13 The Amplitude of Convection

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13.1 Introduction

As a ubiquitous source of motion, both astrophysical and geophysical, convection has attracted theoretical attention since the last century. In the ocean, many different scales are called convection; from the deep circulation due to the seasonal production of Arctic bottom water (see chapter 1) to micromixing of salt fingers (see chapter 8). In the atmosphere, convection dominates the flow from subcloud layers to Hadley "cells." It is proposed that convection in the earth's core powers the geomagnetic field. The nonperiodic reversals of that field, captured in the rock, define the evolution of the ocean basin. Recent recognition that this latter process is caused by convection in the mantle has produced a new geophysics.

In the past, understanding the central features of convection has come from the isolation of "simplest" mechanistic examples. Although large-scale geophysical convection never coincides with the idealized simplest problem, these examples (e.g., Lord Rayleigh's study of the Bénard cells) have generated much of the formal language of inquiry used in the field. Students of dynamic oceanography have favored this formal language mixed in equal parts with more pragmatic engineering tongues when interpreting oceanic convective processes.

Speculations beyond these mathematically accessible problems take the form of hypotheses, experiments, and numerical experiments in which one seeks to isolate the central processes responsible for the qualitative and quantitative features of fully evolved flow fields. The many facets of turbulent convection represent the frontier. This chapter reviews only a narrow path toward that frontier. This path is aimed at an understanding of the elementary processes responsible for the amplitude of convection, in the belief that quantitative theories permit the theorist the least self-deception.

Of course the heat flux due to a prescribed thermal contrast, like the flow due to a given stress, has been observed for a century. The relation between force and flux has been rationalized with models emerging largely from linear theory and kinetic theory—in particular, with the use of observationally determined "eddy conductivities" (estimated for the oceans in Sverdrup, Johnson, and Fleming, 1942). Early theoretical interpretations of oceanic transport processes that go beyond these simple beginnings were explored by Stommel (1949), while current usage and extensions of "mixing" theories are discussed in chapter 8.

Central to the most recent of such proposals is the idea that some large scale of the motion or density field is steady or statistically stable, while turbulent transport due to smaller scales can be parameterized. Changing the amplitude of the small-scale transports is presumed to lead to a new equilibrium for the large scale, so that the statistical equilibrium is marginally stable. This view lurks behind most traditional oceanic model building and its quasi-linear form is used on small-scale phenomena as well—from inviscid marginal stability for the purpose of quantifying aspects of the wind mixed layer (Pollard, Rhines, and Thompson 1973) to viscous marginal stability for the purpose of quantifying double diffusion (Linden and Shirtcliffe, 1978).

It has not yet been possible to establish either the limits of validity or generalizability of this quasi-linear use of marginal stability in the geophysical setting. There can be little doubt that it is "incorrect"—that fluids typically are destabilized by the extreme fluctuations—yet it appears to be the only quantifying concept of sufficient generality to have been used in oceanic phenomena from the largest to the smallest scales. Of course, our idealizations in the realm of geophysics are all "incorrect." We turn to observation to establish in what sense and in what degree these idealizations are good "first-order" descriptions of reality.

This chapter explores the hierarchy of quantifying idealizations in convection theory. The quasi-linear marginal-stability problem is drawn from the full formal statement for stability of the flow. A theory of turbulent convection based on marginal stability is presented, incorporating both the qualitative features determined by inviscid processes and the quantitative aspects determined by dissipative processes.

Observations provide better support for both the quantitative and qualitative results from quasi-linear marginal-stability theory than might have been anticipated, encouraging its continued application in the oceanic setting.

13.2 Basic Boussinesq Description

The primary simplification that permitted mathematical progress in the study of motion driven by buoyancy was the Boussinesq statement of the equations of motion. In retrospect, the central problem was to translate the correct energetic statement

$$\langle \mathbf{u} \cdot \nabla P \rangle = \Phi,$$

into the approximate form

$$\langle \gamma \overline{WT} \rangle = \Phi,$$

where $\mathbf{u}$ is the vector velocity of the fluid, P the pressure, γ the coefficient of thermal expansion times the acceleration of gravity, W the vertical component of velocity, T the temperature field, Φ the total dissipation by viscous processes in the fluid, and the brackets a spatial average over the entire fluid. This has been achieved (e.g., Spiegel and Veronis, 1960; Malkus 1964) by recognizing that the Boussinesq equations are the leading terms in an asonic asymptotic expansion away from a basic adiabatic hydrostatic temperature distribution. This expansion is usually made in two small parameters; one is the ratio of the height of the convecting region to the total "adiabatic depth" of the fluid, while the second is the ratio of the superadiabatic temperature contrast across the convecting region to the mean temperature.

In suitably scaled variables, the leading equations of the expansion are

$$\nabla \cdot \mathbf{u} = 0, \tag{13.1}$$

$$\frac{1}{\sigma}\frac{D\mathbf{u}}{Dt} = -\nabla P + \nabla^2 \mathbf{u} + Ra T \mathbf{k}, \tag{13.2}$$

$$\frac{DT}{Dt} = \nabla^2 T, \tag{13.3}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla, \qquad \sigma = \frac{\nu}{\kappa}, \qquad Ra = \frac{\gamma \Delta T d^3}{\kappa \nu},$$

$\mathbf{k}$ is the unit vector in the antidirection of gravitational acceleration, d the depth of the convecting region, ΔT the superadiabatic temperature contrast, κ the thermometric conductivity of the fluid and ν is its kinematic viscosity, Ra the Rayleigh number, and σ the *Prandtl number*. Other symbols are defined above. The Boussinesq equations retain the principal advective nonlinearity, but have no sonic solutions. Higher-order equations are linear and inhomogeneous, forced by the lower-order solutions.

The most accessible problem in free convection has been the study of motion in a horizontal layer of fluid bounded by good thermal conductors at prescribed temperatures. Such a layer is the thermal equivalent of the constant-stress layer in shear flow. This is seen by taking an average over the horizontal plane of each term in the heat equation. One writes from 13.3

$$-\frac{\partial \overline{T}}{\partial t} = \frac{\partial}{\partial z}\left(-\frac{\partial \overline{T}}{\partial z} + \overline{WT}\right), \tag{13.4}$$

where the overbar indicates the horizontal average. For steady or statistically steady convection, $\partial \overline{T}/\partial t$ vanishes, and one may integrate (13.4) twice to obtain

$$Nu = -\frac{\partial \overline{T}}{\partial t} + \overline{WT} = 1 + \langle WT \rangle, \tag{13.5}$$

where the constant of integration Nu is called the Nusselt number and is the ratio of the total heat flux to that due to conduction alone.

Two other integrals of considerable interest can be constructed from the Boussinesq equations. The first of these is the power integral found by taking the scalar product of (13.2) with $\mathbf{u}$ and integrating over the entire fluid. One obtains

$$\langle -\mathbf{u}\cdot\nabla^2\mathbf{u}\rangle = \langle|\nabla\mathbf{u}|^2\rangle = Ra\langle WT\rangle, \tag{13.6}$$

due to the vanishing of the conservative advective terms. Multiplying (13.6) by the fluctuation temperature,

$$\mathrm{T} = T - \overline{T}, \tag{13.7}$$

one obtains an integral similar to (13.6), which may be written by means of (13.5) as

$$\langle -\mathrm{T}\,\nabla^2\mathrm{T}\rangle = \langle|\nabla\mathrm{T}|^2\rangle = \langle -W\mathrm{T}\frac{\partial\overline{T}}{\partial z}\rangle$$

$$= \langle W\mathrm{T}\rangle + \langle W\mathrm{T}\rangle^2 - \langle\overline{W\mathrm{T}}^2\rangle. \tag{13.8}$$

The integrals (13.5), (13.6), and (13.8) are the principal constraints used in the "upper bound" theories of convection (see section 13.3).

This section would not be complete without a statement of those equations that determine the stability of any solution, say $\mathbf{u}_0$, P_0, T_0, of the basic equations (13.1)–(13.3). Consider a general disturbance $\mathbf{v}$, p, θ to the solution $\mathbf{u}_0$, P_0, T_0. Then

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{v}, \qquad P = P_0 + p, \qquad T = T_0 + \theta. \tag{13.9}$$

One concludes from (13.1)–(13.3) that the time evolution of $\mathbf{v}$, p, θ is determined by

$$\nabla\cdot\mathbf{v} = 0, \tag{13.10}$$

$$\frac{1}{\sigma}\left(\frac{D\mathbf{v}}{Dt} + \mathbf{v}\cdot\nabla\mathbf{u}_0 + \mathbf{v}\cdot\nabla\mathbf{v}\right)$$

$$= -\nabla p + \nabla^2\mathbf{v} + Ra\theta\mathbf{k}, \tag{13.11}$$

$$\left(\frac{D\theta}{Dt} + \mathbf{v}\cdot\nabla T_0 + \mathbf{u}\cdot\nabla\theta\right) = \nabla^2\theta, \tag{13.12}$$

where, as before, $D/Dt = \partial/\partial t + \mathbf{u}_0\cdot\nabla$.

For a solution $\mathbf{u}_0$, P_0, T_0 to be stable (hence realizeable), arbitrary infinitesimal disturbances, $\mathbf{v}$, p, θ must eventually decay. For a solution, $\mathbf{u}_0$, P_0, T_0 to be absolutely stable, arbitrary disturbances of any amplitude must eventually decay. This latter problem is tractable in some instances and has been addressed using the "power" integrals for $\mathbf{v}$ and θ. These are written

$$\frac{1}{\sigma}\frac{D}{Dt}\langle\tfrac{1}{2}\mathbf{v}^2\rangle + \langle\mathbf{v}\cdot\mathbf{v}\cdot\nabla\mathbf{u}_0\rangle$$

$$= -\langle|\nabla\mathbf{v}|^2\rangle + Ra\langle\mathbf{v}\cdot\mathbf{k}\theta\rangle, \tag{13.13}$$

$$\frac{D}{Dt}\langle\tfrac{1}{2}\theta^2\rangle + \langle\theta\mathbf{v}\cdot\nabla T_0\rangle = -\langle|\nabla\theta|^2\rangle. \tag{13.14}$$

Equations (13.10), (13.13), and (13.14) constitute a linear problem for $\mathbf{v}$ and θ whose solution can determine a minimum $Ra = Ra(\sigma)$ for which $\mathbf{u}_0$, P_0, T_0 is absolutely stable. In contrast, the solutions to the linear form of (13.10)–(13.12) can determine a maximum $Ra = Ra(\sigma)$ beyond which $\mathbf{u}_0$, P_0, T_0 is assuredly unstable. In principal these linear problems can be solved when $\mathbf{u}_0$, P_0, T_0 is time independent, and solved at least approximately when $\mathbf{u}_0$, P_0, T_0 is periodic in time. However, analytic techniques for determining the conditions leading to eventual decay of $\mathbf{v}$, p, θ on a nonperiodic solution $\mathbf{u}_0$, P_0, T_0 have not been developed.

In concluding this description of the "simple" Boussinesq fluid one should note how many interesting problems lie outside the formalism. Just outside the framework, but capable of incorporation, are porous boundaries, variable viscosity, and nonlinear equations of state. Much farther outside the framework are convection through several scale heights and velocities comparable to the sound speed.

13.3 Initial Motions

The state of pure conduction without motion, $\mathbf{u}_0 = 0$, $T_0 = -z$, where z is the vertical coordinate measured from the lower surface, is a solution to the Boussinesq equations at all Ra. The determination of that critical value of Ra at which this conduction solution is unstable is the classical Rayleigh convection problem (e.g., Chandrasekhar, 1961). This problem is the linearized form of (13.10)–(13.12) for $\mathbf{u}_0 = 0$, $T_0 = -z$, and is written

$$\nabla\cdot\mathbf{u} = 0, \tag{13.15}$$

$$\frac{1}{\sigma}\frac{\partial\mathbf{u}}{\partial t} = -\nabla p + \nabla^2\mathbf{v} + Ra\theta\mathbf{k}, \tag{13.16}$$

$$\frac{\partial\theta}{\partial t} = w + \nabla^2\theta, \qquad w = \mathbf{v}\cdot\mathbf{k}. \tag{13.17}$$

If one takes the $\mathbf{k}$ component of the curl of the curl of (13.16) and, using (13.17), eliminates θ, one may write the Rayleigh problem as a constant coefficient, sixth-order partial differential equation in the single variable w:

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial t} - \nabla^2\right)\left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 w = -Ra\,\nabla_1^2 w, \tag{13.18}$$

where ∇_1^2 is the horizontal Laplacian. The problem is separable so that

$$\nabla_1^2 w = -\alpha^2 w \tag{13.19}$$

defines the horizontal wavenumber α^2. Also, it is not difficult to establish that the disturbance w first starts to grow without temporal oscillations; the instability is "marginal." Hence, subject to appropriate boundary conditions one is to find the eigenstructure of the ordinary differential operator

$$\left[\left(\frac{\partial^2}{\partial z^2} - \alpha^2\right)^3 - \alpha^2 Ra\right] w = 0. \tag{13.20}$$

The richness of solutions to this simple problem and its "adjacent" modifications have been explored for more than two generations and now enters a third. In the present generation, formal techniques that had been developed to find the initial postcritical amplitude of convection are used to clarify the finite-amplitude stability problems. These finite-amplitude studies include the resolution of the infinite plan form degeneracy of the classical linear problem and the determination of conditions causing subcritical instabilities (or "snap-through" instabilities). The formal finite-amplitude technique, often called modified perturbation theory, is the subject of a recent lengthy review (Busse, 1978). In brief, one expands $\mathbf{v}$, p, θ in terms of an amplitude ϵ (typically the amplitude of w), and in addition employs a parametric expansion of the controlling parameters, also in terms of ϵ, to permit solvability of the sequence of equations generated by the nonlinear terms. One writes

$$\mathbf{v} = \sum_{n=1}^{\infty} \epsilon^n \mathbf{v}_n, \qquad Ra = \sum_{n=0}^{\infty} Ra_n \epsilon^n, \tag{13.21}$$

with similar expansions for p and θ, and where Ra_0 is the critical Rayleigh number determined from the linear problem (13.20). Here, each of the Ra_n is determined to permit a steady solution to the ϵ^n set of equations generated by inserting (13.20) into (13.10)–(13.12). Paralleling (13.10)–(13.12) and (13.20), an expansion can be constructed for potential disturbances $\mathbf{v}'$, p', θ' in order to determine their stability at each order ϵ^n.

Among the many interesting conclusions reached in these studies is that the amplitude of the stable convection forms are determined principally by a modification of $\overline{T}$ due to the convective flux $\overline{w\theta}$. The finite-amplitude distortions of w and θ from their infinitesimal form also affects their equilibrium amplitude, and although a smaller effect than the modification of the mean, it is always present. Conditions for subcritical instability, from (13.21), are seen to be that either $Ra_1 < 0$, $Ra_2 > 0$ (as is the case when hexagonal cellular convection is observed), or $Ra_2 < 0$, $Ra_4 > 0$ (as occurs in penetrative convection in water cooled below 4°C), or some mixture of these two conditions. It is found to order ϵ^2 in each case that the preferred (stable) convection is that which transports the most heat. However, there is now an example of a convective process in which the maximum heat-flux solution at large amplitude is not the most stable form. No simple integral criterion has been found that assures the stability of a Boussinesq solution at large Ra.

Many papers have been written using modified perturbation theory, and more appear each year. Unusual current studies include nonperiodic behavior of initial convection in rotating systems and the onset of magnetic instabilities due to finite-amplitude convection in electrically conducting fluids. Busse's review exhibits the significant enrichment of our knowledge and language of inquiry of fluid dynamics by these finite-amplitude studies. This same review, however, clarifies the intractable character of convection mathematics beyond ϵ^2.

This section on initial motions will be concluded by noting E. Lorenz's (1963b) minimal nonlinear convection model, first explored by Saltzman (1962), is an abruptly truncated ϵ^2 Rayleigh convection process. This third-order autonomous system exhibits a transition to "convection," then at higher "Rayleigh" number a transition to exact solutions with nonperiodic behavior. A group of mathematicians and physicists has emerged to explore these "strange attractors" and similar models, hoping among other things to find new access to the turbulent process in fluids. One can be confident that the study of these autonomous models will lead eventually to elementary insights of value to the geophysical dynamicist, yet such study is only one facet of the third generation beyond linear convective instability mentioned earlier. The path to be followed here through the ever-growing convection literature must include the theory of upper bounds on turbulent heat flux, for this theory deductively addresses convection amplitude. The blaze mark along this path, however, continues to be stability theory, and the final section explores its relation to the observed fully evolved flow.

13.4 Quantitative Theories for High Rayleigh Number

Beyond modified perturbation theory one finds hypotheses, speculation, and assorted ad hoc models related to aspects of turbulent convection or designed to rationalize a body of data. In this section three unique theories that predict an amplitude for convection are discussed. The first of these is the mean-field theory of Herring (1963). The second is the theory of the upper bound on heat flux, first formulated by Howard (1963), and its "multi-α" extension by Busse (1969). The third theory, by Chan (1971), is also a heat-flux upper-bound theory, but for the idealized case of infinite Prandtl number. The link between the first and last theories proves to be most informative.

The mean-field theory is equivalent to the first approximation of many formal statistical closure proposals, and is the only contact this study will make with such proposals. As implemented by Herring (1963), the equation describing the motion and temperature field are obtained by retaining only those nonlinear terms in the basic equations that contain a nonzero horizontally averaged part. From (13.1)–(13.4) and (13.7) one writes the mean-field equations thusly:

$$\nabla \cdot \mathbf{u} = 0, \tag{13.22}$$

$$\frac{1}{\sigma}\frac{\partial \mathbf{u}}{\partial t} = -\nabla P + \nabla^2 \mathbf{u} + Ra\mathrm{T}\mathbf{k}, \qquad (13.23)$$

$$\frac{\partial \mathrm{T}}{\partial t} = \nabla^2 \mathrm{T} + W(Nu - \overline{W\mathrm{T}}). \qquad (13.24)$$

The only nonlinearity retained is $W \cdot \overline{W\mathrm{T}}$ in (13.24). Hence the full problem is separable in the horizontal (e.g., $\nabla_1^2 W = -\alpha^2 W$ as in the linear problem), and (13.22)–(13.24) reduce to an ordinary nonlinear equation. Herring solved this nonlinear problem by numerical computations for several increasingly large Rayleigh numbers and for both one and two separation wavenumbers α. He found that the solutions settled to time-independent cellular forms with sharp boundary layers. The predicted heat flux is several times larger than the observed flux in laboratory experiments and the predicted mean-temperature gradient exhibits reversals not seen in high Rayleigh-number data. This relation between theory and experiment will be discussed again in connection with the study by Chan. It was anticipated that neglect of the fluctuating thermal nonlinear terms $[\mathbf{u}\cdot\nabla\mathrm{T} - (\partial/\partial z)\overline{W\mathrm{T}}]$ would increase the predicted heat flux by some significant, but plausibly constant, fraction. But the effect of neglecting the momentum advection terms $\sigma^{-1}[\mathbf{u}\cdot\nabla\mathbf{u}]$ could have both a stabilizing and destabilizing effect on convection amplitude. If the disturbances in the velocity field are sufficiently large scale compared to the thickness of the thermal boundary region, then one could anticipate from the finite-amplitude studies that $\sigma^{-1}(\mathbf{u}\cdot\nabla\mathbf{u})$ would decrease the amplitude of the convection. But if the local momentum-transporting disturbances are small scale compared to the thermal boundary region, they could strongly enhance the heat transfer. The latter process is typical of geophysical-scale shear flow plus convection, and is not addressed by this mean-field theory. A joint mean-field theory including a mean horizontal velocity $U(z)$ might capture some of this important process of a mean shear flow, enhancing its convective energy source. The upper-bound theory automatically includes this possibility.

The theory of upper bounds on the convective heat flux is based on optimizing a vector field $\mathbf{u}$ and scalar field T that are less constrained than the actual velocity and temperature fields. The only constraints placed on $\mathbf{u}$ and T are that they satisfy the boundary conditions, the continuity condition (13.1), and the three integral conditions (13.5), (13.6), and (13.8). Howard was the first to perform this optimization, and has written a review (1972) of the most certain results of the theory. Busse was the first to discover that the optimal solutions for $\mathbf{u}$ and T contain many (nested) scales of steady convecting motions. Although these results are the only formally correct deductions in the literature applicable to turbulent flows, they are rather far off the mark. The heat flux is much higher than any observed for pure convection, and the optimal solutions seem more elegant than predictive. Their quantitative prediction is that Nu will vary as $Ra^{1/2}$. The highest Ra laboratory data reaches $Ra = 10^{10}$ and appears at most to approach $Nu \sim Ra^{1/3}$. However, careful use of mixing-length theory by Kraichnan (1962) suggests that heat transport in a boundary region caused by momentum advection can occur for $Ra > 10^{24}$, leading to $Nu \sim Ra^{1/2}/(\ln Ra)^{3/2}$. Howard's formal bound includes this possibility.

Hopes to reduce these extreme heat transports by the addition of further integral constraints have not been realized. All other integrals of the basic equations appear either to be trivial in content or to introduce inseparable cubic nonlinear terms into the analysis. The one exception is the work of Chan, which now will be discussed in some detail.

Chan (1971) sought an upper bound on convective heat flux with the power integral constraint (13.6) replaced by the linear Stokes relation,

$$0 = -\nabla P + \nabla^2 \mathbf{u} + Ra\mathrm{T}\mathbf{k}. \qquad (13.25)$$

This is not only significantly more restrictive on the class of possible fields $\mathbf{u}$ and T, but is an exact statement, from (13.2), in the limit as σ approaches infinity. As in the mean-field theory, no formal expansion is proposed that could reincorporate the nonlinear momentum advection. Yet one might anticipate that the upper bounds on the amplitude of convection will be much closer to the laboratory observations, or to oceanic observations where shear instabilities play a small role.

In addition to (13.25), Chan used the continuity condition (13.2), the thermal integral (13.8), and appropriate boundary conditions. The Euler-Lagrange equations for the optimal relation between W and T have the form

$$\nabla^6 \mathrm{T} + (Nu - 1) \times \left[\nabla^4\left(1 - \overline{W\mathrm{T}} - \frac{2\lambda}{Nu - 1}\right) W + (1 - \overline{W\mathrm{T}})\nabla^4 W\right] = 0, \qquad (13.26)$$

$$\nabla^4 W = Ra\,\nabla_1^2 \mathrm{T}, \qquad (13.27)$$

where λ is a constant Lagrange multiplier. These equations can be compared with the equivalent mean-field equations, from (13.22)–(13.24), which are

$$\nabla^6 \mathrm{T} + \nabla^4[(Nu - \overline{W\mathrm{T}})W] = 0, \qquad (13.28)$$

$$\nabla^4 W = Ra\,\nabla_1^2 \mathrm{T}, \qquad (13.29)$$

indicating both the similarity and difference of the two problems.

In constrast to Herring's numerical solutions at moderately high Ra, Chan used Busse's multi-α asymptotic technique to determine an optimal solution and a mean-field solution approached at very high Ra. Perhaps the most significant conclusion was that, in this asymptotic limit, the upper-bound problem and the mean-field problem lead to identical results. This result confirms the expectation that the fluctuating thermal terms reduce the convective heat flux by a fraction of about one-half from currently available high Ra data. This certainly represents a remarkable achievement for a theory of turbulence free of empirical parameters. Such quantitative agreement lends support to the idea that the statistical stability condition for turbulent convection in the absence of strong shearing flow is close to the condition of maximum heat transport. Yet, when the possibility of momentum transport due to shear flow is again included in the problem, what extreme should be sought? Here is the arbitrary element in the formal upper-bound theory—what upper bound best reflects the real statistical stability problem? This question is addressed in the following section.

13.5 The Amplitude of Turbulent Convection from Stability Criteria

The idealization of turbulent convection to be explored in this section is similar in spirit to the optimal-transport theories previously discussed. Optimal properties of vector and scalar fields **u** and T compatible with the boundary conditions and several other constraints are to be compared with the observed averages of the velocity and temperature fields. Here, however, it will be the stability of the flow that will be optimized. At high Ra both theoretical considerations and observations suggest that large-scale flows in the interior of the region are essentially inviscid in character. In keeping with this classical view, the interior fields of this theory are permitted to approach, but not exceed, the inviscid-stability conditions. These conditions alone can determine many of the qualitative features of the interior flows, but the amplitude of these flows remains undetermined. The goal of this theory is to find those amplitudes that lead to maximum stability for the small-scale, dissipative motions near the boundary. In this view the tail wags the dog, for only the tail is in contact with the dissipative reality that modulates amplitudes. Comparison of the predictions of this quantitative theory with observations can determine the extent to which the real flow approaches the freedom of amplitude selection granted the trial fields of the theory.

The linearized forms of (13.10)–(13.12) constitute a complete statement of the necessary stability conditions that must be met by a realizable Boussinesq solution $\mathbf{u}_0$, T_0. In this theoretical proposal one pictures $\mathbf{u}_0$, T_0 at a particular Ra and σ as composed of the finite-amplitude forms of all fields that were unstable at smaller values of Ra. Subject to the inviscid-stability conditions, the amplitudes of these previous instabilities are to be chosen to make the disturbances $\mathbf{v}$, θ as stable as possible. When that Ra is reached at which the stability of $\mathbf{v}$, θ is no longer possible by amplitude adjustment of $\mathbf{u}_0$, T_0, then the unstable $\mathbf{v}$, θ join the ranks of the previously unstable motions that make up $\mathbf{u}_0$, T_0, and a new stability problem for a new $\mathbf{v}$, θ is posed.

Unfortunately, the linear-stability problem posed above involves fluctuating coefficients that would defy analysis even if they were known. Hence, as promised in the introduction, the proposal is weakened to consider only the stability problem on the mean fields $\bar{\mathbf{u}}_0$, $\bar{T}_0$. Indeed, the fluctuations are observed to be only a fraction of the mean values; yet it is during the destabilizing period of the fluctuations that the significant instabilities occur. If the effects of stabilization and destabilization due to the fluctuations around the mean roughly cancel, then the stability of the mean field is a good measure of the overall stability of the flow. This idealization is explored in the following paragraphs, primarily for the "pure convection" case of infinite Prandtl number. The finite Prandtl-number problem is posed and the extreme case of shear-flow-dominated transport discussed.

The mean-field-stability problem, when $\bar{\mathbf{u}}_0 = 0$, involves only the term $\partial\bar{T}_0/\partial z$. Hence the partial differential equations (13.10)–(13.12) are separable and reduce to a form similar to (13.20):

$$\left[\left(\frac{\partial^2}{\partial z^2} - \alpha^2\right)^3 + \alpha^2 Ra\,\frac{\partial \bar{T}_0}{\partial z}\right] w = 0. \tag{13.30}$$

The principal constraint to be imposed on the averaged interior flow is that it approach from the viscously stable side, but not exceed, the inviscid stability condition. For convection without a mean shear flow, this condition is that

$$-\frac{\partial \bar{T}_0}{\partial z} \geq 0. \tag{13.31}$$

It is observed that high-Rayleigh-number convection is very close to this stability boundary. Before establishing the quantitative features of the convection amplitude from (13.30), the qualitative consequences of (13.31) will be explored. One may write (13.31) as

$$-\frac{\partial \bar{T}_0}{\partial z} = I^* I \tag{13.32}$$

where I is any complex function of z and I^* is its complex conjugate. It was shown by Fejer (1916), and we shall see shortly, that a complete representation of an everywhere positive function can be written

$$I(\phi) = \sum_{k=0}^{\infty} I_k e^{ik\phi}, \tag{13.33}$$

where $\phi = 2\pi z$, $0 \leq \phi \leq 2\pi$.

The relation between the representation for I in (13.33) and a normal Fourier representation can be established straightforwardly. Let

$$I_m = A_m + iB_m \tag{13.34}$$

where the A_m and B_m are all real. Then

$$(I^*I)(\phi) = \sum_{k=0}^{\infty} (2 - \delta_{k,0}) \sum_{m=0}^{\infty} (A_m A_{m+k} + B_m B_{m+k}) \cos k\phi + \sum_{k=1}^{\infty} 2 \sum_{m=0}^{\infty} (A_{m+k} B_m - A_m B_{m+k}) \sin k\phi. \tag{13.35}$$

For symmetric $(I^*I)(\phi)$ one may write

$$(I^*I)(\phi) = \sum_{k=0}^{\infty} C_k \cos k\phi,$$
$$C_k = (2 - \delta_{k,0}) \sum_{m=0}^{\infty} I_m I_{m+k}, \qquad I_m \text{ real.} \tag{13.36}$$

The C_k are uniquely determined by a given set I_k, but a given set C_k determines unique I_k only under special circumstances.

The qualitative behavior of $-\partial \overline{T}_0/\partial z$ emerges from the weak assumption that, at high Ra, I_k is some "smooth" function of k, an assumption to be borne out in the quantifying second step of this theory. Of course, at some very small scale, say k_ν, one expects viscosity and thermal diffusion to reduce I_{k_ν} to a vanishingly small value. Then when $I_k \simeq 0$ for $k > k_\nu$, one writes

$$I(\phi) = \sum_{k=0}^{\infty} I_k e^{ik\phi} \simeq \sum_{k=0}^{k_\nu} I_k e^{ik\phi}. \tag{13.37}$$

To explore the consequence of "smoothness" it is convenient to sum (13.37) by parts. First one defines

$$(\Delta I)_k = I_{k+1} - I_k, \qquad (\Delta^2 I)_k = (\Delta I)_{k+1} - (\Delta I)_k,$$
$$F_k = (e^{ik\phi} - 1)/e^{i\phi} - 1. \tag{13.38}$$

Hence $(\Delta F)_k = e^{ik\phi}$ and

$$\sum_{k=0}^{k_\nu} I_k e^{ik\phi} = + \frac{I_0}{1 - e^{i\phi}} + \frac{e^{i\phi}}{1 - e^{i\phi}} \sum_{k=0}^{k_\nu} (\Delta I)_k e^{ik\phi}. \tag{13.39}$$

Repeating this summation by parts on the final sum in (13.39), one may write

$$\sum_{k=0}^{k_\nu} I_k e^{ik\phi} = \frac{1}{1 - e^{2\phi}} \left\{ I_0 + \frac{e^{i\phi}}{1 - e^{2i}} \left[(\Delta I)_0 + e^{i\phi} \sum_{k=0}^{k_\nu} (\Delta^2 I)_k e^{ik\phi} \right] \right\}. \tag{13.40}$$

One now observes that if I_k is "smooth" in the sense that

$$(\Delta I)_k = O(I_0/k_\nu), \qquad (\Delta^2 I)_k = O(I_0/k_\nu^2), \tag{13.41}$$

then from (13.40)

$$I(\phi) = \frac{I_0}{1 - e^{i\phi}} + O(I_0/k_\nu) \tag{13.42}$$

for all angles $\phi \gg k_\nu^{-1}$. Hence a unique and simple form for $I(\phi \gg k_\nu^{-1})$ exists if the weak condition (13.40) is met. From (13.32) and (13.42), the interior mean temperature field is

$$\overline{T}_0(\phi) \simeq I_0^2 \tan\left(\frac{\phi - \pi}{2}\right). \tag{13.43}$$

This is the only law whose qualitative behavior is insensitive to the features of the underlying spectrum, yet reflects the stability conditions presumed responsible for maintaining the negative gradient.

The field equation (13.43) is also independent of the cutoff wavenumber k_ν; yet the assumption of spectral smoothness may seem less plausible at those wavenumbers where viscous effects first become as important as the nonlinear advection. A requirement placed on this "tail" region of the transport spectrum is that it drop off faster than any power of k in order that all moments of the flow be finite. A second requirement is that the "tail" region be continuous with and match the smoothness condition at the wavenumber where the viscous tail joins the inertially controlled lower-wavenumber spectrum. The simplest tail to meet these requirements is a modified exponential. Hence, one explores the consequence of the tail

$$I_{k>k_0} = I_{k_0}[1 + \alpha(k - k_0) + \beta(k - k_0)^2] e^{-\gamma(k-k_0)}, \tag{13.44}$$

where the wavenumber k_0 ($<k_\nu$) marks the low-wavenumber end of the "tail," γ characterizes the degree of abruptness of the spectral cutoff, and α, β are chosen to match smoothness conditions at $k = k_0$. For $\gamma \ll 1$, $\alpha \simeq \gamma$ and $\beta \simeq \frac{1}{2}\gamma^2$. The tail can be summed and leads to the general spectrum

$$I(\phi) = \sum_{k=0}^{k_0} I_k e^{ik\phi} + I_{k_0} e^{i(k_0+1)\phi} e^{-\gamma} \times \left[\frac{1}{1 - e^{-a}} + \alpha \frac{1}{(1 - e^{-a})^2} + \beta \frac{1 + e^{-a}}{(1 - e^{-a})^3} \right], \tag{13.45}$$

where $a = \gamma - i\phi$. As it stands, with k_0, γ, and all the I_k unspecified, (13.45) can describe any plausible turbulent mean-temperature profile of negative slope at any Ra. At this point one seeks the asymptotic consequences of the smoothness hypothesis

$$(\Delta I)_k = O(I_0/k_0),$$
$$(\Delta^2 I)_k = O(I_0/k_0^2), \qquad 0 \leq k \leq k_0, \tag{13.46}$$

and from (13.40) concludes that for $\phi \gg k_0^{-1}$, to $O(I_0/k_0)$,

$$I(\phi) = \frac{1}{1 - e^{i\phi}} \left\{ I_0 - I_{k_0} e^{i(k_0+1)\phi} \left[\frac{1 - e^{-\gamma}}{1 - e^{-a}} + \alpha \frac{e^{-a} - e^{-\gamma}}{(1 - e^{-a})^2} + \beta \frac{(e^{-a} - e^{-\gamma})(1 + e^{-a})}{(1 - e^{-a})^3} \right] \right\}. \quad (13.47)$$

If, then, $\gamma = O(k_0^{-1})$, (13.47) reduces to (13.42) and the interior temperature field (13.43). The novel aspect of the temperature profiles determined by (13.47) is the emergence of the double-tangent structure. This is most easily seen from the leading term of the inner bracketed expression in (13.47) for ϕ, $\gamma \ll 1$. One writes

$$[\,] \simeq \left[\frac{\gamma}{\gamma - i\phi} + \gamma \frac{i\phi}{(\gamma - i\phi)^2} + \gamma^2 \frac{i\phi}{(\gamma - i\phi)^3} \right]. \quad (13.48)$$

Then for $\gamma \gg \phi \gg k_0^{-1}$ the bracket expression approaches the value 1 and the resulting profile has the amplitude $|I_0|^2 + |I_{k_0}|^2$. In contrast, when $\phi \gg \gamma$, the bracket approaches 0 and the temperature profile has the "outer" amplitude $|I_0|^2$. Hence when the transport spectrum has a cutoff sufficiently more abrupt than k_0^{-1}, an "inner" inertial boundary region is predicted.

A γ large compared to k_0^{-1} is deduced in the quantitative work of the following paragraphs. However, it is likely that the predicted transport tail will be quite sensitive to the neglect of the fluctuation term in the stability problem. Unfortunately, it will be seen that present convection data is not sufficiently precise to test this speculation.

In the analogous shear-flow problem two logarithmic regions of different slopes are found (Virk, 1975). Drag-reducing additives, which appear to sharply increase γ, also cause the "inner" logarithmic region to extend much further into the flow. Theoretical studies (Malkus, 1979) predict this behavior, but indicate that for shear turbulence the mean-field-stability theory gives a γ that is larger than observed. In both the case of pure shear flow and the case of pure convection, the quantity I_0 that determines the outer-flow amplitude seems to be the most imperturbable feature of the mean-field-stability computations.

Turning now to these stability computations for convection without mean shear, one is blessed with a problem that, for free boundary conditions, can be cast in variational form. Hence the extensive numerical computations needed to implement this theory for shear flow can be replaced by a sequence of analytic approximations. The variational form of (13.30) is written

$$Ra^* = -\frac{1}{\alpha^2} \left[\int_0^1 w \left(\frac{\partial}{\partial z^2} - \alpha^2 \right)^3 w \, dz \right] \div \left[\int_0^1 \left(-\frac{\partial \overline{T}_0}{\partial z} \right) w^2 \, dz \right] \geq Ra \quad (13.49)$$

Trial forms for w lead to an Ra^* bounding the critical (experimentally given) Ra from above, and insensitive to first-order error in the trial form. Simultaneously, the amplitudes of all previously unstable modes are to be adjusted to change $-\partial\overline{T}_0/\partial z$ so that the marginal stability of any particular w occurs at minimum $Ra^* = Ra$. The free or "slippery" boundary conditions are

$$w = \frac{\partial^2 w}{\partial z^2} = \theta = 0 \quad \text{at} \quad z = 0, 1. \quad (13.50)$$

As these are also the boundary conditions for the full field $\mathbf{u}_0$, T_0, it follows that

$$\overline{W_0 T_0} = \frac{\partial \overline{W_0 T_0}}{\partial z} = 0 \quad \text{at} \quad z = 0, 1. \quad (13.51)$$

Hence, for symmetric $-\partial\overline{T}_0/\partial z$, from (13.5), (13.32), (13.33), and (13.51) I_k is real and

$$Nu = \left(\sum_{k=0}^{\infty} I_k \right)^2, \quad (13.52)$$

while

$$1 = \sum_{k=0}^{\infty} I_k^2. \quad (13.53)$$

Now starting the sequence of computations from the conductive state of no motion ($I_0 = 1$, $I_{k\neq 0} = 0$), one recovers from (13.49) the classical Rayleigh solution

$$w_1 = A_1 \sin\phi/2, \qquad \theta_1 = B_1 \sin\phi/2,$$
$$\alpha_1^2 = \pi^2/2, \qquad Ra_1 = (27/4)\pi^4, \quad (13.54)$$

where A_1 and B_1 are infinitesimal amplitudes and $\phi = 2\pi z$. The next step is to determine that amplitude of $\langle w_1\theta_1 \rangle \equiv C_1$ as a function of Ra that will maintain marginal stability against any disturbance, including w_1. C_1 can be increased until $-\partial T_0/\partial z$ reaches 0 at some point, but no further. There is evidence (Chu and Goldstein, 1973) that the fluid does not quite respect this limitation from inviscid theory in the initial postcritical range of Ra, but at higher Ra no positive mean gradients are observed. From (13.5) and (13.54)

$$-\frac{\partial \overline{T}_0}{\partial z} = 1 + \langle w_1\theta_1 \rangle - \overline{w_1\theta_1} = 1 + C_1 \cos\phi. \quad (13.55)$$

Therefore, from (13.49), marginal stability for w_1 requires that

$$C_1 = 2\left(1 - \frac{Ra_1}{Ra}\right), \quad (13.56)$$

with the limiting value $C_1 = 1$ att $Ra = 2Ra_1$. In the observed initial stage of convection, amplitude increases with Ra as in (13.56), but without stopping at $C_1 = 1$. Instead, the form of the disturbances is altered by finite-amplitude effects (e.g., Malkus and Veronis, 1958). However, at very high Ra, observations suggest that the energetic "overtones" of finite disturbances are not of smaller scale than the marginal disturbances in the boundary region.

By accident perhaps, the higher-wavenumber eigenfunctions of the conductive problem

$$w_n = A_n \sin n\phi/2, \qquad \alpha_n^2 = n^2\pi^2/2,$$

$$Ra_n = (27/4)\pi^4 n^4 \tag{13.57}$$

are optimal forms for the stability problem (13.49), even when the distorted $-\partial\overline{T}_0/\partial z$ of (13.55) is included. That is,

$$\int_0^{2\pi} (1 + C_1 \cos\phi) \sin^2 n\phi/2\, d\phi = \int_0^{2\pi} \sin^2 n\phi/2\, d\phi, \tag{13.58}$$

for $n > 1$. Hence the next instability occurs at $Ra_2 = 2^4 Ra_1$ and has the form $w_2 = \sin\phi$. If this were to continue, the gross dependence of the Nusselt number on Ra would be

$$Nu = (Ra/Ra_1)^{1/4}, \tag{13.59}$$

which is the law observed in the early stages of convection.

In this free boundary condition case, however, a new kind of disturbance leads to a lower critical Ra [greater stability, from (13.49)] beyond the second transition. If one calls the first disturbances above *body* disturbances (that is, $w_n = \sin n\phi/2$ is large throughout the whole fluid and "senses" both boundaries), then the new disturbance is a *boundary* disturbance, and is large only near one boundary. One may presume a statistical symmetry for these disturbances to maintain the observed symmetry of the mean.

Consider a trial form for such a disturbance of

$$w_{k_0+1} = \sin(k_0 + 1)\phi/2,$$

$$0 \le \phi \le \frac{2\pi}{k_0 + 1}, \qquad 2\pi - \frac{2\pi}{k_0 + 1} \le \phi \le 2\pi.$$

Then the trial $-\partial\overline{T}_0/\partial z$ is to consist of (the previous) k_0 modes, and each with arbitrary amplitude. The full problem posed is to choose those k_0 amplitudes, subject to the constraint $-\partial\overline{T}_0/\partial z \ge 0$, in order to minimize Ra in (13.49). A first and second approximation will be reported here.

To anticipate an appropriate trial form for the temperature gradient, it is of value to note the links this problem has with the search for a maximum heat flux. One sees in (13.49) that if $w_{k_0+1}^2$ is large only near the boundary, then minimum Ra requires that $-\partial\overline{T}_0/\partial z$ be large near the boundary also. If the temperature gradient resulting from the k_0 previously unstable modes can be adequately represented by the truncated spectrum

$$I(\phi) = \sum_{k=0}^{k_0} I_k e^{ik\phi}$$

subject to the definitional constraint (13.53),

$$\sum_{k=0}^{k_0} I_k^2 = 1,$$

then, from (13.52) the maximum possible

$$-\frac{\partial\overline{T}_0}{\partial z}(0) = Nu = \left(\sum_{k=0}^{k_0} I_k\right)^2$$

is

$$I_k = \frac{1}{(k_0 + 1)^{1/2}}, \qquad \text{and} \qquad Nu = k_0 + 1. \tag{13.61}$$

This exactly smooth I_k with its sharp truncation assures the general internal temperature field found from (13.42) and (13.47):

$$\overline{T}_0(\phi) = \frac{2}{k_0 + 1} \tan\left(\frac{\phi - \pi}{2}\right). \tag{13.62}$$

A complete description of this $\overline{T}_0(\phi)$ valid right to the boundary can be written in terms of sine integrals. Near the boundary this description is

$$\overline{T}_0(\phi) = \frac{1}{2}\left\{1 - \frac{2}{\pi}\left[S_1(2\xi) - \frac{\sin^2\xi}{\xi}\right]\right\}, \tag{13.63}$$

where $\xi = (k_0 + 1)\phi/2$, which merges into (13.62) for $\xi \gg 1$. If this choice (13.61) is made for the first trial I_k used to determine a minimum Ra for the disturbance w_{k_0+1} of (13.60), then one finds from (13.49) that

$$Ra = (k_0 + 1)^3 \frac{27}{4}\pi^4 \Big/ \frac{1}{\pi}[2S_i(2\pi) - S_i(4\pi)]$$

or

$$Ra = (kc1_0 + 1)^3 Ra_c, \qquad Ra_c = 1{,}533. \tag{13.64}$$

Hence this first trial form for boundary functions predicts

$$Nu = (Ra/Ra_c)^{1/3} \tag{13.65}$$

and a field $\overline{T}_0(\phi)$ determined from boundary to boundary. This result parallels, but is roughly 15% above, the experimental data (Townsend, 1959). The theory is for free boundary conditions, however, and the data for rigid boundaries. An estimate of the theoretical reduction in Nu for rigid boundaries (Malkus, 1963) is 13%. Precise theoretical results for that case will require tedious numerical computations. It might be easier to obtain good data for turbulent convection over a slippery boundary (e.g., silicon oil over mercury).

A second approximation to the most stabilizing spectrum (which, of course, will reduce the heat flux) is also based on a spectrum for $I(\phi)$ truncated at k_0. The optimal stability problem so posed, from (13.49) and (13.60), is to find the I_k that minimizes Ra in

$$Ra = \frac{27}{4}\pi^4 (k_0+1)^4 \Big/ \sum_{k=0}^{k_0}\sum_{m=0}^{k_0} I_k I_m C_{km}, \tag{13.66}$$

where

$$C_{km} = C_{|k-m|} = \frac{(k_0+1)}{\pi}\int_0^{2\pi/(k_0+1)} \cos|k-m|\phi \sin^2\frac{k_0+1}{2}\phi\, d\phi,$$

and, from (13.53),

$$\sum_{k=0}^{k_0} I_k^2 = 1.$$

One may write, for $\eta = |k-m|/(k_0+1)$,

$$C_\eta = \frac{1}{2\pi}\left(\frac{1}{\eta} + \frac{\eta}{1+\eta^2}\right)\sin 2\pi\eta, \tag{13.67}$$

and note that

$$C_0 = 1, \qquad C_{1/2} = 0, \qquad C_{k_0/(k_0+1)} = -\tfrac{1}{2}.$$

The maximum eigenvalue of the matrix C_η, say λ_{max}, determines the minimum value of Ra in (13.66). Since

$$\lambda_{max} \le \mathrm{trace}\, C_\eta = \sum_{k=0}^{k_0} C_0 = k_0 + 1, \tag{13.68}$$

then from (13.66)

$$(k_0+1)^3\frac{27}{4}\pi^4 \le Ra_{min} \le (k_0+1)^3(1{,}533), \tag{13.69}$$

where the latter bound is for the trial I_k [(13.61)]. Since the heat flux varies as $Ra_{min}^{-1/3}$, the maximum possible reduction permitted by (13.67) is 30%.

Last, a first estimate of γ can be made from the second approximation to the boundary eigenfunction (which can increase the heat flux!). If, for I_k = constant, the trial form

$$w = \sin\xi + A\sin 2\xi$$

is chosen, containing an asymmetric part of arbitrary amplitude to reflect the asymmetry of $-\partial\overline{T}_0/\partial z$ in the boundary region, then one finds from (13.49) the optima

$$A = 0.0226, \qquad \frac{\alpha^2}{(k_0+1)^2\pi^2} = 0.5069, \qquad Ra_c = 1513.$$

The small harmonic distortion (2.26%) suggests that the "inner" tangent law may persist to more than ten times the boundary layer thickness. Present data offer no hope of detecting a change in slope at such distance from the boundary. Better data may permit a determination of γ from the "inner" to "outer" law transition region found from (13.48). In any event, considerably more experimental effort is required to test critically the limits of validity of the hypothesis advanced in this section. On the positive side this study rationalizes the observed change of the $Nu = Nu(Ra)$ law from $Ra^{1/4}$ to $Ra^{1/3}$; it predicts a double-z^{-1} law for the mean field, the "inner" part observed; its first approximate convection amplitudes provide solid support for the usefulness of the concept of marginal stability of the mean; last, a theoretical program incorporating marginal stability on both a shear flow and thermal gradient is computationally practical, offering the hope of discovering unanticipated relations between observables in the geophysical setting.