Combinatorics: The Fine Art of Counting

Week One Lecture Notes

A **regular polyhedron** is a three dimensional solid figure whose faces are all congruent regular polygons. The regular polyhedra are also known as Platonic solids. Each vertex of a regular polyhedron has the same number of edges (and faces) incident to it (touching it), and this number is called the **vertex degree** or just the **degree** of the regular polyhedron. Each face has the same number of edges (and vertices) incident to it, and this number is called the **face degree** of the regular polyhedron. There are exactly five regular polyhedra and they are listed in the table below:

Name	**Faces F**	**Vertices V**	**Edges E**	**vertex degree d**	**face degree c**
tetrahedron	4	4	6	3	3
cube	6	8	12	3	4
octahedron	8	6	12	4	3
dodecahedron	12	20	30	3	5
icosahedron	20	12	30	5	3

By counting incidence relationships we find that for any regular polyhedron $\mathbf{dV = 2E = cF}$

Any polyhedron can be embedded in a sphere preserving all vertices, edges, and faces and without crossing any edges. Any graph which can be embedded in the sphere can also be embedded in the plane (with no edges crossing) by picking a point on the sphere which is not on any edge and either puncturing the sphere at this point and then stretching it flat, or by projecting from the point onto a plane below the sphere. This transformation is reversible, so any graph embedded in the plane can also be embedded on a sphere. Such a graph is said to be **planar**.

For any graph, the **degree** of a vertex is the number of edges incident to it. By counting incidence relationships we can see that the sum of the degrees of the vertices of a graph must be equal to twice the number of edges. i.e. $\mathbf{\Sigma d_i = 2E}$ where d_i is the degree of the i^{th} vertex. This can also be proven inductively.

For the rest of this lecture we will only be considering graphs which are finite, connected, and in which every vertex has degree at least two, i.e. there are no edges "poking out", ever edge and every vertex is part of a polygon (also called a cycle).

Any planar graph when embedded in the plane partitions the plane into regions which we can identify as the faces of the graph. Note that one of these regions is unbounded and is referred to as the **outer face**. By mapping to the sphere and back, we can choose to make any face the outer face.

The **dual** of a planar graph is defined by placing a vertex inside of each face and then connecting vertices in adjacent faces by edges. This process interchanges vertices and faces and leaves the number of edges unchanged. The dual of a planar graph is also planar.

Among the regular polyhedra, the cube and octahedron are duals of each other, as are the dodecahedron and icosahedron, while the tetrahedron is self-dual.

For finite planar graphs where every vertex has degree at least 2 the sum of the degrees of the faces is equal to twice the number of edges, i.e. $\mathbf{\Sigma c_i = 2E}$ where c_i is the degree of the i^{th} face. This can proven by considering the dual graph and summing the vertex degrees, or directly by counting incidence relationships.

Euler's formula for connected planar graphs: $\mathbf{V + F - E = 2}$

Proof: The proof is by induction on the number of edges in the graph. A graph with one edge contains two vertices and one face which satisfies the formula. Given a connected planar graph with E edges which satisfies the formula we can add an edge to produce a connected planar graph with E+1 edges in one of two ways, either we connect two existing vertices in which case we don't change the number of edges but we create one new face (we split an existing face in two), or we add one new vertex and don't create a new face. In either case the quantity V+F increases to V+F+1 and the formula is satisfied.

Note that any connected planar graph can be constructed by successively adding edges to a connected planar graph starting with a single edge. To see this imagine working backward and deleting one edge at a time, always choosing an edge with a vertex of degree one if such a vertex exists. This completes the proof.

Corollary 1: For any connected planar graph, $\mathbf{E \leq 3V - 6}$

Proof: This follows from the fact that every face has degree at least 3 so $\mathbf{2E = \Sigma c_i \geq 3F}$ which implies that $\mathbf{F \leq 2/3\ E}$. Plugging this into Euler's formula yields the desired result.

Corollary 2: For any triangle-free connected planar graph, $\mathbf{E \leq 2V - 4}$

Proof: If every face has degree at least 4, then using the same argument as in corollary 1 we get $\mathbf{F \leq 1/2\ E}$ and then plug this into Euler's formula to get Corollary 2.

Theorem: There are only five regular polyhedra.

Proof: First note that for any regular polyhedra the vertex degree d and face degree c must both be at least 3. By Corollary 1 $d \leq 5$ since $\mathbf{dV/2 = E \leq 3V - 6}$ is not true for $d \geq 6$. By duality $c \leq 5$ also since otherwise the dual would have $d > 5$. If c is not 3 than Corollary 2 applies and d must be 3 since $\mathbf{dV/2 = E \leq 2V - 4}$ is not true for $d \geq 4$. Therefore c and d must both be integers between 3 and 5 and one of them must be 3. This leaves exactly five possibilities all of which occur in the table above.

The **complete graph** on n vertices, denoted by $\mathbf{K_n}$ is a graph which contains every possible edge between n vertices, i.e. an edge for each pair of vertices.

A **bipartite graph** is a graph in which the vertices can be divided into two sets such that every edge connects vertices in different sets. The complete bipartite graph on m+n vertices, denoted by $\mathbf{K_{m,n}}$, is a graph which contains every possible edge between a set of m vertices and a set of n vertices.

Theorem: $K_{3,3}$ is not a planar graph.

Proof: $K_{3,3}$ has 3 + 3 = 6 vertices and 3 * 3 = 9 edges and does not contain a triangle (or any cycle of odd length) since it is bipartite – if we begin at a vertex in one set and then follow edges along a cycle, we can only return to the starting vertex after an even number of edges since each edge takes us to the opposite set. By Corollay 2 $K_{3,3}$ cannot be planar since 9 is greater than 2 * 6 – 4 = 8.

The n-dimensional **hypercube**, denoted by $\mathbf{H_n}$ is defined inductively as follows: H_1 is a single edge with two vertices. H_{n+1} is obtained by taking two copies of H_n and connecting corresponding vertices with edges. Thus H_2 is a square, and H_3 is a standard cube.

A **semi-regular polyhedron** is a polyhedron with faces that are all regular polygons and which is **vertex uniform**, i.e. every vertex has the same degree and the same configuration of incident faces, but the faces need not all have the same degree.

One way to obtain a semi-regular polyhedron is by truncating a regular polyhedron, i.e. slicing off the vertices of a regular polyhedron to create a new face where there used to be a vertex. There are two different ways to do this, both of which are explored in the homework problems.

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