} \cdot \nabla \rho$ and the divergence term $\rho \nabla \cdot \mathbf{V}$ must have been negative.

The one-dimensional flow and density of Section 2.2 make a useful example; recall that the trajectory and the Lagrangian density were, repeating Eqs. (16) and (27),

$$\xi(\alpha, t) = \alpha(1 + 2t)^{1/2} \quad \text{and} \quad \rho(\alpha, t) = \frac{\rho_c + \Gamma \alpha}{(1 + 2t)^{1/2}}$$

and given these it is easiest to calculate the Eulerian density by eliminating α in favor of x ,

$$\rho(x, t) = \frac{\rho_c}{(1 + 2t)^{1/2}} + \frac{\Gamma x}{(1 + 2t)}, \quad (69)$$

³³The Dubois-Reymond lemma: it is given that an integral of the sort Eq. (66) but here in R^1 , $\int_{x_1}^{x_2} \sigma(x) dx = 0$, vanishes for any x_1, x_2 and that σ is smooth. If we now suppose that $\sigma(x_a) > 0$ this leads to a contradiction. Given that σ is smooth, then there will be some neighborhood around x_a where $\sigma(x) > 0$. Choose x_1 and x_2 to lie within this neighborhood, and apply the mean value theorem to the integral to find that $\sigma(\bar{x})(x_2 - x_1) \geq 0$ since $\sigma(\bar{x}) > 0$. This contradicts what we know about this integral, and the same holds if $\sigma(\bar{x}) < 0$. Hence the only possibility is that $\sigma(x)$ must be zero at every point. In other words, the only smooth function whose integral is zero on every interval is the zero function.

that is plotted in Fig. (10, right). This is clever, but bypasses the Eulerian density equation that we need to examine,

$$\frac{\partial \rho}{\partial t} = -u \frac{\partial \rho}{\partial x} - \rho \frac{\partial u}{\partial x}.$$

In the present case, the local rate of change of density at a fixed point, the left hand side,

$$\frac{\partial \rho}{\partial t} = \frac{\rho_c}{(1+2t)^{3/2}} - \frac{2\Gamma x}{(1+2t)^2}$$

is given by the sum of an advection term, which is negative,

$$-u \frac{\partial \rho}{\partial x} = -\frac{\Gamma x}{(1+2t)^2},$$

and a divergence term that is also negative,

$$-\rho \frac{\partial u}{\partial x} = -\left(\frac{\rho_c}{(1+2t)^{3/2}} + \frac{\Gamma x}{(1+2t)^2}\right).$$

The divergence term is negative because this flow is divergent, $\frac{\partial u}{\partial x} > 0$, and hence fluid volumes are stretched or elongated by the flow. This divergence process is in common with the Lagrangian density equation, (25), where it appears as the linear deformation, i.e., as the time-integrated divergence. In fact, this divergence term is the only other term in the Lagrangian density equation besides the rate of change following a parcel. The advection term is negative because in this case the density increases toward positive x and because $u > 0$. Thus the effect of translating or 'advecting' this density profile is also to cause density observed at a fixed point to decrease with time.

It is very important understand that the density observed at a fixed point can change due solely to this advection process even while the density observed on moving parcels may remain constant. The only thing required in one dimension is a spatial variation of density combined with a nonzero velocity. In a more realistic multi-dimensional space, the condition is that $\nabla \cdot \rho$ should be nonzero. In a later Section 6.1 this kind of balance between local rate of change and advection will be referred to as a 'frozen field' balance. Thus Eulerian observations of density are effected not only by changes due to stretching, which are significant also from the fluid or Lagrangian perspective, but also by advection, which is irrelevant from the fluid perspective. Very often the advective term is large and significantly complicates the inference of what one might regard as the important forcing terms in an Eulerian budget equation.³⁴

Incompressible flow: for many purposes, fluids may appear to be infinitely stiff The velocity divergence term is clearly necessary in the mass conservation equation, and plays a prominent role in our little one-dimensional flow. We should take note here that for most phenomenon of the atmosphere or ocean and for many engineering flows the fluid velocity associated with the velocity divergence is very small compared

³⁴We verified the mass conservation equation using our one-dimensional flow and density in the previous section, Eqs. (60) and (61). However, it would still be useful to go back and make a sketch of the density and velocity profiles (assume $\Gamma > 0$ and $x > 0$) and explain graphically why both the divergence and the advection terms are negative. What if $\Gamma = 0$? How does that case compare with the Lagrangian density? At a given time, how can you interpret the area under the curve of the Eulerian density, $\rho(x)$, defined by the three parcels plotted in Fig. (10, right)? Finally, can you show that the Lagrangian density equation (27) can be put into a form exactly like Eq. (67) by taking the time derivative and then eliminating the initial density ρ_0 in favor of ρ ? (Hint, recall the steps leading to Eq. 57).

to other fluid velocities and may often be ignored with no appreciable error.³⁵ Under this so-called incompressibility assumption, the velocity is assumed to follow

$$\nabla \cdot \mathbf{V} = 0, \quad (70)$$

which in effect says that the volume (rather than the mass) of fluid parcels is constant in time. In one-dimension this requires the velocity to be spatially uniform, which is not an interesting case (and so our one-dimensional flow is highly and unrealistically divergent). In two dimensions or three dimensions this need not be the case. For example, in two dimensions, a gentle linear deformation in say the x-direction will be almost exactly compensated by a linear deformation of the opposite sign in the y-direction, with the result that the area of a two-dimensional material volume will remain nearly constant and so the fluid will appear to be very nearly incompressible.

Even with the incompressibility assumption in place, it is not inconsistent to have a model in which density may nevertheless change, and indeed density changes may be of primary importance in causing the flow of an incompressible fluid. Under the incompressibility assumption, density may be computed from an equation of state given the pressure, temperature, salinity, etc., with no reference made to the divergence of the fluid velocity. That we can ignore fluid divergence is a useful mathematical approximation that is contingent upon the physical phenomenon under consideration. One important class of phenomena, acoustic waves, owe their entire existence to velocity divergence and associated pressure changes. Once a few more pieces are in place, we will be able to appreciate that the incompressibility assumption for the velocity field, Eq. (70), can be made with negligible error provided that the fluid velocity is much less than the speed of sound, which holds well for most natural flows of the atmosphere and ocean.

3.4.2 The flux form of the Eulerian equations; the effect of fluid flow on properties at a fixed position

A third way to write the Eulerian density equation is to collect terms under the gradient operator,

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0} \quad (71)$$

An Eulerian budget equation in this precise form is said to be in ‘flux’ or ‘conservation’ form and has some important properties. To make it clear that these properties follow from the conservation form of the equation it is helpful to consider the same equation for an arbitrary, intensive scalar b ,

$$\frac{\partial b}{\partial t} + \nabla \cdot \mathbf{b} = 0 \quad (72)$$

which could be mass per unit volume, ρ , or momentum per unit volume, $\rho \mathbf{V}$. Thus $b \times Volume = B$ is an extensive property, e.g., mass or momentum. The vector \mathbf{b} is the flux of b and could be due to transport by the moving fluid, e.g., $\mathbf{b} = b \mathbf{V}$ obtains for any fluid property, which is the reason that fluid flow is of first importance in so many applications. The flux could also arise from molecular diffusion, $\mathbf{b} = -K \nabla b$, if a Fourier diffusion law is appropriate. In either event, \mathbf{b} is a vector with dimensions $b \doteq b \text{ length } time^{-1}$ so that $b \times area \doteq B \text{ time}^{-1}$. For the moment it will be assumed that \mathbf{b} depends upon b , so that $b = 0$ over some neighborhood implies that $\mathbf{b} = 0$ as well.

³⁵A question for you: Suppose that a volume of air, 10 km on a side by 1 km thick, is heated at constant pressure by 5 C in a period of an hour (as might occur in a vigorous sea breeze circulation). What is the magnitude of the associated (divergent) velocity, assuming that it appears on one vertical face only of the volume?

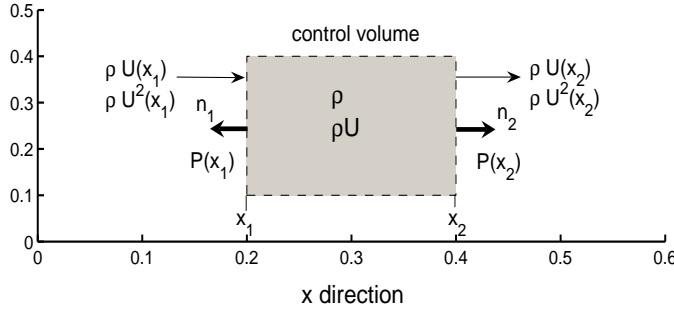


Figure 13: A schematic of a control volume (fixed in space) set in an ideal fluid in order to evaluate the Eulerian budgets of mass and momentum. The face of the control volume that is normal to the x -direction has an area A (not shown). In this one-dimensional Eulerian system the independent coordinates are the x -position, and the time, t . The dependent variables are the fluid density, $\rho(x, t)$, the momentum, $\rho u(x, t)$, and the pressure, $P(x, t)$. The one-dimensional fluxes of mass, ρu , and momentum, ρu^2 , are implied by the arrows coming into or leaving the control volume.

Now imagine a volume integral taken over a fixed portion of a domain, i.e., a control volume (Fig. 14). The total amount of b within the control volume is

$$B = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} b \, dx dy dz.$$

The same volume integral over the left and right sides of Eq. (72) gives

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} \frac{\partial b}{\partial t} dx dy dz = - \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} \nabla \cdot \mathbf{b} \, dx dy dz. \quad (73)$$

The time derivative can be moved outside the volume integral over fixed limits, x_1, x_2 , etc., of the control volume and Gauss' divergence theorem used on the right side,

$$\frac{\partial}{\partial t} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} b \, dx dy dz = \frac{\partial B}{\partial t} = - \oint \mathbf{b} \cdot \mathbf{n} \, da. \quad (74)$$

Thus B can change only because of a flux across the surface of the control volume, Figs. (13) and (14). If it happens that the flux is uniform in space and so is non-divergent, $\nabla \cdot \mathbf{b} = 0$, Eq. (73), then B will remain constant in time. Thus flux divergence, even more than the flux itself, is the quantity of interest for Eulerian budgets.

Now imagine that we let the control volume expand to infinity. We can assume without loss of generality that the variable b and the flux \mathbf{b} are vanishing at infinity, i.e., that the flow is bounded in space, and hence the flux on the surfaces vanishes as well. Over this control volume

$$\oint \mathbf{b} \cdot \mathbf{n} \, da = 0 = \frac{\partial B}{\partial t}$$

and hence B is constant in time; the total amount of b can not change. This holds even though the flux \mathbf{b} may act to redistribute b within the domain so that at a given point in space b could be highly time-dependent. Nevertheless, the conservation form of the governing equation for b assures that the total amount of b must remain constant. The same thing would result if the flux \mathbf{b} vanished on the boundary of a finite domain, e.g., if the flux was due to fluid motion only and the domain boundary was an impermeable surface.

Clearly b could be the mass density, ρ , and the flux \mathbf{b} could be the mass density flux due to the fluid velocity, $\rho \mathbf{V}$, and B could be the net mass, M . In that case we would expect these conservation properties on

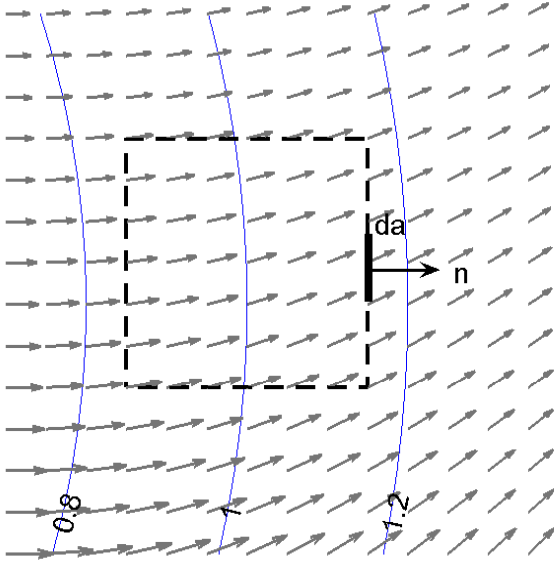


Figure 14: A control volume is fixed in space and has imaginary sides (dashed lines) and a two-dimensional slice through a velocity field, \mathbf{V} , and isolines of a fluid intensive property, b . The flux of b due to transport by the fluid velocity alone is $\mathbf{b} = b\mathbf{V}$; the flux of b through a differential surface element of the control volume is then $\mathbf{b} \cdot \mathbf{n} da$, where \mathbf{n} is the outward unit vector normal to the surface. A question for you: given what you can tell from this figure alone, how will the total amount of b in this control volume, B , change with time?

physical grounds alone, there being no source or sink for mass in the classical physics that we have presumed. But we will soon see that the same integral conservation relation holds for a few other important physical quantities, notably net momentum and total energy, provided the appropriate momentum and energy balances can be written in the conservation form of Eq. (72), i.e., with no external or source terms.

3.4.3 Momentum conservation represented in field coordinates

A second important application of the Reynolds Transport Theorem arises on consideration of momentum balance. The momentum of a moving, three-dimensional volume of fluid can be written

$$N = \iiint_{\text{material}} \rho \mathbf{V} dVol. \quad (75)$$

Because this is a material volume we can assert Newton's Second Law, that the momentum of this specific volume of fluid can change only if there is a net force,

$$\frac{DN}{Dt} = \iiint_{\text{material}} \mathbf{F} dVol, \quad (76)$$

where \mathbf{F} is the sum of all forces acting on the fluid. These forces could include inertial forces, such as gravitational mass attraction or the Coriolis force that act throughout the volume, and stresses (Sections 1.1 and 2.2) that act on the surface of the volume. By means of the Reynolds Transport Theorem and the mass conservation relation we can write the left side of Eq. (76) in field coordinates as

$$\frac{DN}{Dt} = \frac{D}{Dt} \iiint_{\text{material}} \rho \mathbf{V} dVol = \iiint_{\text{field}} \rho \frac{D\mathbf{V}}{Dt} dVol. \quad (77)$$

A term that might have been expected, $\frac{D\rho}{Dt}V$, has dropped out by application of the mass conservation requirement. Thus the momentum of a fluid parcel (or marked fluid volume) can change only because of a change of the velocity.³⁶ The right side of Eq. (76) is a volume integral over field coordinates that is performed over the volume occupied instantaneously by the moving fluid,

$$\iiint_{\text{material}} \mathbf{F} dVol = \iiint_{\text{field}} \mathbf{F} dVol, \quad (78)$$

and thus from Eqs. (77) and (78),

$$\iiint_{\text{field}} \rho \frac{D\mathbf{V}}{Dt} dVol = \iiint_{\text{field}} \mathbf{F} dVol. \quad (79)$$

The volume considered here is arbitrary, i.e., the integral relation holds for every volume, and so the differential form of the momentum balance for a fluid continuum is

$$\frac{D\mathbf{V}}{Dt} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = \mathbf{F}/\rho \quad (80)$$

The volume of the integrals in Eq. (79) is, as we noted before, the volume that is instantaneously coincident with the material volume indicated in Eq. (75). For the purpose of finding the form of the stress we can imagine that this is a fixed control volume. Thus we will say that the stress acts upon the control volume, but in full we mean that the stress acts upon the fluid that is instantaneously contained within the control volume. This focus upon a fixed and imaginary control volume vs a moving material volume is at the heart of the difference between the Eulerian and the Lagrangian systems; the latter point of view is more in tune with physical intuition and yet the former (Eulerian) view leads to what are generally more useful governing equations, especially in three-dimensions.

3.4.4 Fluid mechanics requires a stress tensor (which is not as difficult as it first seems)

Specification of the force \mathbf{F} proceeds in two steps; the first is a kind of mathematical bookkeeping in which we learn to manipulate the stress tensor, essentially a 3x3 matrix of stress components that may be used to evaluate the stress vector acting on any face of a control volume. The second step is to specify the stress components. Forces within a fluid are generated in a way that depends upon the physical properties of the fluid and so this involves something more than kinematics and bookkeeping.

As we noted in Section 1.1, the specification of the stress on a given surface requires specification of a stress tensor, a 3x3 object, because the surface and the stress vector are both 3-dimensional (Fig. 1). That is, the stress vector is defined by three components, and the surface upon which the stress acts is defined by the unit normal vector, and thus another three components. Thus a full representation of the stress within a fluid requires nine components at each point in space. It is often helpful to think of these grouped into a single entity, a 3x3 object called the stress tensor, \mathbb{S} ,

$$\mathbb{S} = \begin{pmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{pmatrix}, \quad (81)$$

³⁶You should fill in the steps of Eq. (77) to verify that this is true whenever the extensive property, say H , is the volume integral of ρh , where ρ is mass density and h is the corresponding intensive property.

that we can add and multiply with the same rules that apply to matrices. By our convention (Section 1.1), the first subscript on a stress component indicates the direction of the stress (the projection of the stress vector on to the e_x unit vectors), and the second subscript indicates the orientation of the surface via the direction of the outward unit normal vector of the surface. We will see later that three pairs of the stress tensor components are equal, $S_{yx} = S_{xy}$, $S_{xz} = S_{zx}$ and $S_{zy} = S_{yz}$, at least for all ordinary fluids such as air and water, leaving six unique components and a stress tensor that is symmetric, $\mathbb{S} = \mathbb{S}'$.

A handy property of the stress tensor is that the matrix product $\mathbb{S} \cdot \mathbf{n}$ picks out the components of the stress vector acting on the face whose unit normal vector is \mathbf{n} . For example, the unit normal for the upper face, \mathbf{n}_u , has components (0; 0; 1) (the semicolon delimiters indicate that these elements are arranged in a column vector), and so $\mathbb{S} \cdot \mathbf{n}_u = (S_{xz}; S_{yz}; S_{zz})$, is a three element (column vector) of the components of the stress vector acting upon the upper face of the parcel. The differential force associated with this stress acting upon a differential area of the surface, da , is then

$$d\mathbf{F} = \mathbb{S} \cdot \mathbf{n} da, \quad (82)$$

where the stress tensor is evaluated at the position of the differential area. The unit normal of the lower (downward looking) face of the control volume of Fig. (1) has components (0; 0; -1) and so the stress vector on that face has components $\mathbb{S} \cdot \mathbf{n}_l = -(S_{xz}; S_{yz}; S_{zz})$, where these components are evaluated at the position of the lower face. If it happened that the stress components were the same on the upper and lower faces, then the stress exerted by the overlying fluid on the upper face of the volume would be equal in direction and magnitude to the stress exerted by the fluid within the volume on the underlying fluid, in which case the net stress on the control volume, $\mathbb{S} \cdot \mathbf{n}_u + \mathbb{S} \cdot \mathbf{n}_l = \mathbb{S} \cdot (\mathbf{n}_u + \mathbf{n}_l) = 0$, would vanish.

Given Eq. (82) for the differential force, the net force on the fluid within the control volume may be computed by summing over the entire surface,

$$\mathbf{F} = \oint \mathbb{S} \cdot \mathbf{n} da, \quad (83)$$

where da is the differential area associated with the unit normal \mathbf{n} . Both \mathbb{S} and \mathbf{n} will, in general, vary over the surface. It is often desirable to express the net force as an integral over the volume of the object, rather than its surface, and this transformation is made by application of the tensor form of the divergence theorem,³⁷

$$\mathbf{F} = \oint \mathbb{S} \cdot \mathbf{n} da = \iiint \nabla \cdot \mathbb{S}^t dv. \quad (84)$$

Stress components in an ideal fluid: The stress tensor has to be defined at every point in space and thus a tensor stress field is coincident with the velocity field. At first this seems a bit daunting; we had the goal of solving for the pressure and the velocity fields, and now it seems that we have to solve for a tensor field as well. However, as will see shortly, the stress tensor depends upon the velocity and the pressure in a straightforward way, so that nothing beyond the velocity and pressure fields will be required.

³⁷This is probably not familiar, so I'll try to make it plausible. Each row of the stress tensor may be considered to be a vector, S_{1j}, S_{2j}, S_{3j} , so that there are three vectors whose components are numbered by the index j . Each of these vectors can be treated by way of the familiar divergence theorem for vectors, $\oint S_{ij} n_j da = \oint \frac{\partial S_{ij}}{\partial x_j} dv$ with $i = 1, 2$ or 3 . Thus each row (or vector) of the tensor goes through separately. When we represent the divergence operator using symbolic notation, with ∇ a row vector operator that will be multiplied on the left of \mathbb{S} , then to operate on rows (each vector separately) we must operate on the transpose of \mathbb{S} , and thus \mathbb{S}^t arises in Eq. (84). Later it will turn out that $\mathbb{S}^t = \mathbb{S}$.

We noted already in Section 1.1 that the pressure-induced stress vector is just $-P\mathbf{n}$. Hence the stress tensor for a fluid that sustains only pressure forces, often called an ideal fluid or an Euler fluid, is simply

$$\mathbb{P} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix}, \quad (85)$$

an isotropic tensor (the same in all directions). The net pressure force is then

$$\boxed{\mathbf{F}_{pressure} = \oint\!\!\!\oint \mathbb{P} \cdot \mathbf{n} \, da = \iiint \nabla \mathbb{P} \, dv = - \iiint \nabla P \, dv} \quad (86)$$

where the divergence theorem was applied to convert the surface integral into a volume integral and the last step holds only for isotropic tensors. The pressure force per unit volume is then

$$\mathbf{F}_{pressure} / Vol = -\nabla P = -\frac{\partial P}{\partial x} \mathbf{e}_x - \frac{\partial P}{\partial y} \mathbf{e}_y - \frac{\partial P}{\partial z} \mathbf{e}_z, \quad (87)$$

a result that we will use repeatedly.

Stress components in a viscous fluid: In Section 1.1 we claimed that the viscous shear stress in a simple shear flow in which the u component of velocity varied in the z -direction only (Figure 4) was just $S_{xz} = \nu \frac{\partial u}{\partial z}$. There is nothing special about the z direction and if the u component of velocity varied in the y -direction, then there should arise a viscous stress in the x -direction that is exerted on the y -face of the parcel in exactly the same way. That viscous shear stress component would then be labelled

$$\tau_{xy} = \nu \frac{\partial u}{\partial y},$$

where we have switched from S , indicating a stress generally, to τ to indicate a viscous stress in particular (you should make a sketch that shows this kind of shear flow and check whether the sense of the stress is given correctly by this equation). An x -directed stress acting upon the x -face might be different, however, since this would involve a linear deformation rate rather than a shear deformation rate. For some fluids it is found experimentally that the viscosity for a linear deformation rate is not equal to the viscosity associated with shearing deformation, in much the same way that the bulk modulus of a solid is generally not equal to the shear modulus (Table 1). While acknowledging that this is plausible, we will nevertheless presume that there is only one viscosity, ν . Thus a linear deformation rate will produce or require a normal viscous stress,

$$\tau_{xx} = \nu \frac{\partial u}{\partial x}.$$

In most cases this normal viscous stress will be very much less than the normal stress associated with pressure.³⁸ With these three examples in hand, we are ready to write down the viscous stress tensor for Newtonian fluids in which the stress and rate of deformation are related linearly (Eq. 4),

$$\boxed{\text{viscous stress for a Newtonian fluid: } \mathbb{S} = \nu \mathbb{G}} \quad (88)$$

³⁸A viscous normal stress is somewhat harder to envision than is a viscous shear stress. But imagine a fluid column that is falling under the influence of gravity while restrained by a normal viscous (tensile) stress associated with the linear deformation rate of the elongating fluid column, e.g., the pitch drop experiment, again! (footnote 3). We are making the implicit assumption that the average of the three normal viscous stresses is much less than the pressure, and hence that viscous stresses and pressure are decoupled. This is exact if the fluid is incompressible, and is a very good (accurate) assumption for geophysical flows.

where

$$\mathbb{G} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix} \quad (89)$$

The tensor \mathbb{G} , called appropriately enough the velocity gradient tensor, is of fundamental importance and arises again on two separate occasions later in this essay.

To find the force per unit volume we can apply the tensor equivalent of the divergence theorem, Eq. (84),

$$\mathbf{F}_{viscous} = \oint \mathbb{S} \cdot \mathbf{n} \, da = \iiint \nabla \cdot \mathbb{S}' \, dv. \quad (90)$$

The force per unit volume, written in vector and component form is then

$$\mathbf{F}_{viscous}/Vol = \nu \nabla^2 \mathbf{V} = \nu \begin{pmatrix} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \\ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \end{pmatrix}. \quad (91)$$

(Note that ∇^2 operating on a vector yields another vector, whose components are the Laplacian of the components of the original vector.) In all of the above it has been assumed that the viscosity is a constant.

The expression (91) for the viscous force per unit volume may at first look formidable because of the second derivatives and the large number of terms involved. But notice that it is linear and does not couple the components together, i.e., in the x -component equation there appears only the x -component of velocity, albeit differentiated with respect to all three spatial coordinates. The viscous force per unit volume may be familiar to you as the diffusion term of the elementary heat diffusion equation, and in fact momentum components are diffused through a fluid in laminar flow just the way that thermal energy is diffused through a solid.³⁹ Diffusion, or viscosity in the case of fluid momentum, acts to smooth out lumps and bumps in the velocity profile (in any direction) with the rate given by the viscosity times the Laplacian.⁴⁰

³⁹It bears emphasis that we considering molecular diffusion in a laminar flow in which Eq. (9) applies. Turbulent motion may cause mixing in a fluid that is often parameterized as a diffusion process, Fig. (4), in which case the equivalent viscosity (stress/shear) is likely to be a complex function of the flow, and not just a physical property of the fluid.

⁴⁰A problem for you: after deriving the vector form of the pressure gradient force per unit volume we went back and derived the z -component of the pressure force without the use of vector or tensor notation. You should do the same for the viscous force per unit volume in the case shown in Fig. 4, a flow in the x -direction only, and with shear in the z -direction only. Find the viscous force on the upper and lower faces of a parcel, then the sum, and allow the dimensions to shrink to infinitesimal. Locate the resulting term in the full 3-dimensional equation, (91). Suppose that the flow has reached a steady state, as in Fig. 4. What is the profile of viscous stress throughout the fluid? Now imagine that the stress has just been imposed at the surface and the flow is developing. Qualitatively, what is the stress profile in this transient case? The lower boundary condition in Fig. (4) is presumed to be no-slip, so that $u(z=0) = 0$, which is appropriate for real fluids having a finite viscosity. What is the equivalent boundary condition for a heat diffusion problem? What are the corresponding, plausible surface boundary conditions?

Navier-Stokes equations: Given this specification of pressure and viscous forces,

$$\mathbf{F} = -\nabla P + \nu \nabla^2 \mathbf{V},$$

the Eulerian form of the momentum balance equation is then, including the body force due to gravity,

$$\boxed{\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} = \frac{-1}{\rho} \nabla P + \frac{\nu}{\rho} \nabla^2 \mathbf{V} - g \mathbf{e}_z} \quad (92)$$

If we regard the density as variable, then we can not write the velocity of the Euler fluid in flux form directly, we have to use instead the momentum density, i.e., $\rho \mathbf{V}$. By adding the mass conservation equation to the momentum equation and minor rearrangement (that you should be sure to verify) we can write that

$$\boxed{\frac{\partial}{\partial t} \begin{pmatrix} \rho u \\ \rho v \\ \rho w \end{pmatrix} + \frac{\partial}{\partial x} \begin{pmatrix} \rho u^2 + p - \nu \frac{\partial u}{\partial x} \\ \rho uv - \nu \frac{\partial v}{\partial x} \\ \rho uw - \nu \frac{\partial w}{\partial x} \end{pmatrix} + \frac{\partial}{\partial y} \begin{pmatrix} \rho vu - \nu \frac{\partial u}{\partial y} \\ \rho v^2 + p - \nu \frac{\partial v}{\partial y} \\ \rho vw - \nu \frac{\partial w}{\partial y} \end{pmatrix} + \frac{\partial}{\partial z} \begin{pmatrix} \rho wu - \nu \frac{\partial u}{\partial z} \\ \rho wv - \nu \frac{\partial v}{\partial z} \\ \rho w^2 + p - \nu \frac{\partial w}{\partial z} + \Phi \end{pmatrix} = 0} \quad (93)$$

where the gravitational potential $\Phi = \rho g z$.⁴¹

There is a lot of symmetry in the momentum equations, and they can be written in a compact form by introducing momentum flux tensors for the advection terms and gravity, added to the frictional and pressure stress tensors, \mathbb{T} and \mathbb{P} (Section 2.3). The momentum flux tensor

$$\mathbb{A} = \rho \begin{pmatrix} uu & uv & uw \\ vu & vv & vw \\ wu & wv & ww \end{pmatrix},$$

can be written as the direct vector product

$$\mathbb{A} = \mathbf{V} \mathbf{V}^t \quad (94)$$

where \mathbf{V} is a 3x1 column vector and the transpose \mathbf{V}^t is a 1x3 row vector; hence \mathbb{A} is 3x3. The gravitational tensor

$$\mathbb{Y} = \rho g z \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

⁴¹Most arbitrary quantities are *not* subject to a strict conservation law. For example, in a two particle collision, the kinetic energy will be conserved only in the special case that the collision is 'elastic', so that no energy is lost to deformation, acoustic waves, etc. On the other hand, if the particles stick together after the collision, then the kinetic energy will decrease by an amount that depends upon the particular conditions of mass and initial velocity. Thus the kinetic energy and higher moments of the velocity are not conserved in most collisions or during mixing events in a fluid. More often a conservation law will not obtain because of the presence some external source, e.g., gravity, that does not vanish with the fluid velocity and hence the global integral need not be conserved. In that case we should probably call the governing equation the momentum 'balance' or 'budget' rather than 'conservation', though this distinction is often ignored. Two questions for you: Can you show the relationship between the RTT, and the conservation form of the differential balance? Discuss the case Eq. (93) in which a gravitational potential is present, and specifically, does the conservation property hold in that case?

The tensor form of the Eulerian momentum balance is then

$$\boxed{\frac{\partial(\rho V)}{\partial t} + \nabla \cdot (\mathbb{A} + \mathbb{P} - \mathbb{T} + \mathbb{Y}) = 0} \quad (95)$$

where ∇ is a three element row vector, $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$. Admittedly, there is little gain in this for the pressure and gravitational terms. But having written the momentum balance equation in tensor form we now can assert two important properties: 1) The tensor equation (95) is valid in any (Cartesian) coordinate system, assuming that it is correct in one coordinate system, and 2) The components of the tensors transform under a rotation of the coordinate axes as

$$\mathbb{A}' = \mathbf{V}'(\mathbf{V}')^t = \mathbb{R} \mathbb{A} \mathbb{R}^t, \quad (96)$$

where \mathbb{R} is the rotation tensor.⁴²

An application of the flux form of the momentum equation to an oceanic flow: The flux form of the momentum equations is often utilized as the means for finding the differential budget equations for instances of unusual geometry and/or flow conditions. We will consider a very simple example; suppose that we aim to keep account of the x -component of momentum within a control volume of x -length l , and width and height w and h , and suppose too that y and z variations of the fluxes can be ignored (Fig. 13). The x -component of the flux of ρu is then from Eq. (93),

$$b = \rho u^2 + p - v(\partial^2 u / \partial^2 x), \quad (97)$$

and the integral momentum budget for this fixed control volume is

$$\begin{aligned} lwh \frac{\partial \rho u}{\partial t} &= wh((b(x=l) - b(x=0))) \\ &= wh((- \rho u^2 - p + v \frac{\partial u}{\partial x})|_{x=l} - (- \rho u^2 - p + v \frac{\partial u}{\partial x})|_{x=0}). \end{aligned} \quad (98)$$

Dividing through by the volume, lwh , taking the limit that l becomes small, application of mass conservation, and *voilà*, out pops the differential form of the momentum balance,

$$\frac{\partial u}{\partial t} = -u \frac{\partial u}{\partial x} - \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{v}{\rho} \frac{\partial^2 u}{\partial x^2}. \quad (99)$$

The only slightly disagreeable part of this procedure is that the forcing terms are evidently applied to points in space, i.e., the control volume, rather than to material fluid volumes. However, we have seen that the RTT amounts to the same thing.

The value of this kind of derivation is that it may be most natural to make assertions about the physical properties of a flow at the first stage of this procedure, Eq. (97), where the flux is prescribed. Here we have kept all three of the terms that contribute to momentum flux, in general, but often this is not necessary or appropriate. In most flows the pressure gradient is of leading importance, and the interesting question is likely

⁴²We can readily infer this last result from the form of the momentum flux tensor, Eq. (94). If the velocity vector (components) transform under rotation of coordinate axes as $\mathbf{V}' = \mathbf{R}\mathbf{V}$, and given that $(\mathbf{V}')^t = (\mathbf{R}\mathbf{V})^t = \mathbf{V}^t \mathbf{R}^t$, then by substitution into the middle term of Eq. (96) and using the associative property of matrix multiplication we verify the transformation rule for Cartesian tensors. (A good review of tensor algebra is Ch. 2 of Kundu and Cohen (2001) noted in footnote 17.)

to be how the advective and diffusive terms compare to one another. To make a rough estimate of these terms we have to make an estimate of the length scale, L , over which the current component varies by about 100%. That is, we seek to write

$$\frac{\partial u}{\partial x} \text{ is } O\left(\frac{U}{L}\right) \text{ and } \frac{\partial^2 u}{\partial x^2} \text{ is } O\left(\frac{U}{L^2}\right)$$

where by the big O notation we mean the order of magnitude without regard to the sign. For example, if $u = A \sin(kx)$, then we would know quite precisely that $\partial u / \partial x = Ak \cos(kx)$ is $O(Ak)$ and so $U = A$ and $L = k^{-1} = \lambda / 2\pi$. If our aim was a rough estimate, then we might ignore the factor 2π , and say that L is $O(\lambda)$. Of course, when factors of 2π or 2 cascade, as they do for the second derivative, then this could eventually lead to trouble. But with that in mind, let's proceed to estimate the ratio of the advective term to the diffusion term as

$$\frac{u \frac{\partial u}{\partial x}}{\nu \frac{\partial^2 u}{\partial x^2}} \text{ is } O\left(\frac{u^2/L}{\nu u/L^2}\right) = O\left(\frac{UL}{\nu}\right) = O(Re), \quad (100)$$

where Re is the Reynolds number. The Reynolds number is nondimensional, and serves here as a measure of the ratio of the advection term to the diffusion term. We have seen the Reynolds number before, Eq. (10), and will see it again whenever viscosity is at issue, but notice that the length scale will be different from case to case. For the wave-like motion shown in the cover graphic, we can estimate that $U \approx 0.1 \text{ m s}^{-1}$, and that $L \approx 100 \text{ km}$, roughly. Using the known viscosity of water, $\nu = 0.001 \text{ m}^2 \text{ s}^{-1}$, then $Re \approx 10^7$. No matter how crude the estimate of L , it is unmistakable that the viscous diffusion term is much, much smaller than the advection term in this instance. After making a few such estimates, you will be entirely justified in dropping (without even mentioning) the viscous, diffusive contribution to the momentum flux in the case that the momentum budget is evaluated for large scale motions of the atmosphere or ocean.

You might well wonder if the viscosity term is ever important. Yes, but the viscous term is of leading importance only for scales of motion that are very, very small compared to basin scale oceanic or atmospheric motions. Recall the three-dimensional flow in a tea cup; the smallest scales of motion are vortices having $L = 0.005 \text{ m}$, roughly, and a typical speed $U = 0.05 \text{ m s}^{-1}$, and hence $Re \approx 1/2$. You may have noticed that these small scales of motion are damped rather quickly, within a few seconds. Can this be attributed to viscosity? The time scale required for diffusion to propagate a signal over a distance L is roughly $t_{diff} = L^2/\nu$, deduced from dimensional analysis or from a solution of the diffusion equation, while the rotation time for a vortical motion is $t_{adv} = L/U$. The ratio of these time scales is $t_{diff}/t_{adv} = Re$, once again the Reynolds number, and viscous diffusion of momentum is evidently sufficient to damp the smallest vortical motions in a time that is the order of several rotations, a few seconds. Larger vortices last longer.

3.4.5 Energy conservation; the First Law of Thermodynamics applied to a fluid

As a final application we will consider the balance of thermal energy in a compressible fluid. The First Law of Thermodynamics keeps track of several kinds of energy storage and energy exchange that may occur between a given fluid volume and the environment, schematically,

$$\frac{dE}{dt} = \text{WorkRate} + \text{HeatSource} \quad (101)$$

where $E = \rho C_v T$, is the internal energy, and C_v is the heat capacity of the fluid in an isovolume process, taken to be a known physical property of the fluid, and a constant for this purpose. WorkRate is the rate at

which mechanical work is done by the environment on the fluid volume. In the case of an Euler fluid, this can be due only to the rate of work by pressure, $-P\mathbf{n} \cdot \mathbf{V}$, evaluated over the surface of the volume. *HeatSource* is the divergence of heat fluxes due to radiation, $-\nabla \cdot \mathbf{Q}$, and conduction, $-\kappa \nabla T$, where κ is the thermal conductivity of the fluid, and to a body source due to the dissipation of kinetic energy to internal energy, ϵ . In most cases ϵ is negligibly small compared to the heat fluxes or *WorkRate* and will be omitted from here on. Our final result will be appropriate for geophysical flows but not completely general. The approximate balance of thermal energy is then:

$$\frac{DE}{Dt} = C_v \frac{D}{Dt} \iiint_{material} \rho T dVol = -\oint P\mathbf{n} \cdot \mathbf{V} da - \oint \kappa \nabla T \cdot \mathbf{n} da + \iiint_{material} -\nabla \cdot \mathbf{Q} dVol. \quad (102)$$

Applying the divergence theorem to the surface integrals and the RTT and then collecting terms under the volume integral yields the differential form of the thermal energy balance in field coordinates:

$$\rho C_v \frac{DT}{Dt} = -P\nabla \cdot \mathbf{V} + \kappa \nabla^2 T - \nabla \cdot \mathbf{Q}. \quad (103)$$

To simplify this a little further, the fluid will be presumed to be an ideal gas described the equation of state $P = \rho RT$, where $R = C_p - C_v$ is the universal gas constant and C_p is the heat capacity at constant pressure (Table 1). The divergence of fluid velocity may be eliminated by use of the mass conservation equation (67) and then the equation of state,

$$-P\nabla \cdot \mathbf{V} = -\frac{P}{\rho} \frac{D\rho}{Dt} = \frac{P}{\rho} \frac{\partial \rho}{\partial T} \Big|_P \frac{DT}{Dt} = -\frac{P}{\rho} \frac{P}{RT^2} \frac{DT}{Dt} = -\rho R \frac{DT}{Dt}.$$

When this is substituted into Eq. (103) the result is

$$\boxed{\text{for an ideal gas: } \rho C_p \frac{DT}{Dt} = \kappa \nabla^2 T - \nabla \cdot \mathbf{Q}} \quad (104)$$

where notice the heat capacity is now C_p in place of C_v . For most liquids and solids the distinction between C_p and C_v is negligible under geophysical conditions, since there is usually little volume change.

3.5 A few remarks on the Eulerian equations

The momentum equations (92) have two terms that are characteristic of fluid flows, the pressure gradient term and the advection terms. Pressure is a scalar, and the gradient of the pressure appears in each of the component equations. A localized pressure perturbation in a three-dimensional flow is thus likely to induce motion in all three components. The pressure gradient thus acts to couple the component equations, and is generally the physical process that allows for (or enforces) mass conservation; as fluid converges into a given volume the pressure will rise and so produce a compensating pressure-driven divergence, e.g., acoustic and gravity waves if high or low frequency. The pressure gradient term is linear, and does not, in and of itself, present any special mathematical difficulty.

From a physical (and Eulerian) perspective, it is the process of advection that endows many fluid flows with rich spatial structure and complexity and it is advection that most distinguishes a fluid flow from solid mechanics. The advection terms are nonlinear, in general, and hence the advective terms stymie most of the familiar PDE solution techniques that require superposition of solutions (more about advection in Section 6).

It is not uncommon that the advection terms are demonstrably much, much smaller than the pressure gradient term, and so they may be omitted to leave a linear balance equation. A linearized system of that sort clearly omits some of the distinctive character of fluid flow but may be quite useful for the analysis of wave-like motions, especially.

The (Eulerian) momentum, mass and energy equations are impressive, and it might seem that writing them out in full would be a significant step towards the solution of a fluid mechanics problem. Well, yes and no. An understanding of the origin and the meaning of these equations is certainly a vital step toward understanding fluid mechanics, generally. But it is essential to understand that these equations are extremely general, and that the definition of a specific problem requires a number of additional steps and likely some important simplifications. In fact, the content of these equations is just what their name implies - that momentum, mass and energy are conserved (though we omitted the dissipation of mechanical energy to thermal energy) and, here's the punch line, there is *nothing more implied than that*. Momentum conservation may occur by virtue of wave motions or swirling vortices or random turbulence, or more likely all three at once (as in the tea cup).

A complete model of a fluid flow problem will likely require all three of the Eulerian equations derived above together with careful consideration of the boundary and initial conditions that serve to define a specific problem. The solution of such a model system is often a substantial task requiring a great deal of time and effort and resort to numerical and sometimes experimental methods; most of your study of fluid mechanics will be aimed at just that kind of task. Here and now, though, we are now going to make the generous assumption that we have analytic or numerical solutions of the velocity field, so that we can continue on with the development of Lagrangian/Eulerian kinematics.

4 Depictions of fluid flows represented in field coordinates.

It may be apparent from observing the flow in a tea cup or in an ocean circulation model that displaying or depicting a fluid flow can be a significant task in cases where the domain is multi-dimensional and the flow is time-dependent. A variety of methods are used to show the flow dependence upon one or more of the independent variables, and some of these are a direct application Lagrangian-Eulerian transformation problem considered in Sections 2 and 3. ⁴³

4.1 Trajectories (or pathlines) are important Lagrangian properties

An important fluid property is the parcel trajectories, often called pathlines (and we have already treated one-dimensional examples of trajectories). In this section we will consider position and velocity in a two-dimensional space, \mathbb{R}^2 , and \mathbf{x} and \mathbf{V} indicate vector position and velocity. From here on out we are going to drop the subscripts L and E that have been used to emphasize Lagrangian and Eulerian velocity. The kind of velocity should be clear from the context, or from the list of independent variables.

⁴³An excellent web page that shows the practical reasons and methods for computing the trajectories of air parcels using Eulerian data from large scale numerical models of the atmosphere is at http://ready.arl.noaa.gov/READY_traj_neus.php

We can compute parcel trajectories from the Eulerian velocity field via

$$\boxed{\frac{d\mathbf{x}}{dt} = \mathbf{V}(\mathbf{x}(t), t)} \quad (105)$$

provided we recognize that \mathbf{x} on the right side is the moving, time-dependent, parcel position. The appropriate initial condition is just

$$\mathbf{x}(t = 0) = \mathbf{A}. \quad (106)$$

Note that (105) is in the form of the FPK, or Eq. (14), except that here the parcel position is \mathbf{x} from the beginning. In component form this may be written out

$$\frac{dx}{dt} = u(x, y, t); \quad \frac{dy}{dt} = v(x, y, t) \quad (107)$$

and with the initial conditions (ICs)

$$x(t = t_0) = \alpha; \quad y(t = t_0) = \beta \quad (108)$$

which makes clear that we have two first order ODEs. On first sight these trajectory equations (107) could be deceptive; as here written they are quite general and applicable to any fluid motion in R^2 . Thus it should not be surprising if on most occasions they prove intractable by elementary methods. If u depends upon y or v , or if v depends upon x or u , then these are coupled equations that have to be solved simultaneously; if u or v are nonlinear then they are nonlinear equations. Either way their solution may have to be sought with numerical techniques. What is surprising about Eq. (107), even after several encounters, is that what can seem to be simple velocity fields can yield complex and interesting trajectories (one example is in Section 4.2).

We can best illustrate these diagnostic quantities with a two-dimensional velocity field,

$$\mathbf{V} = x\mathbf{e}_x + \frac{y}{1+2t}\mathbf{e}_y, \quad (109)$$

which is plotted for two times in Fig. 15. The component equations are then

$$\frac{dx}{dt} = x; \quad \frac{dy}{dt} = \frac{y}{1+2t}, \quad (110)$$

and with ICs as above. The dependent variables are uncoupled, and moreover, within each component equation the independent variables can be readily separated,

$$\frac{dx}{x} = dt; \quad \frac{dy}{y} = \frac{dt}{(1+2t)}.$$

These can then be integrated over the limits α to x (β to y) and t_0 to t to yield the trajectory

$$x(\alpha, t_0, t) = \alpha \exp(t - t_0); \quad y(\beta, t_0, t) = \beta \frac{(1+2t)^{1/2}}{(1+2t_0)^{1/2}}. \quad (111)$$

Notice that the y -component is just as before, Eq. (16), except that we have retained the initial time as a parameter (we will need it below). Trajectories starting from a few different α , β are in Fig. 16. In this case the trajectories are roughly in the direction of the flow as seen in Fig. 15, and appear to bend over in time, consistent with the temporally-decreasing y -component of the velocity field. There is nothing surprising in this case, but in a later section, 5.2, we will see an example where the trajectories could not have been anticipated in advance of an integration, and the Eulerian and Lagrangian mean flows are qualitatively different.

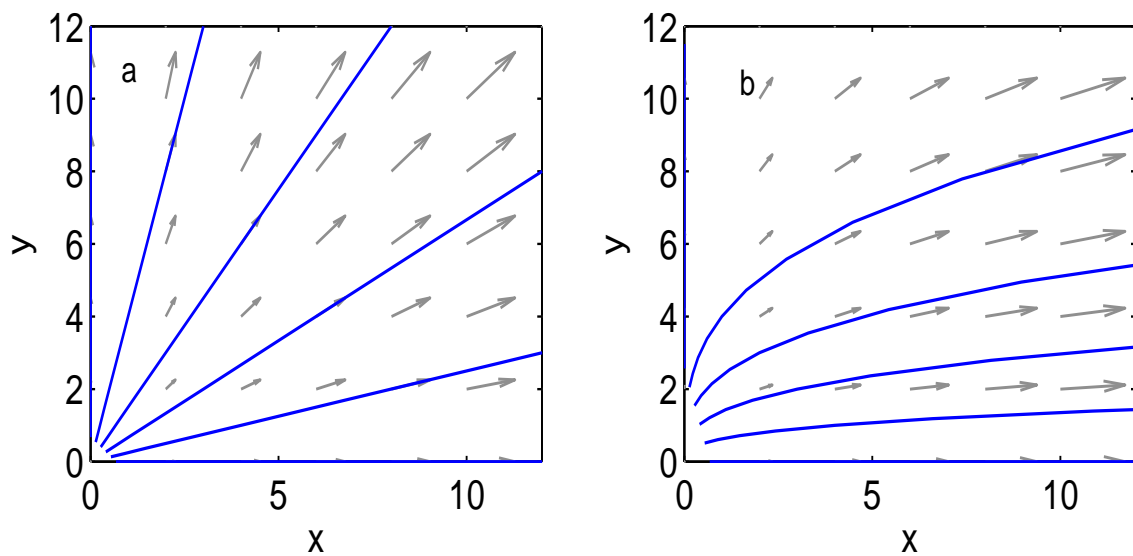


Figure 15: (a) Velocity field and streamlines (the family of solid lines) for Eq. (110) at $t = 0$. (b) At $t = 1$. Notice that the velocity at a given point turns clockwise with time as the y -component of the velocity decreases with time.

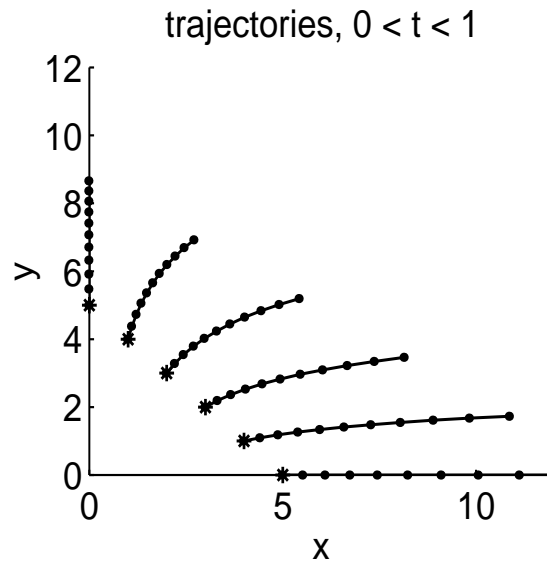


Figure 16: Trajectories of six parcels that were released into the flow given by Eq. (110) at the same time, $t_0 = 0$, and tracked until $t = 1$. The sources are shown by asterisks. Dots along the trajectories are at time intervals of 0.1

4.2 Streaklines are a snapshot of parcels having a common origin

Another useful characterization of the history of parcel positions is the so-called streakline, which shows the positions, at a fixed time, of all of the parcels which at some earlier time passed through a given point. An example of this would be the plume of smoke coming from a point source located at \mathbf{x}_p and recorded, say by a photograph taken at a time, t_p . The information needed to construct a streakline is contained within the trajectory, Eq. (111). To see this we will construct a streakline by releasing parcels one after the other from a fixed source. The first parcel is released at time $t_0 = 0$, and we let the trajectory run until $t = t_p$, the time we make the photograph. The only data we retain from this trajectory is the position at time $t = t_p$, i.e., we record $\mathbf{x}(t_p, \mathbf{x}_p, t_0 = 0)$. A second parcel is released a little later, say at $t_0 = \frac{1}{4}$, and again we let the trajectory run until $t = t_p$, where we retain only the last position, $\mathbf{x}(t_p, \mathbf{x}_p, t_0 = \frac{1}{4})$. A third parcel is released at $t_0 = \frac{2}{4}$, and again we record its position at $t = t_p$, $\mathbf{x}(t_p, \mathbf{x}_p, t_0 = \frac{2}{4})$. It appears, then, that a recipe for making streakline from a trajectory is that we treat the initial time, t_0 , as a variable, while holding t constant at t_p , and also the initial position at $\mathbf{x} = \mathbf{x}_p$. Several streaklines are in Fig. 17. Notice that in this time-dependent flow, trajectories and streaklines are not parallel.

4.3 Streamlines are parallel to an instantaneous flow field

Still another useful method for depicting a velocity field is to draw the streamlines, a family of lines that are everywhere parallel to the velocity. Time is fixed, say at $t = t_f$, and thus streamlines portray the direction field of a velocity field, with no reference to parcels or trajectories or time-dependence of any sort. There is more than one way to construct a set of streamlines, but a method that lends itself to generalization is to solve for the parametric representation of a curve, $\mathbf{X}(s)$ that is everywhere parallel to the velocity;

$$\boxed{\frac{d\mathbf{X}}{ds} = \mathbf{v}(t_f, x, y)} \quad (112)$$

or in components;

$$\frac{dX}{ds} = u(t_f, x, y); \quad \frac{dY}{ds} = v(t_f, x, y). \quad (113)$$

A suitable initial condition is $X(s_0) = X_0$, etc. Notice that s is here a dummy variable; we could just as well have used any other symbol but s is conventional.⁴⁴ \mathbf{X} is the position of a point on a line, where just above \mathbf{x} meant the position of a parcel. This reuse of symbols is certainly a risky practice, but it is also almost unavoidable. Given the velocity components Eq. (110), these equations are also readily integrated to yield a family of streamlines:

$$X = X_0 \exp(s - s_0); \quad Y = Y_0 \exp\left(\frac{s - s_0}{1 + 2t_f}\right), \quad (114)$$

and recall that t_f is the fixed time that we draw the streamlines. We are free to choose the integration constants so that a given streamline will pass through a position that we specify. There is no rule for choosing these positions; in Fig. 15 we arbitrarily picked five positions and then let s vary over sufficient range to sweep through the domain. Other streamlines could be added if needed to help fill out the picture. No

⁴⁴In fact, s could be regarded as time, provided we make certain not to confuse this use of time with the time-dependence of the velocity field (which is suppressed while we draw a given map of streamlines). This helps make clear that streamlines are parallel to parcel trajectories in steady flows. If we marked off equal increments of time along a streamline we could depict the speed of the flow.

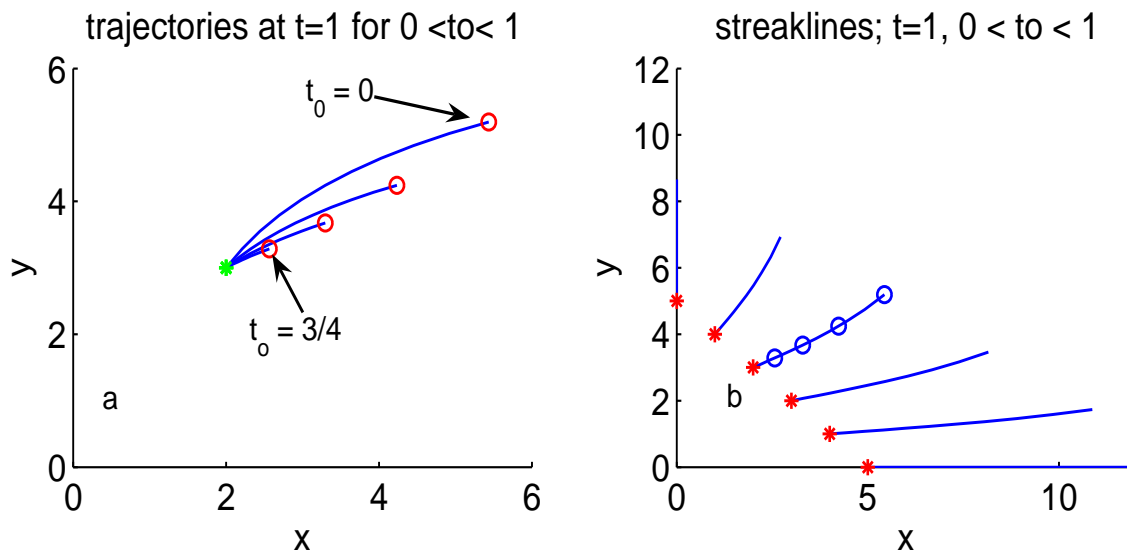


Figure 17: (a) Trajectories of five parcels that were released from a common source, $(x,y) = (2,3)$, and tracked until $t = 1$. The parcels were released at different initial times, $t_0 = 0, 1/4, 2/4, 3/4,$ and 1 . The latter trajectory has zero length. The end points of the trajectories are the open circles, the locus of which forms a streakline. (b) Streaklines from several different sources. These streaklines start at $t_0 = 0$ and the 'photograph' was taken at $t_p = 1$. Notice that these streaklines end at the endpoint of the trajectories of Fig. 16 (they have that one point in common) but that streaklines generally have a different shape (different curvature) from the trajectories made over the same time range. In this figure and in previous ones (Fig. 2b) the two quantities being compared were only slightly different and one might well wonder if, for example, streaklines are some kind of approximation to trajectories. The answer is no, in general, they are qualitatively different.

particular value is attached to a given streamline. In the future we will consider the streamline's sophisticated cousin, the streamfunction, which has isolines that are also parallel to velocity, but which assigns values that are related to the speed of the flow.

5 Eulerian to Lagrangian transformation by approximate methods.

The previous Section 4 developed the formal methods needed to transform Eulerian velocity into several forms, including (Lagrangian) trajectories, that help depict properties of a fluid flow. An understanding of the formal steps is important, of course, but the ease with which we could make the transformation in those cases resulted from the assumption of a very simple closed solution for the Eulerian velocity — not something you are likely to encounter outside of a homework problem. In practice, partial and incomplete information is the norm, and in this section we will consider an approximate analysis method based upon an expansion of the velocity field in Taylor series. This yields results that are interesting and important of themselves, especially the notion of Stokes drift, and introduces some new tools, e.g., the velocity gradient tensor, that are widely useful.

5.1 Tracking parcels around a steady vortex given limited Eulerian data

The power and the limitations of the Taylor series method can be appreciated by analysis of parcel motion in a steady, irrotational vortex in R^2 . The radial and azimuthal velocity components of the vortex are taken to be

$$\mathbf{V} = (u_{rad}, u_{azi}) = (0, C/2\pi r), \quad (115)$$

where r is the distance from the vortex center. The $1/r$ dependence of azimuthal speed is the distinguishing feature of an irrotational vortex. C is a constant, termed the circulation,

$$C = \oint \mathbf{V} \cdot d\mathbf{s},$$

where $d\mathbf{s}$ is the vector line segment along a path that encloses the vortex center and that is traversed in an anti-clockwise direction. C measures the vortex strength, and without loss of generality we can set $C = -2\pi$ to define a vortex that rotates clockwise, Fig. (18). It is apparent that parcel trajectories in this steady vortex will be circular, and that a parcel will make a complete orbit in time $T = (2\pi r)^2/C$. The Cartesian velocity components are

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{C}{2\pi r} \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} \quad (116)$$

where $r = (x^2 + y^2)^{1/2}$ and the angle $\theta = \text{atan}(y/x)$ is measured counter-clockwise from the x-axis.

An irrotational vortex is an idealization of the vortex flow produced by the convergent flow into a drain, for example, and has several interesting properties that we will consider in later sections (including why it is said to be irrotational). For now it makes a convenient flow into which we can insert floats and current meters to investigate kinematics.

5.1.1 The zeroth order approximation, or PVD

For the sake of this development we are going to imagine for the moment that the only thing we know about this vortex is the velocity observed at one fixed site, say $(x_s, y_s) = (0, 1)$.⁴⁵ The velocity observed at this

⁴⁵When we write the Cartesian components of a vector within the body of the text as here, they will be written as if they were in a row matrix, i.e., (x, y) ; when they are written as a separate equation they will be written as a column matrix, Eq. (116), which is the form in which they actually appear in tensor equations.

fixed site, $V_0 = V(x_s, y_s)$, an Eulerian velocity, would then be a steady, uni-directional flow having Cartesian components $(U_0, V_0) = -(C/2\pi r, 0) = (1, 0)$. And, of course, if that was all we that knew, we wouldn't know that this was just one point in a vortex. Here's the question — what, if anything, can be inferred about parcel trajectories from this very limited data? If no other information was available, then out of desperation we could estimate a trajectory by integrating this Eulerian velocity in time as if it were the Lagrangian velocity, i.e.,

$$\delta X_0 = \int V_0 dt = \begin{pmatrix} U_0 t \\ 0 \end{pmatrix}. \quad (117)$$

It is essential to understand that such a procedure is wrong, formally, as we have completely missed the distinction between Lagrangian and Eulerian (it happens!). But, that aside, it is fruitful to see this estimate, termed a progressive vector diagram, or PVD, as the lowest or zeroth-order approximation of the trajectory.⁴⁶ The PVD in this case indicates a linear (pseudo-)trajectory, consistent with the uni-directional velocity observed at the fixed site, Fig. 6. A PVD is a useful way to visualize a current meter record or a wind record insofar as it gives a direct measure of how much fluid has gone past the observation site. But does a PVD show *where* fluid parcels will go after they pass through the observation site? Generally, no. Is there any flow condition under which we could we interpret a PVD as if it were a parcel trajectory? Yes, a PVD would represent a true trajectory if the Eulerian velocity field was spatially uniform. Observations made at any position would then be equal to observations made anywhere else, including at the moving position of a parcel. A spatially uniform flow is a degenerate case of little practical interest, but this helps us to see that the issue insofar as this Eulerian to Lagrangian transformation is concerned is the spatial variation of the flow.

If we consider the example of a steady vortex flow, then it would appear that the PVD is an acceptable trajectory estimate only for (pseudo-)displacements that are much less than the horizontal distance over which the flow changes significantly. By inspection, the horizontal scale of this vortex is estimated to be the radius (at a given point), and so this condition could be written $\delta X_0 \ll r$. When the displacement is greater than this, the velocity at the position of the parcel (the Lagrangian velocity that we should be integrating) will begin to differ significantly from the velocity observed back at the fixed site (the Eulerian that we are integrating in this PVD-approximation). Once this discrepancy is evident, the PVD will soon fail to make a good approximation to the actual trajectory.

5.1.2 A first order approximation, and the velocity gradient tensor

Evidently we can not get very far from our starting point with velocity only, so let's give ourselves another piece of information, the velocity gradients, again at the (fixed) observation point only. With this new piece of data we can take some partial account of the spatial variation of the velocity. To do this we can represent the velocity field in the vicinity of the observation point by expanding in a Taylor series, here for each component

⁴⁶The notation δX would usually mean a displacement vector that is small in some sense, e.g., compared to the radius of convergence of a power series. Here we are going to integrate long enough for the displacement to be substantial, and then we will call δX the trajectory. This abuse of δ is intentional, because we want to see the consequences of violating the small displacement restriction.

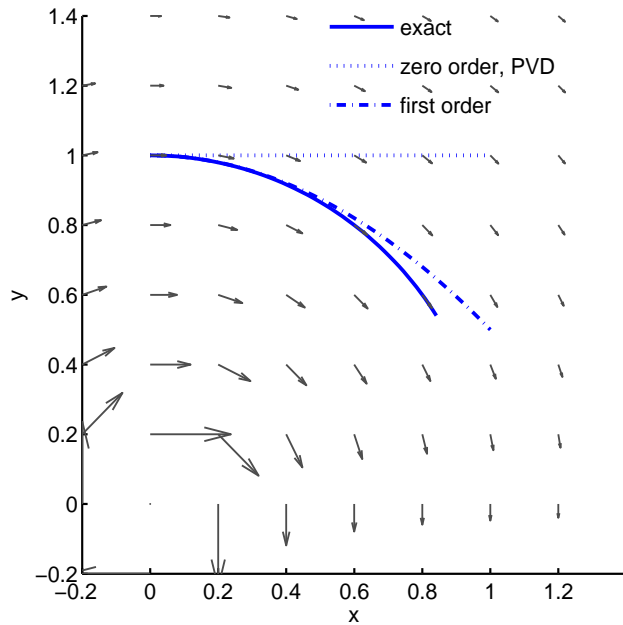


Figure 18: Parcel tracking around a free vortex. Contour lines are of azimuthal speed with arrows indicating direction. Positions of a parcel launched at $(x, y) = (0, 1)$ are shown at times $= 0, 0.2, 0.4, 0.6, 0.8$ and 1.0 . The exact trajectory was easily computed in cylindrical coordinates. The straight path is the zero order approximation, or PVD, in which the Eulerian velocity measured at the fixed observation site has been integrated as if it were the Lagrangian velocity of the parcel. Note that the PVD starts off well enough, but begins to accumulate a noticeable error as the displacement becomes even a small fraction of the radius. The first order trajectory accounts approximately for the spatial variation of the velocity and is somewhat better than the PVD, but it too becomes rather inaccurate as the displacement becomes comparable to the radius.

separately,

$$u(x, y) = u_0 + \frac{\partial u}{\partial x} \delta X + \frac{\partial u}{\partial y} \delta Y + HOT, \quad (118)$$

$$v(x, y) = v_0 + \frac{\partial v}{\partial x} \delta X + \frac{\partial v}{\partial y} \delta Y + HOT,$$

where (u_0, v_0) is the velocity observed at the observation site, the partial derivatives are evaluated at the observation site, and *HOT* is the sum of all the higher order terms that are proportional to δx^2 , δx^3 , etc. In effect, we are now allowing that we know not only the velocity but also the four partial derivatives, though at the one position only. It is convenient to use a vector and tensor notation to write equations like these in a format

$$\mathbf{V}(x, y) = \mathbf{V}_0 + \mathbb{G} \cdot \delta \mathbf{X} + HOT, \quad (119)$$

where \mathbb{G} is the velocity gradient tensor encountered already in Section 2.3, and repeated here for the two-dimensional case at hand,

$$\mathbb{G} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \quad (120)$$

\mathbb{G} recurs in the study of kinematics and we will encounter it again in Section 8.3. For now \mathbb{G} is a device to streamline notation; when we (matrix) multiply⁴⁷ \mathbb{G} into a displacement vector (written as a column vector),

⁴⁷ \mathbb{G} is a Cartesian tensor that can be manipulated as if it were a matrix. \mathbb{G} is a tensor insofar as its elements will transform with a rotation of coordinate axes in a way that leaves tensor equations invariant to any (time-independent) coordinate rotation (footnote 42). No such transformation properties are implied for the elements of a matrix.

we get the velocity difference that corresponds to that displacement vector. It is easy to see that if we doubled or halved the length of the displacement vector we would get twice or half the velocity difference. Thus, multiplication by the velocity gradient tensor serves to make a linear transformation on a displacement vector. In general the result will be a velocity difference vector having a different direction from that of the displacement vector, and of course it has a different amplitude and different dimensions as well. The velocity gradient tensor evaluated at the observation site $(x_s, y_s) = (1, 0)$ has a simple form

$$\mathbb{G} = \frac{C}{2\pi} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

The PVD approximation amounts to an integration of the first term only of the Taylor series, while ignoring the spatial variation of the velocity altogether. Since this approximation omits all terms that are first and higher power in the displacement, the PVD is termed the zeroth order displacement or trajectory. The first order trajectory, by which we mean the first correction to the trajectory, can then be computed by dropping the *HOT* from Eq. (119), by approximating $\delta X = \delta X_0 = \int V_0 dt$ and integrating in time,

$$\delta X_1 = \delta X_0 + \mathbb{G} \cdot V_0 t^2 / 2. \quad (121)$$

This first order trajectory is a considerable improvement upon the PVD (Fig. 18). Nevertheless, after sufficient time has passed and the displacement becomes comparable to the length scale of the flow, the radius, then this first order trajectory also accumulates a noticeable error. Adding an evaluation of the next term of the *HOT* would delay the failure, in general, but in any steady vortex flow the displacement will eventually carry a parcel long distances from its origin. Approximation methods built around a Taylor series expansion are not uniformly valid in time when applied to a steady vortex flow.

5.2 Tracking parcels in gravity waves

This method has somewhat better success when applied to a wavelike motion in which the parcel displacements over a single wave passage are small compared to the wavelength. In this flow there are then two length scales (where in the vortex flow above there was only one, the radius). As an example, we will analyze the parcel motion associated with a surface gravity wave having a surface displacement

$$\eta(x, t) = a \cos(kx - \omega t), \quad (122)$$

where a is the amplitude of the surface displacement, $k = 2\pi/\lambda$ is the wavenumber given the wavelength λ and ω is the wave angular frequency (it is assumed that ω and $k > 0$). The argument of the trigonometric function shows that this surface displacement moves rightward as a progressive wave having a phase speed $c = \omega/k$. The two-dimensional and time-dependent velocity field associated with this wave

$$V(x, z, t) = U e^{kz} \begin{pmatrix} \cos(kx - \omega t) \\ \sin(kx - \omega t) \end{pmatrix}, \quad (123)$$

where z is the depth, positive upwards from the surface. The amplitude or speed at the surface is $U = a\omega$ and decays with depth on an e-folding scale $1/k$. This exponential decay with depth is appropriate for a wave whose wavelength is less than the water depth, a so-called deep water wave. If the wavelength is much greater

than the water depth, a shallow water wave, the x component of the velocity is independent of depth and the z component is linear with depth and vanishes at the (flat) bottom.⁴⁸

5.2.1 The zeroth order approximation, closed loops

The (Eulerian) velocity observed at a fixed point is a rotary current, often called the orbital velocity, of amplitude $a\omega e^{kz}$ that turns clockwise with time at the angular frequency ω (assuming that x increases to the right). Given the known velocity we can readily calculate the PVD-like parcel displacements by integrating $V(x, z, t)$ with respect to time while holding x and z constant,

$$\delta X_0 = ae^{kz} \begin{pmatrix} -\sin(kx - \omega t) \\ \cos(kx - \omega t) \end{pmatrix}. \quad (124)$$

The PVD indicates that parcels move in a closed rotary motion with each wave passage and that the net motion is zero, consistent with the wave-average of the Eulerian velocity.

From the analysis of motion around a vortex we might have developed the insight that this PVD for a gravity wave would probably give a fairly accurate prediction for the actual parcel displacements provided that the parcel displacements were much, much less than the scale over which the wave orbital velocity varies. In this case the scale is k^{-1} in either direction, so that this condition is equivalent to requiring that the wave steepness, $ak = 2\pi a/\lambda$ must be much less than 1. This is also the condition under which the linear solution gives an accurate waveform of the surface displacement, a pure sinusoid, which we have assumed with Eq. (122).

5.2.2 The first order approximation yields the wave momentum and Stokes drift

By now we should be dubious that this zeroth order approximation of the trajectories can tell us anything reliable about the actual trajectories, and so we proceed to calculate the first order velocity via Eq. (119). The velocity gradient tensor for this wave is just

$$\mathbb{G} = a\omega k e^{kz} \begin{pmatrix} -\sin(kx - \omega t) & \cos(kx - \omega t) \\ \cos(kx - \omega t) & \sin(kx - \omega t) \end{pmatrix}$$

and matrix-multiplying into the zeroth order displacement given by Eq. (124) gives the first order velocity (insofar as we have dropped the *HOT*),

$$\begin{aligned} V_1(z) &= a^2 \omega k e^{2kz} \begin{pmatrix} \sin^2(kx - \omega t) + \cos^2(kx - \omega t) \\ 0 \end{pmatrix} \\ &= U a k e^{2kz} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \end{aligned} \quad (125)$$

⁴⁸An excellent introduction to surface gravity waves is 'Water Wave Mechanics for Engineers and Scientists', by R. G. Dean and R. A. Dalrymple, World Scientific Pub. Co., 1991. For Stokes drift in particular see also Longuet-Higgins, M. S., 1969. 'On the transport of mass by time-varying ocean currents'. *Deep-Sea Res.*, **16**, 431-447 and Longuet-Higgins, M. S., 1986. 'Eulerian and Lagrangian aspects of surface waves', *J. Fluid Mech.*, **173**, 683-707.

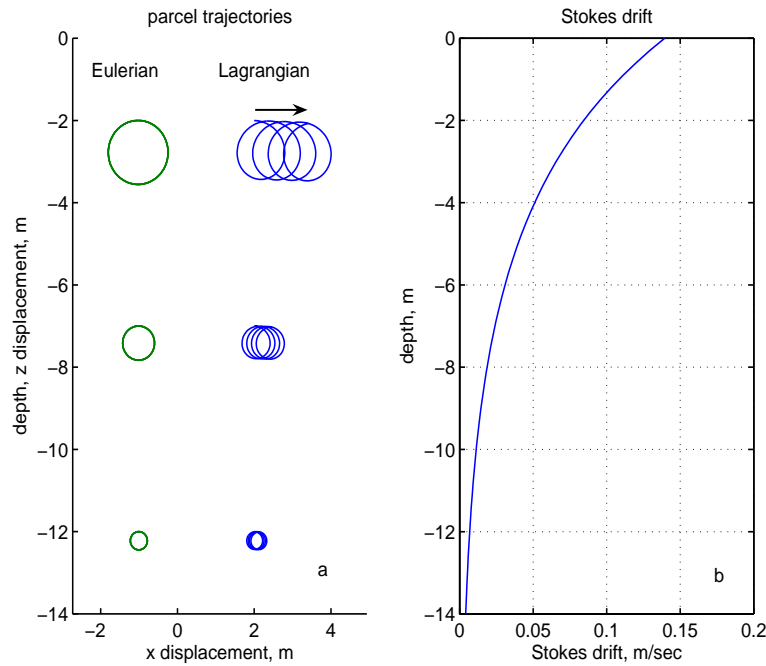


Figure 19: (a) Parcel trajectories underneath a deep-water, surface gravity wave that is presumed to be propagating from left to right. The amplitude of the surface displacement was taken to be $a = 1$ m and the wavelength $\lambda = 50$ m. The Eulerian trajectories (or PVDs) are closed circular orbits around which the parcels move clockwise. The Lagrangian trajectories were computed by integrating numerically for four periods ($= 22.6$ sec) and are open loops indicating there is a net drift of fluid parcels from left to right (shown as an arrow). (b) Stokes drift for this wave computed from Eq. (125). An animation of a gravity wave and parcel trajectories is in the next figure.

where recall $U = a\omega$ is the amplitude of the orbital velocity at the surface. The coefficient $Uak = a^2k^2c$ is the wave steepness squared times the phase speed. This velocity is independent of time and x , and so the first order displacement is easily computed,

$$\delta X_1(z) = V_1(z)t, \quad (126)$$

where there is no particular initial position. Notice that this formula relates the displacement of a parcel to a depth, i.e., a field coordinate, so that it gives a field representation of a Lagrangian property.

Quite unlike the PVD, Eq. (126) indicates that fluid parcels have a substantial net motion in the direction of the wave propagation, often called Stokes drift or mass transport velocity, that is a fraction $2\pi \frac{a}{\lambda} e^{kz}$ of the orbital motion amplitude. For example, for a wave having an amplitude of $a = 1$ m and wavelength of $\lambda = 50$ m, the orbital motion at the surface is about 1.11 m s^{-1} where the Stokes drift is about 0.14 m s^{-1} . The Stokes drift decreases rapidly with depth; the length scale in depth is half of that of the wave orbital velocity.

The vertically integrated Stokes drift is the mass transport per unit length,

$$M = \rho \int_{-\infty}^0 V_1(z) dz = \begin{pmatrix} \rho U a / 2 \\ 0 \end{pmatrix},$$

and also the momentum per unit area associated with the wave. The kinetic energy per unit area of the wave

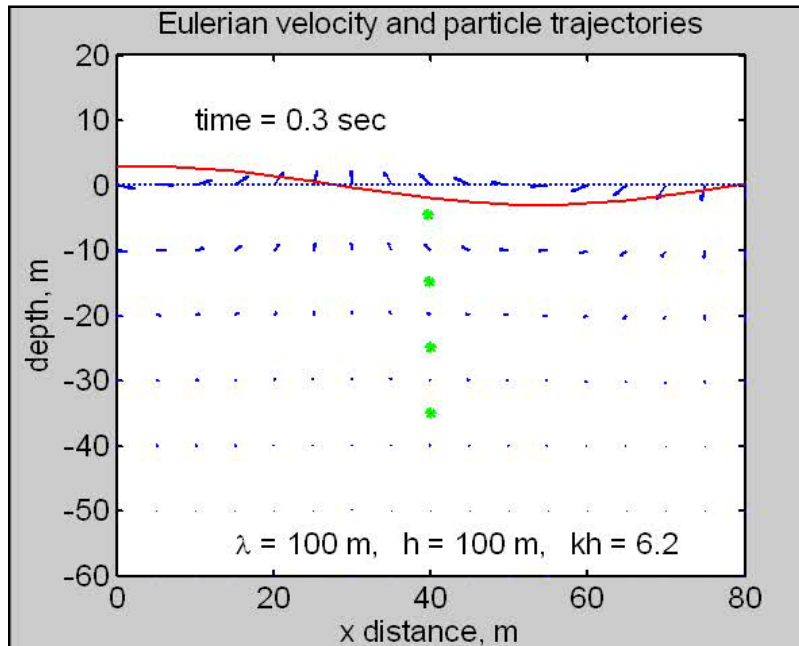


Figure 20: A surface gravity wave propagating from left to right as computed on linear theory. The Eulerian (orbital) velocities are the array of vectors, and parcel trajectories are the red, growing lines that were computed numerically. This wave has an amplitude of 3 m, and a wavelength of 100 m in a water depth of 100 m and so is rather steep. The wave angular frequency $\approx 1 \text{ s}^{-1}$ so that the PVD trajectories have an amplitude almost equal (numerically, in these MKS units) to that of the Eulerian velocity.

motion is

$$K = \rho \int_{-\infty}^0 V(z)^2 dz = \rho Uca/4.$$

Thus the momentum in the direction of gravity wave propagation and the kinetic energy of gravity waves are related by the particle-like relation

$$M = \frac{2K}{c},$$

which holds for other kinds of waves, e.g., electromagnetic waves, that have genuine momentum.⁴⁹ Thus, the Stokes drift turns out to be much more than a residual effect of switching from an Eulerian to a Lagrangian coordinate system and indeed it is one of the most important means by which surface gravity waves interact with other scales of motion.⁵⁰ It is notable that all of the information needed to calculate the Stokes drift was present in the Eulerian velocity field, Eq. (123). However, to reveal this important phenomenon, we had to

⁴⁹Waves that exist on a physical medium need *not* have a genuine momentum of this sort, though they will have a momentum flux, see M. E. McIntyre, ‘On the wave momentum myth’, *J. Fluid Mech.*, **106**, 331-347, 1981. An excellent recent reference on the topic of wave momentum is by D. Rowland, ‘Comments on “What happens to energy and momentum when two oppositely-moving wave pulses overlap?”’, *Am. J. Phys.*, **72**(11), 1425-1429, 2004. Surface gravity waves have momentum by virtue of the displaced free surface. Here’s a small problem for you: show that the Eulerian mean momentum over the water column is equal to the mass transport associated with Stokes drift by making the same kind of (bilinear) approximation for transport that we made for Stokes drift, i.e., assume that the wave velocity V under the displaced surface is given by Eq. (26) evaluated at $z = 0$, and compute the wave mean of ηV .

⁵⁰Stokes drift will transport fluid properties as well as any material, e.g., plankton, that are suspended in the fluid. The role of Stokes

carry out an analysis that was explicitly Lagrangian, i.e., that tracked parcels over a significant duration.⁵¹ Two comments regarding the Stokes drift. First, it has been estimated as the product of two terms that are derived from a linear solution, the velocity gradient tensor and the PVD displacement, and is thus 'bilinear'. The amplitude of the Stokes drift is proportional to the wave amplitude squared. This seems to imply that the Stokes drift estimated from a nonlinear wave might be rather different from that calculated here. We don't intend to carry this calculation any further, but will simply note that this is not the case, assuming only that the nonlinear wave is not grossly distorted from the linear solution. A nonlinear correction to the linear solution gives only a nonlinear correction squared (and probably very small) to the Stokes drift. Second, the Stokes drift can be envisaged as the difference between the time-integral of the Eulerian and Lagrangian velocities in the sense that, averaged over a wave, the various fluid velocities can be imagined related by Euler + Stokes = Lagrange (see Longuet-Higgins of footnote 48, and in the surface wave case, Euler = 0). This stands in marked contrast to the FPK which, recall, applies to the *instantaneous* velocity. Thus Stokes drift arises from time-averaging or integrating, which most observation techniques necessarily do, of course, if only to yield data sets of a manageable size. In other cases averaging may arise as an inevitable byproduct of the sampling procedure. In either event, the strict, formal FPK may not apply.⁵²

6 Aspects of advection, the Eulerian representation of fluid flow.

As we noted at the close of Section 3, the most distinctive feature of the Eulerian equations is the occurrence of the advection terms. These terms represent the process (or at least the effect of) fluid flow, and contribute most of the physical and mathematical complexity of fluid mechanics. In this section we will begin to

drift in ocean circulation and ocean ecology is an active topic in ocean science. An entry to some of this research can be found at http://www.aslo.org/lo/toc/vol_49/issue_4/1214.pdf. Other, more advanced studies are by Leibovich, S., 'The form and dynamics of Langmuir circulation.' *Ann. Rev. Fluid Mech.*, **15**, 391-427, 1983 and by McWilliams, J.C. and J.M Restrepo, 'The wave-driven ocean circulation.' *J. Phys. Oceanogr.* **29**, 2523-2540, 1999.

⁵¹Stokes drift is a robust phenomenon that can be produced and observed with simple means: fill a flat container with water to a depth of about 2-4 cm. A bath tub works well, but even a large cake pan will suffice. To make gravity waves use a cylinder having a diameter of roughly the water depth and a length that is about half the width of the tank. Oscillate the wave maker up and down with a frequency that makes gravity waves and observe the motion of more or less neutrally buoyant particles; some that float and others that sink to the bottom. You can easily vary three things, the amplitude of the waves, the depth of the water, and the width of the wave maker. Are the waves in your tank shallow water or deep water waves? Describe the mean flow (if any) set up by the oscillating wave maker, and how or whether it varies with the configuration of the tank and wave maker.

Suppose that the one-dimensional velocity in a progressive wave is given as $U \cos(kx - \omega t)$. Calculate the Stokes drift approximation of the mean parcel motion in this wave, and compare the result to the numerical integration of the full trajectory equation, i.e., $dx/dt = U \cos(kx(t) - \omega t)$. This requires a small program or script and can be accomplished with a rather crude numerical method. For what range of wave steepness does the Stokes drift estimate give an accurate estimate of the mean flow? Why do parcels have a Stokes drift in this wave?, i.e., explain why parcels in this wave velocity field have a net motion. What happens at large steepness? How does this compare with your observations from the bath tub? How does this compare with the Stokes drift of a deep water gravity wave?

⁵²We will not go through a comparable, lengthy discussion of the complementary transformation from Lagrangian velocity measurements to an Eulerian velocity field. Here are some things for you to think about. Suppose that we can make perfect Lagrangian measurements at arbitrarily fine temporal resolution. How would you use these data to construct the corresponding Eulerian velocity field? Now suppose, more realistically, that our Lagrangian measurements are averages over some finite time interval, say many wave periods in a case where surface gravity waves are present. How would you (or could you) construct the corresponding Eulerian velocity field from these data?

consider ways to set some bounds upon what advection alone can do in a fluid flow, and just as important, to understand what advection can not do. There are three topics in this section. Of these, the first, and third are essential elements of fluid kinematics: Modes of an advection equation (6.1), and the Cauchy-Stokes Theorem (6.3). The third topic, the Method of characteristics (6.2), is perhaps a little less so, but is (almost) irresistible given the Lagrangian/Eulerian theme of this essay.

6.1 The modes of a two-dimensional thermal advection equation

The thermal energy balance equation is a good example of a forced, time-dependent advection equation, viz., (104),

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \mathbf{V} \cdot \nabla T = -\nabla \cdot \mathbf{Q}^*, \quad (127)$$

where $\mathbf{V} \cdot \nabla T = u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}$ is the two-dimensional advection of internal energy (proportional to temperature), and \mathbf{Q}^* is the heat flux in kinematic units, $\mathbf{Q}^* = \mathbf{Q} / \rho_0 C_p$ where ρ_0 is the nominal density and C_p is the nominal heat capacity at constant pressure. The instantaneous thermal energy balance will generally involve all three terms (taking advection as one term). It may also happen that at some times, or averaged over some time scales, the balance will be dominated by just two terms, which we will dub a ‘mode’ of the thermal energy balance equation. There are three modes inherent in a three term equation like this one, and as part of building a vocabulary for describing fluid mechanics it is useful to note each of the possibilities.

Local balance: The advective term $\mathbf{V} \cdot \nabla T$ will vanish if the fluid is at rest, if the temperature is spatially uniform so that ∇T vanished, or if the fluid velocity is parallel to isolines of temperature. In that case the local time rate of change of temperature would reflect the source term, $\nabla \cdot \mathbf{Q}^*$, in what is called a local balance,

$$\frac{\partial T}{\partial t} = -\nabla \cdot \mathbf{Q}^*,$$

where ‘local’ means at one place, ‘nonlocal’ meaning that spatial gradients are important (the next two modes). For example, on a summer day with strong solar radiation, the air at ground level will likely warm rapidly during the late morning, indicating a local, source-driven thermal energy balance. Much the same thing happens on a seasonal cycle, especially at middle and higher latitudes, when data are averaged over a period of a few months. A local balance of just this kind can occur in a solid as well, e.g., the Earth’s land surface during a diurnal or seasonal cycle. The dynamics of such a local balance are not inherently fluid mechanical since fluid motion need not be involved (though this depends upon the nature of the source term, \mathbf{Q}^* , which may involve a generalized diffusivity analogous to Eq. 8).

Steady balance: The term $\partial T / \partial t$ is the rate of change of temperature observed at a fixed position and a steady flow is one that has $\partial / \partial t = 0$ for all relevant properties throughout the domain. A steady flow may nevertheless be subject to significant external forcing in which the advection term balances the source,

$$\mathbf{V} \cdot \nabla T = -\nabla \cdot \mathbf{Q}^*.$$

For example, if the source was positive or heating and yet we observed that there was little or no local time rate of change (i.e., that the temperature was steady) then we would infer that the advection term was acting to cool the observation site.

Such a steady balance may occur instantaneously, or more often when the thermal energy balance equation is time-averaged over a time scale that spans a full cycle of heating, say diurnal or annual. The

temperature at a given site in the lower atmosphere is likely to go through a nearly closed annual cycle, so that the annual average of the local rate of change will nearly vanish. Nevertheless the local source term $\nabla \cdot \mathbf{Q}^*$ may be significant when averaged over the same period. In that case we can infer that the advection term averaged over the annual cycle must also have been significant. For example, the ocean's overturning circulation transports comparatively warm waters from the tropics to middle and higher latitudes where thermal energy is given up from the ocean to the atmosphere. The ocean at higher latitudes is thus cooled by heat loss to the atmosphere, $-\nabla \cdot \mathbf{Q}^* < 0$, and warmed by advection of thermal energy from lower latitudes, $\mathbf{V} \cdot \nabla T > 0$ so that a more or less closed annual cycle holds (aside from climate drift, of course!). The signs of the local source and the advective term are reversed at lower latitudes; the source term is positive due to an excess of solar radiation and the ocean's overturning circulation acts to bring in cooler waters from higher latitudes. At a given location, the advection term may well be three-dimensional, including an important contribution from vertical motion.⁵³ This advection or transport of thermal energy from lower to higher latitudes, which occurs in both the atmosphere and ocean, makes a significant contribution to the moderation of Earth's meridional temperature gradient.

Frozen field: Finally, the third mode inherent to Eq. (127) is that

$$\frac{\partial T}{\partial t} + \mathbf{V} \cdot \nabla T = 0,$$

while both terms are considerably larger than the source, $\nabla \cdot \mathbf{Q}^*$. Said a little differently, the local rate of change of temperature may be due mainly to advection rather than to a local (heat) source term. This kind of balance is sometimes referred to as a 'frozen field', though the thing imagined to be frozen is the spatial structure of temperature that is embedded within the moving fluid (and not frozen in space). When this spatial structure is carried past fixed points by the fluid flow there is then a local rate of change of the property. For example, the passage of an atmospheric frontal boundary will often cause a rapid and significant change in the locally observed air temperature or humidity.

The object of many (geophysical) field experiments is to observe the source term, here \mathbf{Q}^* , and its relationship to the local flow environment, the topography, etc. When only Lagrangian measurements are available to define the heating rate following fluid parcels, then the estimation of the source term will probably be straightforward and noted in Section 2. Observing the connection with the environment may be challenging, however, since the measurement array will be uncontrolled. When only Eulerian measurements are available, then the inference of a source term from measurements of the local heating rate must account for what are usually important effects of advection, i.e., the frozen field mode. The scope of the observations needed to do this is usually quite significant, but of the kind needed to define the environment, generally.⁵⁴

⁵³The vertical advection term is just $w\partial T/\partial z$. In natural flows of the atmosphere and ocean the vertical velocity w is usually much, much smaller than the horizontal velocity, but then the vertical gradient of most properties is also much, much larger than the horizontal gradient. The vertical advection term is sufficiently different from the horizontal advection terms that it would be reasonable to treat it as a separate, fourth term in a three-dimensional thermal energy budget. We won't do that, however, as three modes are quite sufficient.

⁵⁴Two questions for you: 1) This description of modes has been for the Eulerian balance. Go back and describe the Lagrangian balance for each of these modes. Consider whether the temperature can or must vary with ξ . 2) The temperature T of our little thermal energy budget Eq. (127) could be the temperature of the atmosphere observed near ground level, i.e., your local climate. In that case \mathbf{Q} would be due mainly to solar insolation and radiative cooling, and the advection term would be associated with the advection of differing air masses to your observation site. Over the next week or two, take notice of your local climate and how it varies on a diurnal to weekly basis. What causes the local temperature to change, advection or the local source? Assuming that the latter is mainly

6.2 The method of characteristics implements parcel tracking as a solution method

Advection transports, and in the simplest case, translates, fluid property fields. To begin to examine this, consider the case of a constant (spatially and temporally uniform) velocity, $u \geq 0$, and some scalar property of the fluid, say T , in R^1 . Assuming that there is no body force for T , then the Eulerian (field) equation for the evolution of T reads

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} = 0. \quad (128)$$

In Lagrangian (material) terms this states that T is constant in time on each parcel, though T could well be a different constant on each parcel. If we could specify T for each parcel at some point along its trajectory, and if in addition we could solve for the trajectories of all parcels, we would have a complete solution. Of course, solving for the trajectories is, in general, a formidable task, but there are idealized flows that are both easily solved and illuminating.

For the problem at hand we will assume that $T(x)$ is known along the line $(x, t = 0)$,

$$T(x, t = 0) = H(x) = B \exp(-x^2/2W^2), \quad (129)$$

where $H(x)$ is a Gaussian hump of amplitude $B = 1$ and width $W = 1$. Advection by the uniform velocity U will act to shift each parcel to the right at the rate U and according to our governing equation (128), each parcel retains its initial T . Since all parcels move with the same speed, the initial profile $T(x)$ is unchanged in width or amplitude. Advection by a uniform velocity amounts to a simple translation of the T field, Fig. (21).

The field $T(x, t)$ is constant in the x, t plane along lines given by $x - Ut = \text{constant}$ that are said to be the *characteristics* of the PDE, Eq. (128). If we set

$$T(x, t) = f(x - Ut), \quad (130)$$

where $f(x, t)$ is any differentiable function, this $T(x, t)$ satisfies the governing PDE, Eq. (128). If we further set $f = H$ we can satisfy the initial condition, and thus $T(x, t) = H(x - Ut)$ is the solution to the problem posed (as you should verify by substitution into the governing equation).

This is a nearly trivial example, of course, but it illustrates the basis of a powerful solution technique for first order PDEs, including some nonlinear forms, called the method of characteristics.⁵⁵ The great appeal of this method is that it takes a Lagrangian (parcel tracking) approach to solving the advection equation by recognizing that advection alone does not change the scalar T of a given parcel, though, of course, advection can change T at a fixed point in space. The solution strategy is then to convert the governing PDE into a set of (usually coupled) ODEs; one of the ODEs will describe the rate of change of the property T along the trajectory of a parcel, and the other ODEs will serve to define the trajectory. When combined, these will give

radiative, then it can be inferred roughly from the diurnal variation due to radiative fluxes. To evaluate advection requires that you monitor the mesoscale temperature in your region and the local wind; the necessary data are available from good weather maps in the newspaper or better, from a weather forecasting center such as FNMOC. The angle between the wind and the gradient of temperature may be rather small even when horizontal advection is quite important, and so the inference of advection may be semi-quantitative, at best. We are not expecting precise quantitative estimates so much as a qualitative discussion of the two-dimensional thermal energy balance sorted upon time scale, i.e., hourly, daily, weekly.

⁵⁵The method of characteristics is described well in many textbooks on partial differential equation. An excellent text that emphasizes numerical solution methods built upon the idea of characteristics is by R. J. LeVeque, *Numerical Methods for Conservation Laws* (Birkhauser Verlag, Basel, 1992). A clear and concise online source is at <http://www.scottsarra.org/shock/shock.html>

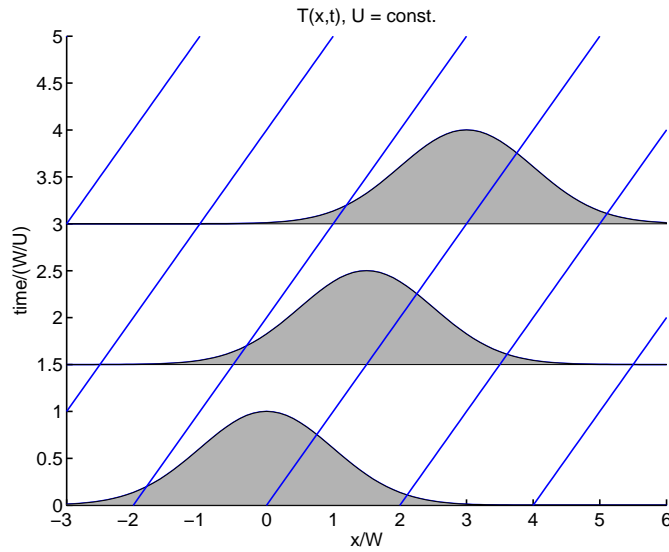


Figure 21: A scalar field, $T(x)$, in a uniform flow $U = 1$; the field T is translated to the right at the rate U . The straight solid lines are the characteristics of the flow, each of which has a slope $dt/dx = 1/U$. Notice that along these lines T is constant.

the full solution. Of course, it isn't every problem that will yield to this method, but some interesting and important ones will.

To start we will seek the path in (x, t) along which the governing equation reduces to an ordinary differential. To do this we will seek the parametric form of a curve $(x(s), t(s))$ where s is distance along the curve. Assuming that there is such a curve, then we can write $T(x(s), t(s))$, and the (directional) derivative along s is just

$$\frac{dT}{ds} = \frac{\partial T}{\partial t} \frac{dt}{ds} + \frac{\partial T}{\partial x} \frac{dx}{ds}. \quad (131)$$

Notice that we can write the differentials of x and t with respect to s as ordinary differentials. Comparing this with Eq. (128), it appears that we can set

$$\frac{dT}{ds} = 0, \quad \text{and thus } T = \text{constant} \quad (132)$$

along this path, provided that the parametric representation of the path satisfies

$$\frac{dt}{ds} = 1 \quad (133)$$

and

$$\frac{dx}{ds} = U. \quad (134)$$

These latter two ODEs define the family of lines along which Eq. (132) holds. The first of these can be immediately integrated to yield

$$t = s, \quad (135)$$

where the integration constant can be set to zero, and the second condition integrates to

$$x = Us + b, \quad (136)$$

where b is the value of x when $t = 0$. Using Eq. (135) this last can be written

$$x = Ut + b. \quad (137)$$

Thus the family of lines along which $T = \text{constant}$ are given parametrically by Eqs. (135) and (136) or by Eq. (137). These lines are called the *characteristics* of the governing PDE, Eq. (128). In this extremely simple problem all of the characteristics have the same slope, $dt/dx = 1/U$, Fig. (21), since we have presumed that U is constant.

Along each characteristic line T remains constant, according to Eq. (132), and to find what constant value holds on a given characteristic we need initial data on each characteristic; here we have Eq. (129), or

$$T(x, t) = H(b).$$

We can then eliminate b using Eqs. (135) and (136), and find the explicit solution,

$$T(x, t) = U_0 \exp(-(x - Ut)^2/2W^2). \quad (138)$$

Thus the field $T(x, t)$ is translated to the right at the constant rate U , as we had surmised already.

The method of characteristics can be applied to real advantage to many first order PDEs, including some that are not linear. For example, consider a problem in which the field being advected is the current U itself,

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0, \quad (139)$$

so that u is now the dependent variable. The initial condition is again presumed to be a Gaussian hump, Fig. (22),

$$u(x, t = 0) = U_0 \exp(-x^2/2W^2). \quad (140)$$

This governing equation, often called the inviscid Burgers' equation, is not linear because the advection term is the product of two unknowns and so the wide array of methods that are available to solve linear PDEs will not be applicable. It is linear in the partial derivatives, however, and the method of characteristics is well-suited for such quasi-linear problems.⁵⁶ We again seek solutions of

$$\frac{du}{ds} = 0$$

and as above find that the characteristics $x(s)$ and $t(s)$ are (straight) lines, Figs. (22) and (23). A significant difference is that the slope of the characteristic lines now varies from characteristics to characteristic since u varies with x . Far from the origin the characteristic lines are nearly vertical in the x, t plane (large slope corresponds to small u) while close to the origin the characteristics have a minimum slope and thus a maximum dx/dt and $1/u$. The solution for this problem can be written,

$$u(x, t) = U_0 \exp(-(x - u(x, t)t)^2/2W^2), \quad (141)$$

which is not separable into an explicit solution (in which u is on the left side and the right side is a function of x and t). It can be readily graphed, however, Fig. (22), and interpretation of the solution is made clear by the characteristics, Fig. (23). The most rapidly moving part of the hump that begins near the origin starts to overtake the slower moving part that starts at larger x ; after some time t_c some characteristics will intersect,

⁵⁶It should be noted that this is not a complete model of any fluid flow in that we have not considered the conservation of volume (or mass) nor the possibility of a pressure gradient. These will be considered in Part II in the context of acoustic and shallow water waves.

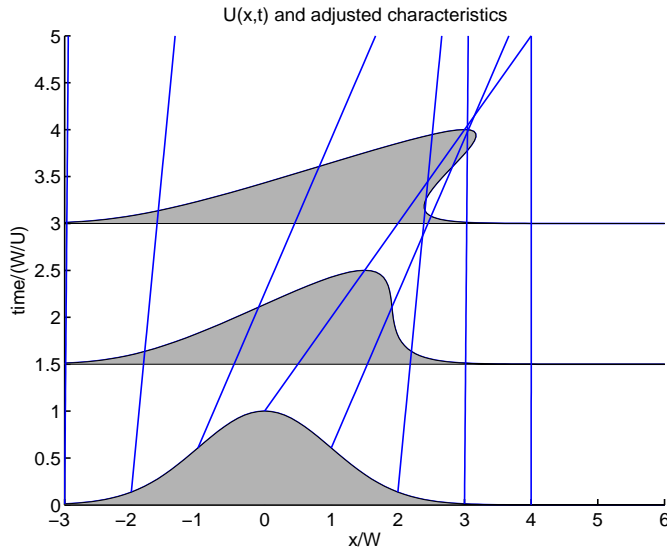


Figure 22: In this case the field being advected is the current itself. Because the initial field is nonuniform (Gaussian), the subsequent advection distorts the pattern and at about $t/(W/U_0) = 1.7$ the field predicted by advection alone becomes multivalued; at that time we could say that the wave has broken. Subsequent evolution has to account for the formation of a shock. Notice that the abscissa in this plot is both time and current amplitude. To show where points of constant current amplitude are found, the lines plotted here are characteristics that have been adjusted upward by the amount $U_0(x, t = 0)$. The next figure shows the characteristics without this adjustment.

and u at that point will there have an infinite derivative; if the method of characteristics is continued, the solution for u will then appear to be triple-valued. In the case that u is the current speed, a multi-valued solution makes no physical sense and has to be rejected.⁵⁷ It isn't that the method of characteristics has failed, but rather that the governing equation (139) has omitted some physical process(es), e.g., diffusion or viscosity, that will become important when the derivative $\partial/\partial x$ becomes large. Even with diffusion present, the derivative may, nevertheless, become large and the flow said to form a shock wave, across which the current speed is nearly discontinuous. The conservation of momentum holds regardless of the details of the field, and the subsequent motion of a shock wave can be determined using fundamental principles.

The method of characteristics can be readily extended to higher dimensions and to problems in which there is some forcing. Suppose that we have a model

$$\frac{\partial T}{\partial t} + u(x, y) \frac{\partial T}{\partial x} + v(x, y) \frac{\partial T}{\partial y} = Q(x, y). \tag{142}$$

The governing equation can be reduced to an ODE:

$$\frac{dT}{ds} = Q \tag{143}$$

⁵⁷To calculate when characteristics will first cross, consider the following problem, the governing equation is the inviscid Burgers' equation, and the initial data is piece-wise linear: $u = U_0$ for $-\infty < x \leq 0$; $u = U_0 - x/L$ for $0 < x \leq L$ and $u = 0$ for $x > L$. Assume that $U_0 > 0$. Sketch the characteristics and the solution at several times, and show that the characteristics starting from $x = 0$ and $x = L$ will cross at $t_c = L/U_0$. In the limit that L is small this can be written $t_c = -1/(\partial U_0/\partial x)$, and for a continuous initial $U_0(x)$ it is plausible that the first crossing will be due to the largest (negative) value of $(\partial U_0/\partial x)$. For the Gaussian of Eq. (140) the crossing is expected at $t_c = (W/U_0) \exp(-1/2)$. Thus, even a small amplitude current pulse will steepen into a shock given enough time, and the practical question is whether other effects — dispersion, spreading and dissipation — will reduce the amplitude sufficiently to avoid a shock. Where in space does the first crossing occur? What would happen in the piece-wise linear case if $U_0 < 0$?

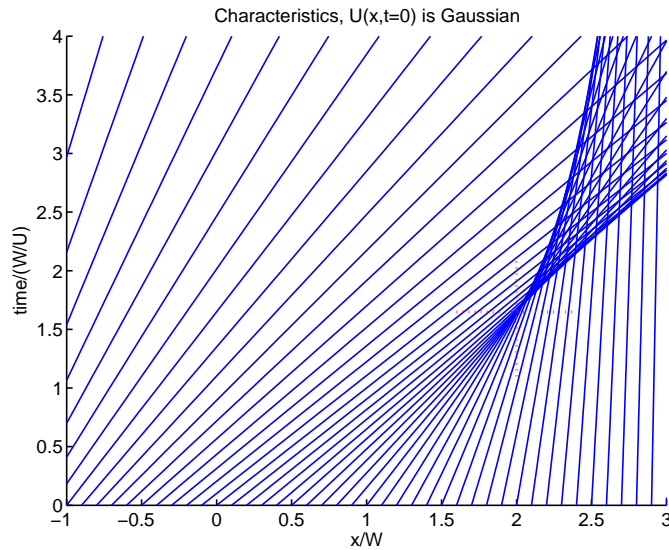


Figure 23: The characteristics of the flow shown in Fig. (22). The short dashed lines denote $x = 2W$ and $t = \frac{W}{U_0} \exp(1/2)$ where characteristics are expected to first cross in time. Since only a few characteristic lines are shown here, the first crossing that actually occurs in this figure is at a slightly later time, and hence is slightly displaced as well.

provided that

$$\frac{dt}{ds} = 1,$$

and

$$\frac{dx}{ds} = u \quad \text{and} \quad \frac{dy}{ds} = v. \quad (144)$$

Notice that Eq. (144) is exactly Eq. (113) for streamlines that we considered in Section 4.3. Thus streamlines and characteristics are one and the same when the velocity is steady and the governing PDE is the advection equation. In more realistic fluid models that include acoustic or gravity waves, the characteristic lines are lines along which certain properties may be constant, but again are not, in general, particle paths (more on this in a later essay).⁵⁸

6.3 A systematic look at deformation due to advection; the Cauchy-Stokes Theorem

You have probably noticed that advection does much more than simply translate parcels from place to place; the flow in a teacup will usually also act to draw out parcels into long, thin streaks (allowing that they have finite size) and will also change their orientation. This second aspect of advection, that it may change the shape and orientation of fluid parcels, is an important part of kinematics that we first considered in Section 1.1, and take up here in some detail.

To pose a definite problem, we will exploit the last result of Section 6.2 to calculate the motion of a parcel (identified with a colored tracer) that is embedded in a steady, clockwise rotating, vortical (circular) flow; either an irrotational vortex, Fig. (24, left), discussed in Section 5.1 and defined by Eq. (115), or a solid

⁵⁸Model PDE systems in which the fields are propagated at a definite, finite rate are said to be *hyperbolic*; in the advective equation (142) the speed is simply the fluid velocity. The elementary wave equation is also hyperbolic, since fields are propagated at the wave phase speed. A system that includes diffusion is said to be *parabolic*; at a given point in space the field will be influenced by the entire domain at all previous times. The identification of a model system as hyperbolic or parabolic is a key step in the design and implementation of efficient numerical schemes (e.g., LeVeque⁵⁵).

body rotation, Fig. (24, right) in which the azimuthal speed increases with radius as

$$\mathbf{V} = (U_r, U_\theta) = (0, \Omega r), \quad (145)$$

where Ω is the uniform rotation rate. It is assumed that $C < 0$ and $\Omega < 0$ so that both vortical flows are clockwise. These are idealizations, of course, and yet with a little effort (imagination?) something akin to both kinds of vortices can be observed in the flow in a teacup: more or less irrotational vortices are observed to spill off the edges of a spoon that is pushed through the fluid, and at longer times the azimuthal motion that fills the teacup will often approximate a solid body rotation (except near the edges).⁵⁹

The first thing to note is that the parcels are transported clockwise with the clockwise flow in either vortex; in these flows there is nothing quite as exciting as Stokes drift (Section 5.2) that can make the Lagrangian mean flow (i.e., what we see as the displacement of the parcel) qualitatively different from the Eulerian mean flow (what you would expect given the field of vector velocity). Aside from that, the effects of advection are remarkably different — the irrotational vortex produces a strong deformation of the parcel, while the solid body rotation leaves the shape of the parcel unchanged. The irrotational vortex leaves the (average) orientation of the parcel unchanged, though this is impossible to verify in Fig. (24) given the large deformation, while the solid body rotation changes the orientation at the rate Ω that characterizes the rotation rate of the vortex. The area of the parcel is unchanged in either case.

These changes in the orientation and shape of a fluid parcel are caused solely by advection and are thus a consequence of the velocity field. Our goal in this section is to find out what specific properties of the velocity field are relevant. Once again we are asking for what amounts to Lagrangian properties — the size, shape, etc. of a fluid parcel — in terms of the Eulerian velocity field, $\mathbf{V}(x, y, z, t)$. To make the analysis tractable we are going to consider flows that are two-dimensional and we will follow the parcel only for short times (unlike the examples of Fig. 24) so that the velocity field can be assumed steady. Given these restrictions, the Eulerian velocity field around a given point, x_0, y_0 can be calculated with sufficient accuracy by the first terms of a Taylor expansion (as in Section 5.1 and repeated here),

$$\mathbf{V}(x, y) = \mathbf{V}(x_0, y_0) + \mathbb{G} \cdot \delta \mathbf{X} + HOT, \quad (146)$$

where $\delta \mathbf{X}$ is a small displacement and

$$\mathbb{G} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

is the velocity gradient tensor, which represents all that we presume to know regarding the velocity field. These restrictions to small displacement and steady flow may seem severe, but in the end we come to results that can be applied to the differential (Eulerian) conservation equations, which is just what we need.

The velocity gradient tensor is the center of attention for now, and we'd like to know what it looks like. We can not make a diagram of a tensor *per se*, but we can show what \mathbb{G} does when it operates on a

⁵⁹The irrotational vortex has a singularity in U_θ at $r = 0$, while in a solid body rotation U_θ grows linearly and without bound with r . A hybrid made from these two idealizations — solid body rotation near the center of a vortex, Eq. (145), matched to an irrotational profile $U_\theta(r)$ from Eq. (115), that continues on for larger r — avoids both problems and can make a convenient, useful approximation to a real vortex, e.g., a hurricane. This kind of hybrid is called a Rankine vortex.

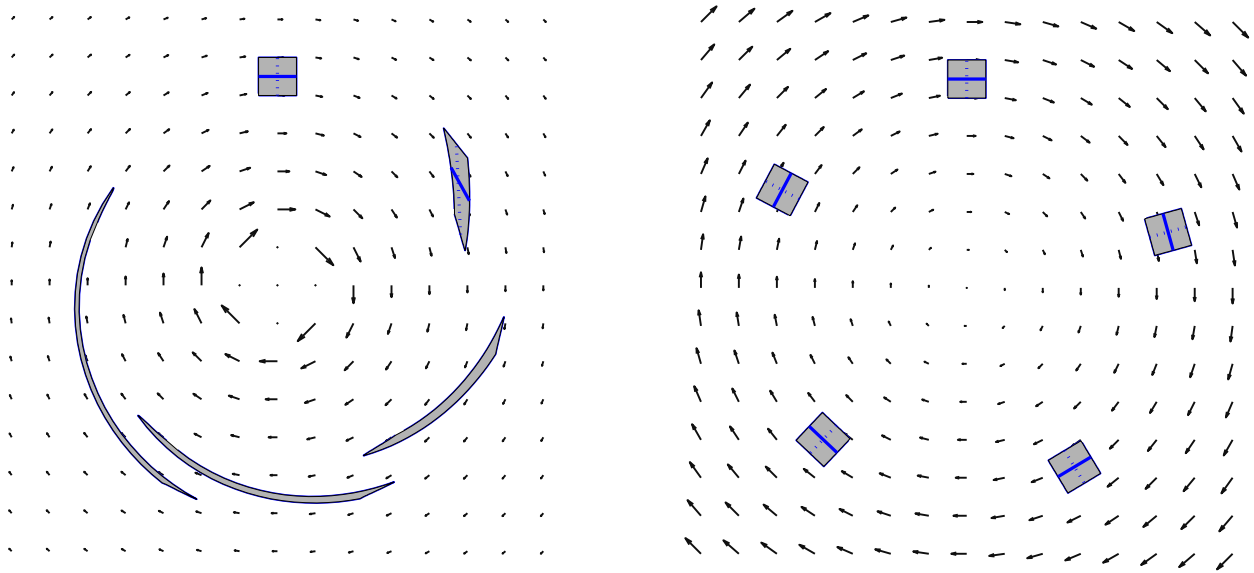


Figure 24: (a) A small patch of colored tracer, or parcel, has been set into a steady, clockwise rotating irrotational vortex and advected through most of one revolution. The largest velocity vectors near the center of the vortex were not plotted. The parcel was square, initially (at 12 o'clock) and then rather severely deformed by this flow. Nevertheless, the area of the parcel was conserved, as was its average orientation (though the orientation is obscured by the large deformation). (b) In this experiment the vortex flow was a solid body rotation. As the name implies, this motion could just as well be that of a solid, rotating object. The orientation of the parcel changes in time, but the area and the shape are conserved, i.e., there is no deformation.

displacement vector, and that is what counts. We will illustrate this with the simplest shear flow,

$$\mathbf{V}(x, y) = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} U_0 + \Delta y \\ 0 \end{pmatrix}, \quad (147)$$

where the spatial mean flow is $U_0 = 0.5$ and the shear is $\partial u / \partial y = \Delta = 0.7$ and constant (notice that the units of shear are time^{-1} , which we can leave unspecified in the present discussion, and recall the example of the viscous, laminar flow sketched in Fig. (4, left)). A fluid parcel embedded in this flow evolves as shown in Fig. (25). The velocity gradient tensor evaluated at any point in this flow is just

$$\mathbb{G} = \begin{pmatrix} 0 & \Delta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0.7 \\ 0 & 0 \end{pmatrix}. \quad (148)$$

To see what \mathbb{G} does, we can multiply \mathbb{G} into a set of unit vectors, \mathbf{e} , that span 360 degrees in direction, and plot the resulting velocity difference, $\delta \mathbf{V} = \mathbb{G} \mathbf{e}$, at the end of each unit vector, Fig. (26a). The velocity difference plotted in this way looks a lot like the velocity field itself since the shear is presumed spatially uniform. However, these diagrams show properties of the velocity field at the specific point where \mathbb{G} has been evaluated, and are not a map of the velocity field *per se*.

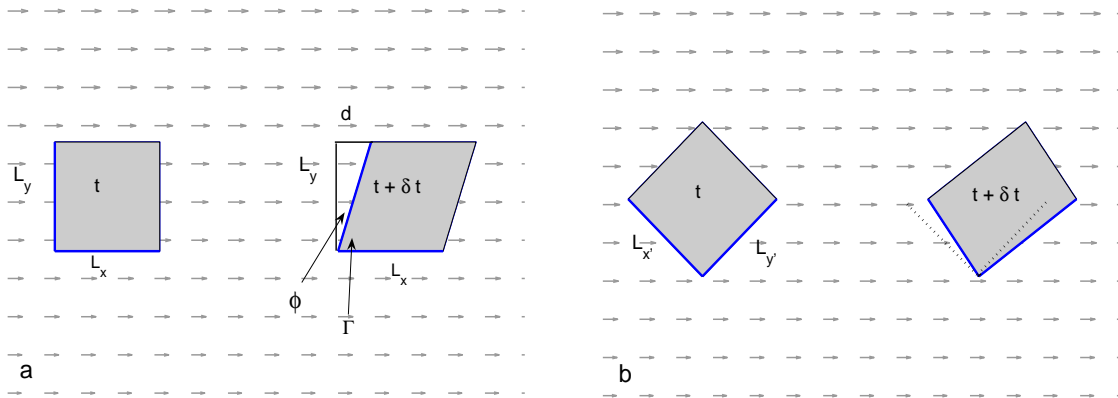


Figure 25: (a) A square (initially) parcel has been set into the shear flow defined by Eq. (147) and advected for a time interval, δt . The sides on the lower and left edges of the parcel were initially orthogonal, and of length L_x and L_y . After a time interval δt , the upper left corner has been displaced a distance $d = \delta t L_y \partial u / \partial y$ with respect to the lower left corner and so the left edge of the parcel has rotated clockwise through an angle $\phi \approx \tan \phi = -d/L_y$. The angle ϕ thus changes at the rate $d\phi/dt \approx \phi/\delta t = -\partial u/\partial y$, as long as ϕ is small. (b) The same flow and the same parcel, but compared to the example at left, the initial orientation of the parcel was rotated by 45 deg. The original lower left and lower right sides are shown at $t + \delta t$ as the dotted lines. In this case the angle defined by the lower left and lower right edges remains 90 deg, while the length of these sides is compressed or stretched. Evidently this particular shear flow has both a shear deformation rate, which is evident at left, and a linear deformation rate that is evident at right. An orthogonal axes pair, e.g., the lower and left sides of this parcel, will sample one or the other (or a little of both) of these deformation rates depending upon their orientation with respect to the flow.

6.3.1 The rotation rate tensor

Consider a spatial variation of the velocity that could cause the sides of the parcel to change orientation: in the example of Eq. (147) and Fig. (25a), the u component of velocity increases with increasing y , and this causes the left and right sides of the parcel to rotate clockwise. If we denote the angle of the left side of the parcel with respect to the y -axis by ϕ , then by simple geometry we can see that

$$\frac{d\phi}{dt} = -\frac{\partial u}{\partial y}.$$

Similarly, if we denote the angle between the lower side of the parcel and the x -axis by ν , then the lower side of the parcel would rotate counter-clockwise if the y -component of velocity increased with x , i.e.,

$$\frac{d\nu}{dt} = \frac{\partial v}{\partial x}$$

(the angle $\nu = 0$ in this figure, and so you should make a sketch to verify this, keeping in mind that the angle ν can be assumed to be small). The angles ν and ϕ may change independently. A sensible measure of the average rotation rate of the parcel, $\bar{\omega}$, also called the physical rotation rate, is the average of these angular rates,

$$\bar{\omega} = \frac{1}{2} \left(\frac{d\nu}{dt} + \frac{d\phi}{dt} \right) = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right). \quad (149)$$

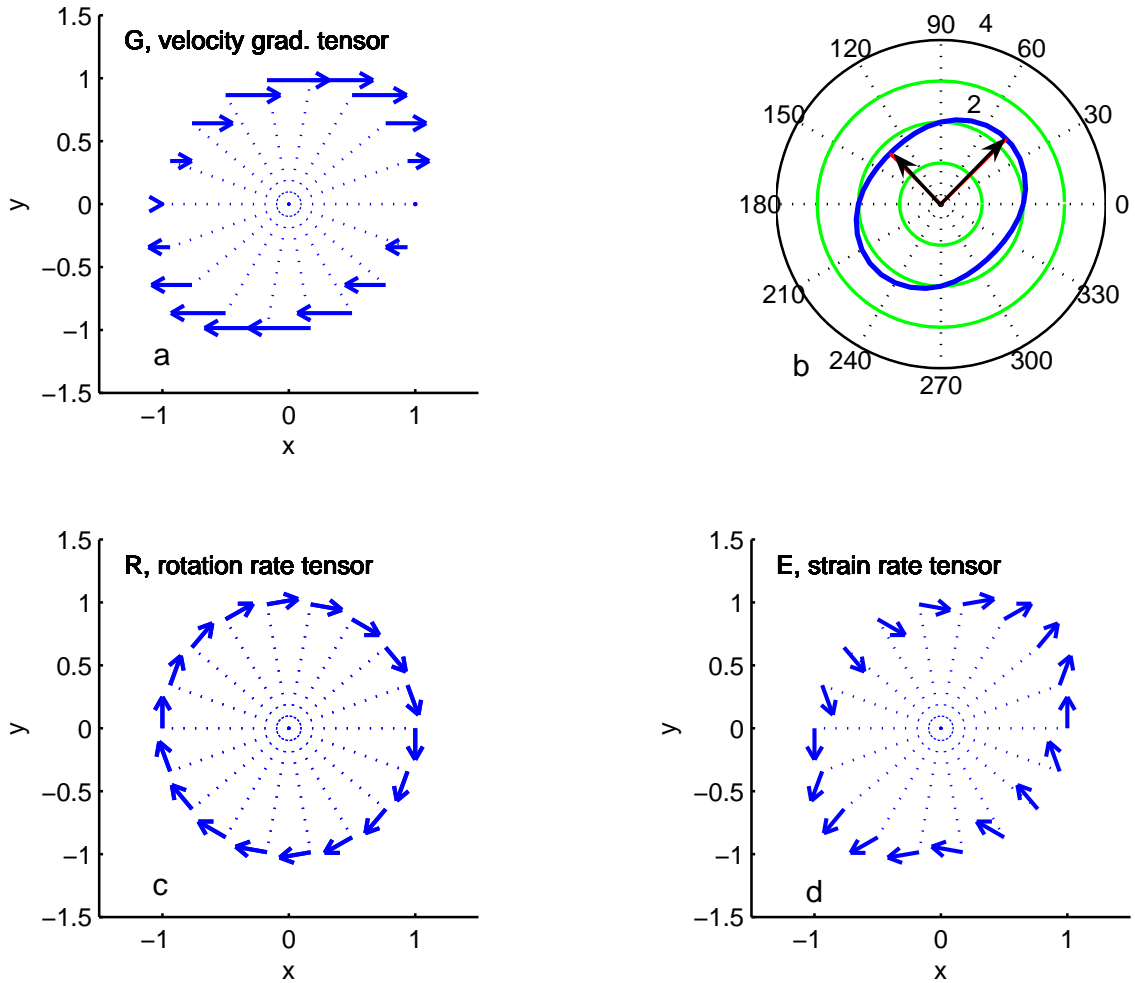


Figure 26: (a) The velocity gradient tensor \mathbb{G} , Eq. (148), has been multiplied into a sequence of unit (displacement) vectors with varying directions (the dotted lines) and the resulting velocity plotted at the end of the unit vectors. This is a useful way to show what the velocity gradient tensor does, but keep in mind that all of this should be envisioned to hold at a single point. (b) The eigenvectors of the deformation rate tensor \mathbb{E} are shown as the vectors, and the linear deformation rate + 2 is the peanut-shaped ellipse plotted in a radial coordinate system. The maximum and minimum values of the linear deformation rate are aligned with the eigenvectors and the value of the maximum and minimum linear deformation rate is equal to the eigenvalues. (c) The rotation rate tensor \mathbb{R} has been multiplied into a sequence of unit vectors. (d) The deformation rate tensor \mathbb{E} multiplied into the unit vectors. Notice how the sum of the velocity vectors in (c) and (d) compares with the velocity vectors of (a), and notice too how $e \cdot \delta V$ compares with the linear deformation rate shown in the upper right panel.

A small, rigid orthogonal vane embedded in the flow would rotate at this rate (assuming that the force on the vanes is linear in the velocity). For many purposes it is convenient to use a measure of the rotation rate called the vorticity, $\chi = 2\bar{\omega}$, which is just twice the physical rotation rate,

$$\chi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \nabla \times \mathbf{V} \quad (150)$$

The vorticity is invariant to the orientation of the parcel with respect to the flow, and hence is invariant to a rotation of the coordinate system. In R^2 vorticity is effectively a scalar, i.e., a single number; in R^3 the vorticity $\nabla \times \mathbf{V}$ is a vector with three components. The vorticity of a fluid is analogous in many respects to the angular momentum of a rotating solid and because it follows a particularly simple conservation law (no pressure effects!) vorticity often makes an invaluable diagnostic quantity.

6.3.2 The deformation rate tensor

In the case that the two angles ν and ϕ change at different rates, then our parcel will necessarily change shape or deform. One of several plausible measures of the shape of the parcel is the angle made by the lower and left sides, Γ , and evidently $\Gamma = \pi/2 + \phi - \nu$, Fig. (25a). If ϕ and ν change by the same amount, then $\Gamma = \text{constant}$, and the parcel will simply rotate without deforming; this is what we could call a solid body rotation, considered just above. But if the angles change at a different rate, then the shape of the parcel will necessarily change, and

$$\frac{d\Gamma}{dt} = \frac{d\nu}{dt} - \frac{d\phi}{dt} = \frac{\partial \nu}{\partial x} + \frac{\partial u}{\partial y}. \quad (151)$$

If the angle ϕ alone changes while the lengths of the parcel sides remain constant, then this kind of time-varying deformation is called a *shear* deformation (or strain) rate. This is the case in Fig. (25a) when the angle ϕ is small, i.e., for small times after the parcel is released in the flow. This particular shear deformation rate is in the plane parallel to the x-axis (and recall the shear deformation noted in Section 1.1; in that case the deformation was due partly to the no-slip lower boundary condition).

But suppose that we rotate the parcel by 45 degrees before we release it into the same shear flow. The result of advection then appears to be quite different if we continue to emphasize an orthogonal axes pair (Fig. 25b); the sides of the parcel remain orthogonal, but now the lengths of the sides are compressed (lower left side, $d(L_{x'})/dt < 0$) or stretched (lower right side, $d(L_{y'})/dt > 0$). A deformation rate that causes a change in the length of a material line is termed a *linear* deformation rate, e.g., in the x'-direction or y'-direction,

$$\frac{dL_{x'}}{dt} = L_{x'} \frac{\partial u}{\partial x'} \quad \text{and} \quad \frac{dL_{y'}}{dt} = L_{y'} \frac{\partial v}{\partial y'}. \quad (152)$$

Divergence, another important quantity that we have encountered already in Section 6.2. is the sum of the linear deformation rates measured in any two orthogonal directions and gives the normalized rate of change of the area of the parcel, $A = L_x L_y$, or rewriting Eq. (62),

$$\frac{1}{dA} \frac{d}{dt}(dA) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \nabla \cdot \mathbf{V}.$$

Unlike the vorticity and the divergence, the deformation rate can not be written as a vector operator on the velocity, and from Fig.(25) it seems that the deformation rate that we diagnose with a given orthogonal

axes pair (the sides of the parcel), is entirely dependent upon the orientation of these axes with respect to the flow. It is highly unlikely that a quantity that depends entirely upon the orientation of the coordinate system can have any fundamental role, and this in turn implies there is more to say about the deformation rate than Eqs. (151) and (152) taken separately.⁶⁰

The deformation rate may change the shape and the size (area) of a parcel, while rotation alone can not. This suggests that there is something fundamentally different in these quantities, and that it may be useful to separate the rotational part of the velocity gradient tensor from all the rest. This turns out to be straightforward because the rotation is associated with the anti-symmetric part of the velocity gradient tensor, and any tensor can be factored into symmetric and anti-symmetric component tensors by the following simple procedure.

Let \mathbb{G}' be the transpose of \mathbb{G} ,

$$\mathbb{G}' = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

We can always subtract and add \mathbb{G}' from \mathbb{G}

$$\begin{aligned} \mathbb{G} &= \frac{1}{2}(\mathbb{G} - \mathbb{G}') + \frac{1}{2}(\mathbb{G} + \mathbb{G}') \\ &= \mathbb{R} + \mathbb{E}, \end{aligned} \tag{153}$$

and thereby decompose \mathbb{G} into two new tensors

$$\mathbb{R} = \frac{1}{2} \begin{pmatrix} 0 & \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} & 0 \end{pmatrix} \tag{154}$$

and

$$\mathbb{E} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \\ \frac{1}{2}(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) & \frac{\partial v}{\partial y} \end{pmatrix}. \tag{155}$$

\mathbb{R} is called the rotation rate tensor and is anti-symmetric, $R_{12} = -R_{21}$ (this is a new and distinct use of the symbol \mathbb{R} compared with that of Section 6.2.1); \mathbb{E} is called the deformation rate tensor and is symmetric, $E_{12} = E_{21}$. In R^2 , \mathbb{R} has only one unique component that is proportional to the rotation rate ω . When \mathbb{R} is multiplied onto a set of unit vectors the result is a velocity difference $\delta\mathbf{V} = \mathbb{R}\mathbf{e}$ that is normal to the unit vector and has the same amplitude (same speed) for all directions of the unit vectors. The rotation associated with $\mathbb{R}\mathbf{e}$ is apparent, Fig. (26, lower left), and the magnitude is just $\omega = -\frac{1}{2}\partial u/\partial y = -0.35$; the vorticity of this shear flow is then $2\omega = -0.7$.

Eigenvectors of the deformation rate tensor. The strain rate tensor \mathbb{E} has three independent components (in general) and is a little more involved. The resulting velocity difference, $\delta\mathbf{V} = \mathbb{E}\mathbf{e}$, varies in direction and amplitude depending upon the direction of the unit vector (though in the specific case shown in Fig. (26d) the amplitude happens to be constant). The linear deformation rate in a given direction is given by the component of $\delta\mathbf{V}$ that is parallel or anti-parallel to the unit vector in that direction. There are two special directions in

⁶⁰It was noted above that the rotation rate defined by Eqs. (149) or (150) is independent of the orientation of the axes with respect to the flow. Can you verify this (semi-quantitatively) from Fig. (25)?

which the linear deformation rate is either a minimum or a maximum. Given the specific \mathbb{G} of Eq. (148), the minimum linear deformation rate is -0.35 and is found when the unit vector makes an angle of 135 (or 315) degrees with respect to the x axis (Fig. 26b); the maximum linear deformation rate is 0.35 and is at 45 (or 225) degrees. Thus a parcel will be compressed along a line that is oriented 135 degrees (with respect to the x axis) and will be stretched along a line normal to this, 45 degrees. Notice too that when the unit vector is pointing in these special directions the velocity difference and the unit vector are either anti-parallel or parallel, and the relationship among \mathbb{E} , e , and δV may be written

$$\mathbb{E}e = \delta V = \lambda e,$$

where λ is a real number. These directions are thus the directions of the *eigenvectors* of the symmetric tensor \mathbb{E} , and the amplitude of the linear deformation rate in those directions is given by the corresponding *eigenvalues*, $\lambda = -0.35$ and $+0.35$ for this particular \mathbb{E} .⁶¹ The divergence, $\nabla \cdot V = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$, is the sum of the eigenvalues, and also the trace of \mathbb{E} ; in the specific case of the shear flow Eq.(147), the divergence happens to be zero though the linear deformation rate in most directions is not. Like the rotation rate, the divergence is invariant to rotation of the coordinate system.

Stirring and mixing. There can be a significant deformation rate even when the divergence vanishes, as it does in the case of the irrotational vortex Fig. (24.a), the plane shear of Fig. (25.a) or to an excellent approximation, the flow in a tea cup. From the ‘conservation form’ of the Eulerian budget equations (Section 6.2) we can be sure that the total amount of a tracer is thus unaffected by advection and deformation rate, and the same holds also for the mean square of the tracer (when divergence vanishes). What has changed, however, is that the mean square of the *gradient* of the tracer field, $\propto \oint (\nabla T)^2 dArea$, with T the tracer density, is likely to be increased dramatically by a sustained or a random deformation rate. Molecular diffusion acting upon the greatly increased surface area of the long, thin streaks produced by a sustained or random deformation rate can then act much more effectively to mix the tracer with the surrounding fluid, and so produce a homogeneous equilibrium more rapidly than would occur in the absence of deformation (or straining). At this level of detail, there is a significant difference between molecular diffusion, which acts to reduce tracer gradients, and a deformation rate, often called ‘stirring’ when it is random, which acts to increase mean square gradients.

6.3.3 The Cauchy-Stokes Theorem collects it all together

In the discussion above we have sketched out The Cauchy-Stokes Decomposition Theorem:

An instantaneous fluid motion may, at each point in space, be resolved into three components;

- 1) a *translation* — the velocity $V(x_0, y_0)$ of Eq. (146),
- 2) a *rigid body rotation* — the amplitude of which is given by the off- diagonal elements of the rotation rate tensor, \mathbb{R} , and,

⁶¹Eigenvectors and eigenvalues are a central theme of linear algebra and will not be reviewed here. Excellent references are by J. Pettofrezzo *Matrices and Transformations* (Dover Pub., New York, 1966) and for undergraduate-level applied mathematics generally, M. L. Boas, *Mathematical Methods in the Physical Sciences, 2nd edition* (John Wiley and Sons, 1983). If you have access to Matlab, a search on ‘eigenvector’ will return several useful, concise tutorials.

3) a linear deformation rate along two mutually perpendicular directions — the directions and the amplitudes of the maximum and minimum linear deformation rate are given by the eigenvectors and eigenvalues of the deformation rate tensor, \mathbb{E} .

In most real fluid flows the rotation rate and the deformation rate will vary spatially and in time, right along with the velocity itself, and each of them have important roles in Eulerian theories that seek to predict the evolution of fluid flows: the deformation rate is proportional to the rate at which adjacent fluid parcels are slide past one another. Since the solid body-rotational part of the velocity gradient tensor does not contribute to linear or shear deformation rate, it may as well be subtracted from the velocity gradient tensor before computing the viscous stress (Section 2.2.3). Thus the viscous stress tensor is often written $\mathbb{T} = \nu\mathbb{E}$, which shows that the stress tensor is symmetric. We will see these tensors and these concepts again.⁶²

7 Lagrangian observation and diagnosis of an oceanic flow.

The ocean current observed by the cluster of SOFAR floats shown on the cover image and in Fig. (27) may be analyzed fruitfully by a macro-scale application of the Cauchy-Stokes Theorem. The east and north gradients of the east and north velocity components were estimated by fitting a 2-dimensional plane to each velocity component at daily intervals.⁶³ The vorticity (twice the rotation rate) and the divergence may then be estimated by sums and differences of the horizontal shears, exactly as noted above in Sections 6.4.1 and 6.4.2. These estimates are Lagrangian in the sense that the floats act as a tag on a specific material volume that we can follow for an extended period of time. As we will describe below, the motion of this float cluster appears to have been representative of the full water column.

The vorticity of a fluid column is analogous to the rotation rate of a spinning, solid column, and the conservation of vorticity is closely analogous to the conservation of angular momentum. Many fluid

⁶²So soon? 1) Go back and take another look at the (stirred) fluid flow in a teacup. Do you see evidence of divergence, rotation or deformation rate? 2) Compute the velocity gradient tensor, and the associated divergence, rotation, and deformation rates for the case of an irrotational vortex, and for the case of a solid body rotation. You may compute the derivatives either analytically or numerically, and use Matlab to calculate the eigenvalues and vectors. Choose several points at which to do the calculation and interpret your results in conjunction with Fig. (24). 3) The Cauchy-Stokes Theorem is useful as a systematic characterization and interpretation of the velocity expansion formula, Eq. (146), and especially of the velocity gradient tensor. A characterization of the deformation rate tensor by its eigenvectors/values is surely the most natural, but suppose that we have a special interest in the shear deformation rate, or, perhaps we just want to be perverse — can you state an equivalent theorem in terms of the shear deformation rate? 4) Familiarity with tensors sometimes breeds a certain affection for them, and we decide to make yet another one: move the trace (the divergence) of the deformation rate tensor into what we might call the divergence tensor, \mathbf{D} , i.e., $\mathbf{D}(1, 1) = \mathbf{D}(2, 2) = \frac{1}{2}(\nabla \cdot \mathbf{V})$, and $\mathbf{D}(1, 2) = \mathbf{D}(2, 1) = 0$. What properties does this new tensor \mathbf{D} have? What simplification results so far as the eigenvectors/values of the deformation rate tensor are concerned? 5) In a similar vein, the velocity gradient tensor in R^2 has four independent elements, $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$ and $\partial v/\partial y$. We have seen that three of the four possible combinations of these terms have real importance, the divergence, $\partial u/\partial x + \partial v/\partial y$, the rotation rate or vorticity, $\xi = \partial v/\partial x - \partial u/\partial y$, and the shear deformation rate, $\partial v/\partial x + \partial u/\partial y$ of Eq. (151). Can you interpret the fourth possible combination, $\partial u/\partial x - \partial v/\partial y$? (Don't expect something of cosmic significance.)

⁶³At a given time we have measurements of say the east or u -component of velocity at $n = 12$ float locations, (x_i, y_i) , where i denotes a specific float. Define the average of u , x , and y over this cluster to be $\bar{u} = \frac{1}{n}\sum u_i$ and similarly for \bar{x} and \bar{y} . Now fit a linear function of x and y to the cluster of float-measured $u(x_i, y_i)$ by minimizing the mean square difference $\sum ((u(x_i, y_i) - (u_0 + \alpha(x_i - \bar{x}) + \beta(y_i - \bar{y})))^2$ to find the best fit values $\alpha = \partial u/\partial x$ and $\beta = \partial u/\partial y$. The same is done for the v component of the velocity. Thus the float cluster is used to make a single estimate of the velocity gradient tensor on each day. The goodness of fit may be diagnosed by the fraction of the velocity variance accounted for by the fit (termed FIT in Fig 29).

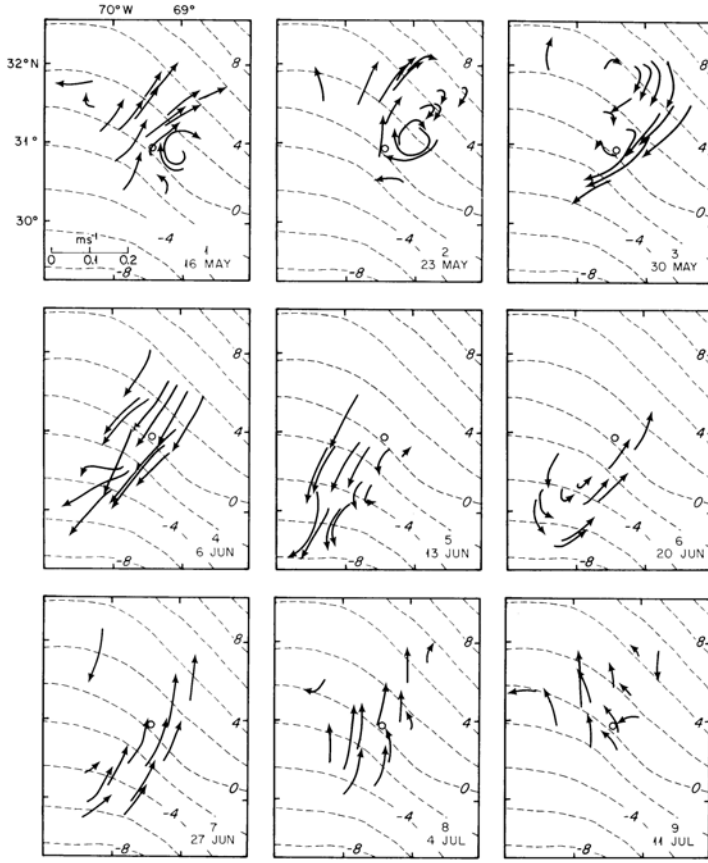


Figure 27: Maps of SOFAR float trajectories at weekly intervals. The region is the central Sagasso Sea, about 700 km west-southwest of Bermuda, and these floats were at a depth of 1300 m. A speed scale is in the upper left panel. The dashed contour lines in the background are lines of constant f/H in units of percent of f . The center of this figure just happened to be the location where the relative vorticity of the motion was approximately zero, and the zero of f/H is assigned there. There are two obvious kinds of motion evident here, the dominant motion is a northeast to southwest oscillation, the potential vorticity aspects of which are described in the main text and also a small, anticyclonic eddy centered on 31 N 69 W (on 16 May) that is not dealt with here. The wave-like aspects of this oscillation are described by Price and Rossby (1982).

phenomenon are illuminated by analysis of vorticity, and nowhere is this more true than in geophysical fluid mechanics. Indeed, most understanding of geophysical flows comes by understanding the distribution and the balance of the potential vorticity, Ψ , which for a layer of thickness H is⁶⁴

$$\Psi = \frac{f + \chi}{H}, \quad (156)$$

where f is the Coriolis parameter. In the present context of vorticity balance, f is dubbed appropriately the ‘planetary vorticity’ because even when a fluid column is at rest with respect to Earth-bound observers, i.e., there is no horizontal wind or ocean current, it will nevertheless be rotating with respect to an inertial observer by virtue of Earth’s rotation. The magnitude of this Earth-induced vorticity is given by the projection of Earth’s rotation vector onto the unit normal of the horizontal face of the column,

$$f = \boldsymbol{\Omega} \cdot \mathbf{n} = 2\Omega \sin(\text{latitude}),$$

where $\Omega = 2.729 \times 10^{-5} \text{ rad s}^{-1}$ is Earth’s rotation rate (Fig. 28). Thus, for a fluid column sitting directly over the north pole, the planetary vorticity is twice the Earth’s rotation rate, $f = 2\Omega$; for a column at 30 N,

⁶⁴This potential vorticity for a single layer may be derived from the momentum and mass conservation equations for a rotating layer in just a few steps, see, e.g., Tritton of footnote 17. Here we will argue the plausibility of Eqs. (156) and (157) by analogy with the conservation of angular momentum of a spinning column.

the planetary vorticity is $f = \Omega$ and for a column on the equator, the planetary vorticity vanishes and $f = 0$ and column sitting over the south pole would have a planetary vorticity $f = -2\Omega$. If f is the planetary vorticity, then it is appropriate to call $\chi = \nabla \times \mathcal{V}$ the ‘relative vorticity’, since \mathcal{V} is the fluid velocity relative to the Earth. An inertial observer will observe that the vorticity of a column is the sum of planetary and relative vorticity, called the ‘absolute vorticity’ $\mathcal{Y} = f + \chi$, while an Earth-bound observer will see only the relative vorticity, χ .

The potential vorticity Ψ is an intensive, scalar property of a fluid, and aside from what are usually small effects of bottom drag or diffusion, the Ψ of a given fluid column is conserved,

$$\frac{d\Psi}{dt} = 0. \quad (157)$$

Often our interest is in the balance between changes in relative vorticity and planetary or stretching vorticity (described below) and so it is useful to consider the difference form;

$$\delta\chi = -\delta f + (f + \chi)\frac{\delta H}{H}, \quad (158)$$

where

$$\delta(\) = \int_{t_0}^t \frac{d}{dt}(\) dt$$

is the change in time following a given float cluster and presumably a material volume. Note that no one part of the expanded potential vorticity is conserved, but only the sum of the three. To make a complete Lagrangian model (as in the method of characteristics, Sec. 6.3) we would next have to write down an equation for the column trajectory. In this case, we observed the float trajectories and so to understand why the relative vorticity changed we need consider only Eq. (158) or equivalent (ah, the joys of Lagrangian *measurement*). Two properties of Ψ make it extraordinarily useful for the analysis of many geophysical flows: (1) The potential vorticity is transported with the fluid velocity (rather than with gravity wave phase speeds, as applies for momentum), so that Ψ serves to tag the fluid, much like temperature or salinity. (2) Ψ is nevertheless closely related to the velocity, and in many cases knowledge of the distribution of Ψ is sufficient to recover the horizontal velocity (though not when gravity waves are present).

Assuming that the moment of inertia (volume times radius²) of a fluid column is constant (it generally will not be) then the absolute vorticity of a column will be conserved, $d\mathcal{Y}/dt = 0$, or $\mathcal{Y} = f + \chi = \text{constant}$, including when a column changes latitude. For example, suppose that a fluid column is at rest at a mid-latitude so that it has a planetary vorticity f_0 but no relative vorticity, and thus its absolute vorticity is just $\Psi = f_0$. If this column is displaced toward the pole, as more or less happened to the float cluster starting on ca. 16 May (Figure 27), then the planetary vorticity of the column will increase. To maintain a constant absolute vorticity, $\mathcal{Y} = f + \chi = f_0$, the column will thus acquire negative or anticyclonic relative vorticity,⁶⁵ $\delta\chi = -\delta f$, the amount given by the change in the planetary vorticity (the Coriolis parameter). Notice that as the float cluster moved northeastward, it seemed to acquire clockwise or anticyclonic relative vorticity, and after a couple of weeks the entire cluster turned and moved back toward the southwest. As the float cluster moved equatorward it acquired positive or cyclonic relative vorticity, which in

⁶⁵Relative vorticity or rotation that is in the same direction as Earth’s rotation is said to *cyclonic*, which comes from the Greek *kyklon* for circular motion. Cyclonic rotation is thus counterclockwise in the northern hemisphere and clockwise in the southern hemisphere. *Anticyclonic* is the reverse.

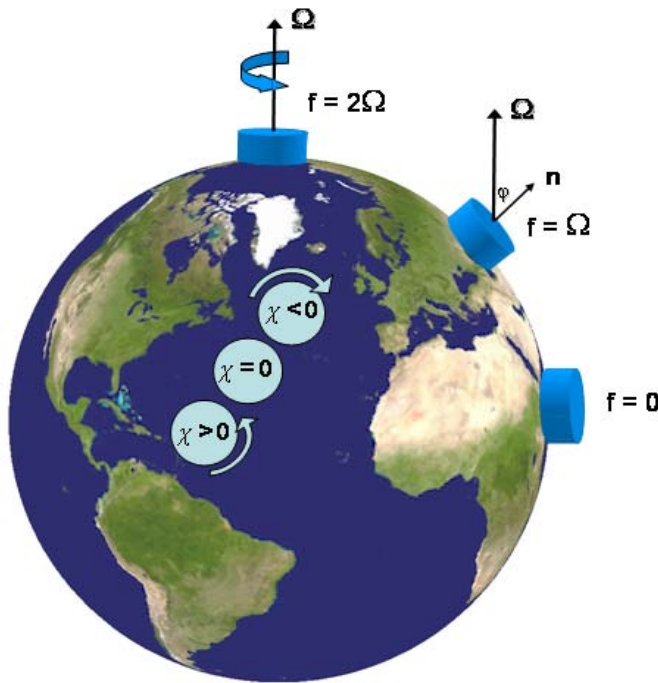


Figure 28: A rotating Earth and schematic fluid columns. The darker blue columns at right show that the horizontal rotation of a column due to Earth's rotation is $f = 2\Omega \sin(\text{latitude})$; the angle ϕ between the unit normal and Earth's rotation vector Ω is the colatitude. The three lighter blue columns show the sense of relative vorticity acquired by a column that is displaced north and south while maintaining constant thickness. It is natural to show a column spinning about its center when it acquires relative vorticity, as here. In fact, the float cluster of Fig. (27) indicates that the relative vorticity was due mainly to horizontal shear rather than curvature, as implied by these spinning columns.

this case served to erase the negative relative vorticity acquired during the poleward displacement. An Earth-bound observer (i.e., what we see in Fig. 27) will note that the cluster lost and gained relative vorticity as it oscillated north to south. An observer who witnessed this motion from an inertial reference frame would note that the absolute vorticity of the cluster did not appear to change (aside from thickness changes, which we will discuss next), though she would be able to tell that the absolute vorticity of this cluster was different from the surrounding water (assuming that it is at rest).

Another and often important aspect of vorticity balance is that column thickness will change if there is a divergence of the horizontal velocity within the fluid column, as would occur if the column moved over variable water depth. For a given column, the moment of inertia is $\propto 1/H$, and the absolute angular momentum is thus $\propto \Upsilon/H$, which is just the potential vorticity noted above. If the thickness changes, then so too does the moment of inertia and the absolute vorticity, exactly as would be expected from angular momentum conservation. Thus when a column is squashed it will slow its rotation rate (aside from changes in latitude and thus a change in planetary vorticity), and the reverse must happen if a column is stretched.⁶⁶ In the example of the float cluster described here, the water depth shoaled gently toward the east and thus when the column moved northeast, it was apparently squashed by about 100 m out of a nominal thickness of 5000 m, or by roughly 2 percent, while at the same time it also went to larger f , by about 3 percent. Thus the changes in thickness and planetary vorticity were roughly comparable and in phase for this particular motion (Fig. 29).

It is noteworthy that a change in f/H of only about 5 percent was sufficient to account for the rather impressive change in the relative vorticity, about 5 percent of f_0 , and by extension, of the current, which had

⁶⁶The conservation of potential vorticity makes such a useful and vivid diagnostic tool that it seems to *govern* these motions. However, potential vorticity conservation is probably better thought of as our guide to geophysical fluid mechanics rather than the governing dynamics *per se*, since it is once removed from the momentum equations, which more nearly govern the motion.

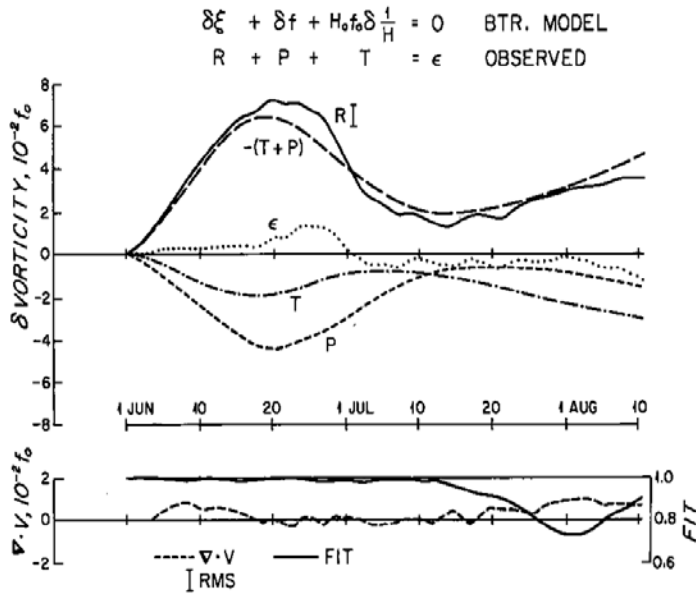


Figure 29: The balance of barotropic potential vorticity estimated from the float data of the previous figure. The estimates begin on 1 June, as the float cluster moved southwestward through the center of the oscillation. The lower panel shows the divergence of the horizontal velocity along with FIT, the fraction of the velocity variance accounted for by the fitted plane; so long as FIT exceeds about 0.9, the velocity field is described well by a mean and constant shears in x and y .

an amplitude of roughly 0.15 m s^{-1} . Thus what would seem to be comparatively small changes in latitude or column thickness result in significant changes in the relative vorticity and the current itself. The reason, hinted at above, is that the planetary vorticity f is generally much larger than the relative vorticity of most ocean currents (or winds) and hence the planetary vorticity represents a large ‘potential’ of the relative vorticity, which can be released by changing either the latitude, and thus the Coriolis parameter, f , or column thickness.⁶⁷

8 Concluding remarks; where next?

The broad goal of this essay has been to introduce some of the central concepts of kinematics applied to fluid flows, and especially to develop an understanding of Eulerian and Lagrangian representations of fluid flow. The starting point is the so-called Fundamental Principle of Kinematics, or FPK, (Section 1.2), which asserts that there is one unique fluid velocity. The fluid velocity can be sampled either by tracking fluid parcels or by placing current meters at fixed locations. We have seen by way of a simple example (Sections 2 and 3) that it is possible to shift back and forth from a Lagrangian to an Eulerian representation provided that we have either (1) a complete knowledge of all parcel trajectories, or, (2) the complete velocity field at all relevant times. In Section 2 we first presumed (1), which happens only in the special world of homework problems. However,

⁶⁷This analysis was carried out in a Lagrangian coordinate or reference frame, in that the floats followed a material volume of fluid. The Eulerian statement of potential vorticity conservation (that we would be much more likely to use for a complete, predictive model) is just $\frac{D\Psi}{Dt} = \frac{\partial\Psi}{\partial t} + \mathbf{V} \cdot \nabla\Psi = 0$. How would the northeast to southwest oscillation appear if it was observed at a fixed site, i.e., from an Eulerian reference frame? Consider three cases: (1) the idealized case that only variations of planetary vorticity are present, and assume that f can be expanded in a Taylor series, $f = f_0 + \beta y$. (2) that only variations of water depth are important, and finally, (3) consider the actual, observed current field of Fig (27) and whether ‘frozen field’ advection of relative vorticity would have been observed (Sec. 6.1). Finally, when the cluster changed directions from northeast to southwest the turn appeared to propagate across the cluster (see Fig. 27 or the animation on the cover page). Can you estimate a phase speed and direction? How about a period and a wavelength?

in the usual course of a numerical model calculation we probably will satisfy (2), and thus can compute parcel trajectories on demand (Sections 4 and 5). Numerical issues and diffusion (numerical and physical) will complicate the process and to some degree the result. Nevertheless, the procedure is straightforward in principle and is often an important step in the diagnostic study of complex flows computed in an Eulerian frame. Approximate methods may be usefully employed in the analysis of some flows, an important example being the time-mean drift of parcels in a field of surface gravity waves (Section 5.2). The Eulerian mean motion below the wave trough is zero on linear theory, while the Lagrangian mean flow may be substantial, depending upon wave steepness.

The statement of conservation laws for mass, momentum, etc., applies to specific parcels or volumes of fluid, and yet to apply these laws to a continuum it is usually preferable to transform these laws from an essentially Lagrangian perspective into Eulerian or field form (Section 3). There are three key pieces in this transformation; the first is the FPK (Section 1.2). The second is the material derivative (Section 3.1); an ordinary time derivative transformed into the Eulerian system is $D()/D = \partial()/\partial t + \mathbf{V} \cdot \nabla()$, the sum of a local time rate of change and an advective rate of change (Eq. (48)). The third is that integrals and their time derivatives can be transformed from material to field coordinates by way of the Reynolds Transport Theorem (Eq. (63) and Section 3.2). Important applications of the RTT yield the mass conservation relation and the momentum balance (Sections 3.2.1 and 3.2.2), which are the foundation of classical fluid mechanics.

The process of advection contributes much of the interesting and most of the challenging dynamics and kinematics of fluid flows. The advection term is semi-linear in that it involves the product of an unknown (generally) velocity component and the first partial derivative of a field variable. There are some important bounds on the consequences of advection. For variables that can be written in a conservation form (e.g., mass and momentum), advection alone can not be a net (globally integrated) source or sink, though it may cause variations at any given point in the domain (Section 6.1). Advection alone transports fluid properties at a definite rate and direction, that of the fluid velocity. The method of characteristics (Section 6.3) exploits this hyperbolic property of the advection equation to compute solutions of nonlinear PDEs; along a characteristic line (which are streamlines in steady flow) the governing equation is exactly as seen by a parcel. In the instances where we can solve for the characteristics, this leads to insightful solutions. The idea of characteristics is the basis of efficient numerical advection schemes⁶⁸ and for the interpretation of many fluid flows. Besides merely transporting fluid properties, advection by a nonuniform velocity field (which is to say nearly all velocity fields) will also cause a rotation of fluid parcels that is akin to angular momentum (Section 6.4). Advection may also cause deformation of fluid parcels that may lead to greatly increased mixing rates in a stirred fluid compared to diffusion alone.

9 Appendix: A Review of Composite Functions

Composite functions are, essentially, functions of functions. As we point out in Section 2, the transformation of velocity from a Lagrangian to an Eulerian system can be thought of as the construction of a composite function, the velocity as a function of position, that we like to call the 'Eulerian' velocity. In this Appendix the salient properties of composite functions are reviewed and demonstrated with a very simple example. No

⁶⁸Staniforth and J. Cote, 'Semi-Lagrangian integration schemes for atmospheric models - a review', *Mon. Wea. Rev.*, **119**, 2206-2223, 1991.

more than algebra is required for the first part of this, and the only issue is that we need to be a bit more explicit than usual about the meaning of equations used to define or evaluate functions.

9.1 Definition

Let f be a function of a single variable, say

$$f(y) = 1 + cy \quad (159)$$

in which case the function notation f means ‘take the variable found in the argument, multiply by c and then add 1’ (we said this would be simple). The parameter c is presumed real, and the domain of y will be restricted to the positive real numbers, $0 < y$. An equation of the sort Eq. (159) does not by itself make it clear whether we mean to define the function f , or whether we intend to evaluate $f(y)$ by assigning a value to y . In this case the context makes it clear enough that this equation defines f , since y as yet has no particular significance and serves only to hold a place, i.e., it is a dummy variable that could just as well have been written x or z .

But now suppose that y can be or must be regarded as a function of an independent variable x , say

$$y = g(x) = x^2$$

with x real and having the domain $0 < x$. By restricting the domain of x and y we insure that the inverse of the function g , written

$$g^I(x) = x^{1/2},$$

is a single-valued function, the positive square root. If we treat the left and right sides of this equation as the argument for g , then $g(g^I(x)) = g(x^{1/2}) = x$. Thus the inverse function undoes the function itself.

A composite function may be constructed from f and g by taking g as the argument for f , i.e., $f(g)$. The function g inside the parentheses is termed the inner function, and f is termed the outer function. In the present example the composite function is

$$f(g(x)) = f(x^2) = 1 + cx^2.$$

Notice that the composite function depends upon the dummy variable x alone, i.e., the independent variable of a composite function is the independent variable of the inner function.

For the f and g of our example, $f(x) = 1 + cx \neq f(g(x)) = 1 + cx^2$, and thus in general, $f(x) \neq f(g(x))$. If we want to use a single symbol to denote the composite function, and we often will, then we should give it a name other than f , say

$$h = f(g), \quad (160)$$

and in the example given, $h(x) = 1 + cx^2$. Thus when we speak of a composite function it is implicit that we have three functions in mind; an outer function f , an inner function g , and the composite function, $h = f(g)$.⁶⁹

⁶⁹Three questions for you: 1) Can you show that what we have called the composite function and what we have called the outer function can just as well be reversed, assuming that the inner function can be inverted? 2) Can you also show that $f(g) \neq g(f)$ and thus that the order of compositing does not commute? 3) Finally, assuming that you have read past Section 2.2, match up the symbols and meaning of Eq. (20) with Eq. (160), and then explain the difference between $V_{\mathbf{L}}(A, t)$ and $V_{\mathbf{L}}(A(\xi), t)$.

In general, $h(x) \neq f(y)$, where the intent here is that that h is to be evaluated at x and f is to be evaluated at y , and as yet no restriction is placed upon x or y . However, it is true that

$$h(x) = f(y) |_{y=g(x)}, \quad (161)$$

where $() |_{y=y(x)}$ means that the argument of f is equal to the inner function evaluated at the x that appears as the argument on the left hand side. In other words, it is true that $h(x) = f(y)$ just in case the values of x and y are related by the inner function. With a moment's reflection this seems fairly obvious, but an equivalent way to state this is not quite so transparent, namely

$$h(x) |_{x=g^I(y)} = f(y), \quad (162)$$

where g^I indicates the inverse of the inner function. In the example noted above, $g^I(y) = y^{1/2}$, the positive square root of y . This last result Eq. (162) is an important one because it is exactly the form we used to state the FPK, Eq. (14), our definition of Eulerian velocity:

the Eulerian velocity $V_E \rightarrow h$, the composite function,
the field coordinate, $x \rightarrow x$, the argument of the composite function
the Lagrangian velocity $V_L \rightarrow f$, the outer function,
the trajectory inverse, $A(\xi) \rightarrow g(x)$, the inner function, and
the material coordinate, $\xi \rightarrow y$, the argument of the outer function.

Thus when we start with a Lagrangian perspective, presuming to know the Lagrangian velocity and the trajectories (and their inverse), then the Eulerian velocity can be constructed as a composite function made up from these Lagrangian data.

9.2 Rules for differentiation and change of variables in integrals

Let $f(g)$ be a composite function of f and g , where g is itself a function of the independent variable x . Thus $h(x) = f(g(x))$ depends only upon the independent variable x . The derivative d/dx of $f(g)$ may be computed as

$$\frac{d}{dx} f(g) = \frac{df}{dg} \frac{dg}{dx} = \frac{dh}{dx}. \quad (163)$$

This is rather tedious to derive, but probably already familiar as the chain rule of differential calculus.

Let F be the antiderivative of f , i.e.,

$$F(y) = \int f(y) dy \quad (164)$$

and so

$$F(y)' = \frac{dF}{dy} = f(y).$$

If it is the case that $y = g(x)$ then $F(g(x))$ is a composite function and by the chain rule,

$$\frac{d F(g)}{dx} = F' \frac{dg}{dx} = f \frac{dg}{dx}.$$

Upon integrating the left and right sides,

$$F(g(x)) = \int f(g) \frac{dg}{dx} dx. \quad (165)$$

Thus if F is the antiderivative of $\int f(y)dy$, then $F(g)$ is the antiderivative of $\int f(g(x)) \frac{dg}{dx} dx$.

We can also transform the independent variable of an integral by direct substitution of $y = g(x)$, and so

$$\int_{y_1}^{y_2} f(y)dy = \int_{x_1=g^{-1}(y_1)}^{x_2=g^{-1}(y_2)} f(g(x)) \frac{dg}{dx} dx. \quad (166)$$

The most interesting part of this is $dy \rightarrow \frac{dg}{dx} dx$, which includes the Jacobian of the transformation and accounts for the stretching or compression of the differential 'length'. (You should verify Eqs. (163) and (166) with simple examples of f and g we have used to now.) Thus we can change the independent variable y of the integral of Eq. (164) to x in Eq. (166), provided that $y = g(x)$ and x can be thought of as the inner function and independent variable of a composite function. This result is invoked in Section 3.3 to change an integral over ξ , which is time-dependent if we are tracking a moving material volume, to the equivalent integral over α , the initial position of the material volume and which is time-independent.

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