15.433 INVESTMENTS

Class 5: Portfolio Theory

Part 3: Optimal Risky Portfolio

Spring 2003

Introduction

- Having determined the appropriate exposure to risk, the investor's next task is to build his risky portfolio r_p.

- This selection will be made from the whole universe of risky assets that are available for investment.

How Big is the Universe of Risky Assets?

We live in an exciting time, with an expansive investment environment. Here is a list of the plain-vanilla instruments reported by RiskMetrics$^©$, $source: RiskMetrics^{TM} - Technical\ Document$

       Equity Indices: 31 countries.

       Foreign Exchange Rates: 31 currencies.

       Money Markets around the world: 111.

       Swaps around the world: 121.

       Gov't Bonds around the world: 153.

       Commodities: 33.

Over 400 different instruments!

This does not include individual equities, mutual funds, venture capital funds, futures, options, and other derivatives ... .

Two Risky Assets

1. r_1 $\mu_1 = 13\%, \sigma_1 = 20\%$
2. r_1 $\mu_2 = 8\%, \sigma_2 = 12\%$
3. *correlated returns* : $\rho = corr\,(r_1, r_2) = 30\%$.

Mixing the Risky Assets

- a fraction of w in equity

- a fraction of 1 - w in debt

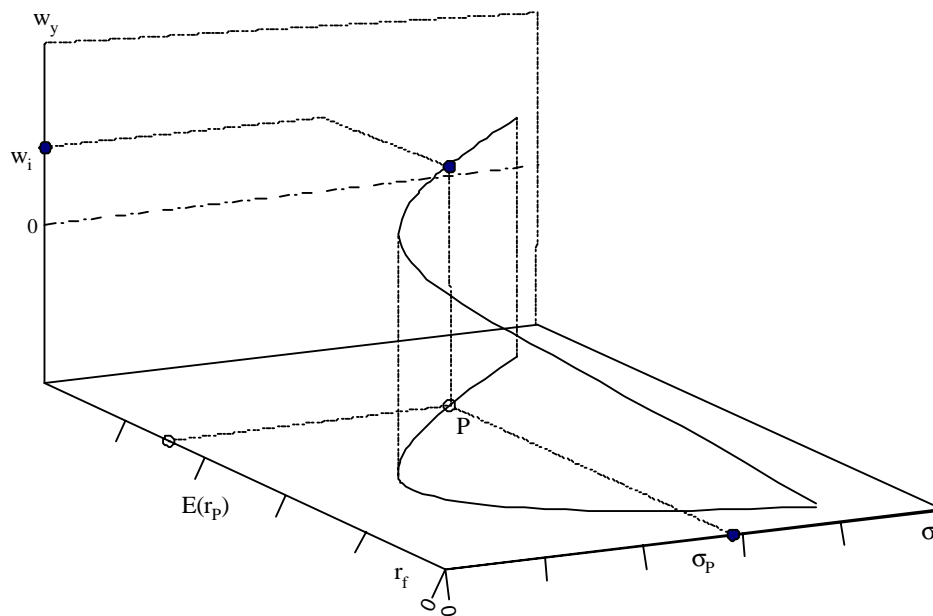- the risky portfolio $r_p = w \cdot r_1 + (1 - w) \cdot r_2$

Figure 4: Efficient frontier.

The Risky Portfolios

Let's first calculate the mean and variance of the risky portfolio:

$$\mu_P \quad = \quad w \cdot \mu_1 + (1 - w) \cdot \mu_2 \tag{1}$$

$$\sigma_p^2 \quad = \quad w^2 \cdot \sigma_1^2 + (1 - w)^2 \cdot \sigma_2^2 + 2 \cdot w \cdot (1 - w) \cdot cov\,(r_1, r_2) \tag{2}$$

$$= \quad w^2 \cdot \sigma_1^2 + (1 - w)^2 \cdot \sigma_2^2 + 2 \cdot w \cdot (1 - w) \cdot \sigma_1 \sigma_2 \rho_{1,2} \tag{3}$$
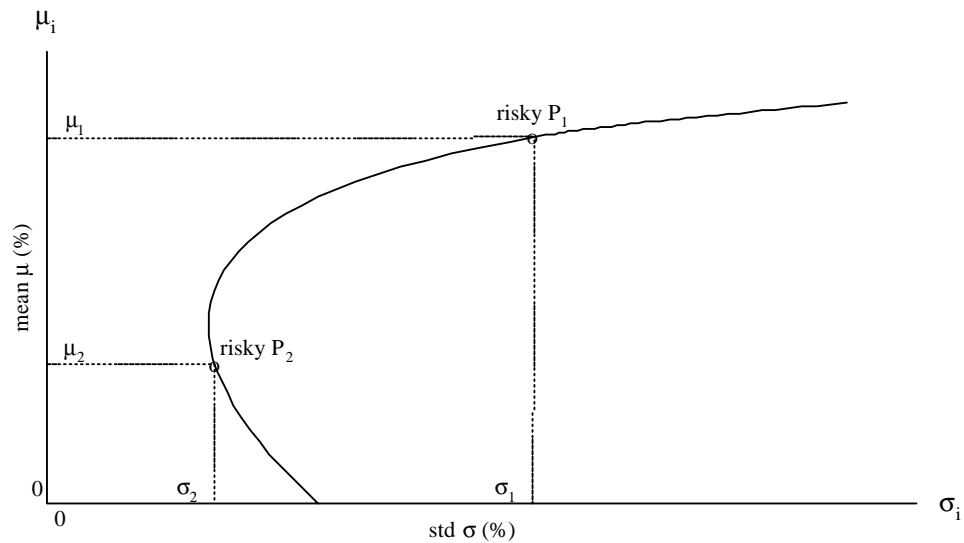
Spanning the mean-std space:

Figure 5: Efficient frontier.

The Optimal Portfolio Problem

the investment opportunity

- one riskfree r_f

- two risky assets r_1 and r_2

a mean-variance investor

$$U\left(r\right) = E\left(r\right) - \frac{1}{2} \cdot A \cdot var\left(r\right) + \frac{1}{6} \cdot B \cdot E\left(r - E\left(r\right)\right)^3 \tag{4}$$

investment decisions:

1. choose the overall exposure to risk:
   y in the risky portfolio r_p and $1 - y$ in r_f.

2. choose the right risky portfolio r_p:
   w in r_1, and $1 - w$ in r_2.

possible portfolios:

$$r_{y,w} = \left(1 - y\right) \cdot r_f + y \cdot \left(w \cdot r_1 + \left(1 - w\right) \cdot r_2\right) \tag{5}$$

the optimal portfolio:

$$\underset{y \in \mathbb{R}, w \in \mathbb{R}}{max} U\left(r_{y,w}\right) \tag{6}$$

The Separation Principle

In addition to the choice variable y, which controls the investor's overall exposure to risk, our current problem also involves the choice variable w, which controls the right mix of the risky assets.

The Complication: Need to solve y and w simultaneously in the optimization problem:

$$\max_{y \in \mathbb{R}, \, w \in \mathbb{R}} U\left(r_{y,w}\right) \tag{7}$$

Solution: The Separation Principle (James Tobin 1958):

- the choice y of the overall risk exposure is investor specific, depending on his degree of risk aversion;

- the choice w of the optimal risky portfolio is the same for all investors.
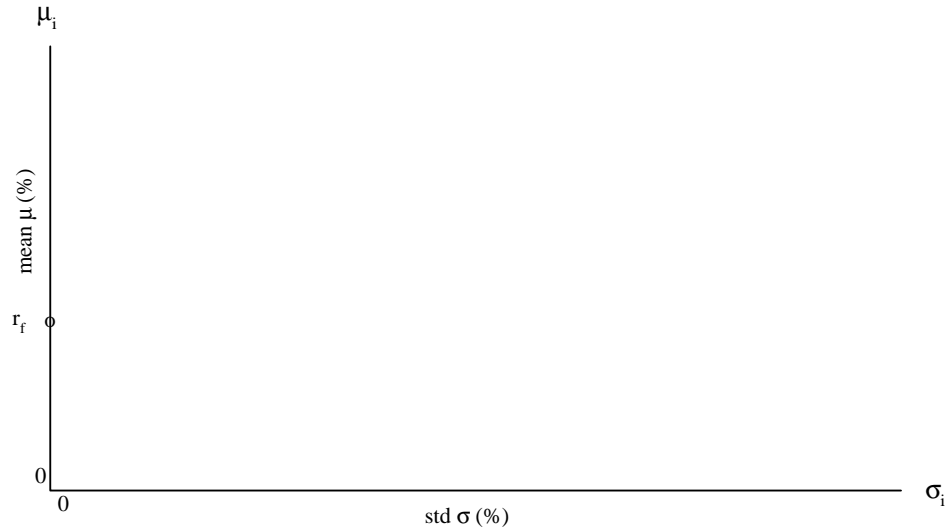
A Brief Review of CAL

Figure 6: Efficient frontier.

Pick any risky portfolio r_p, invest a fraction y in it, leaving the rest in the riskfree account r_f.

What are the possible (E-Std) combinations?

$$E\left(r_y\right) - r_f = \frac{E\left(r_y\right) - r_f}{std\left(r_p\right)} std\left(r_y\right) \tag{8}$$
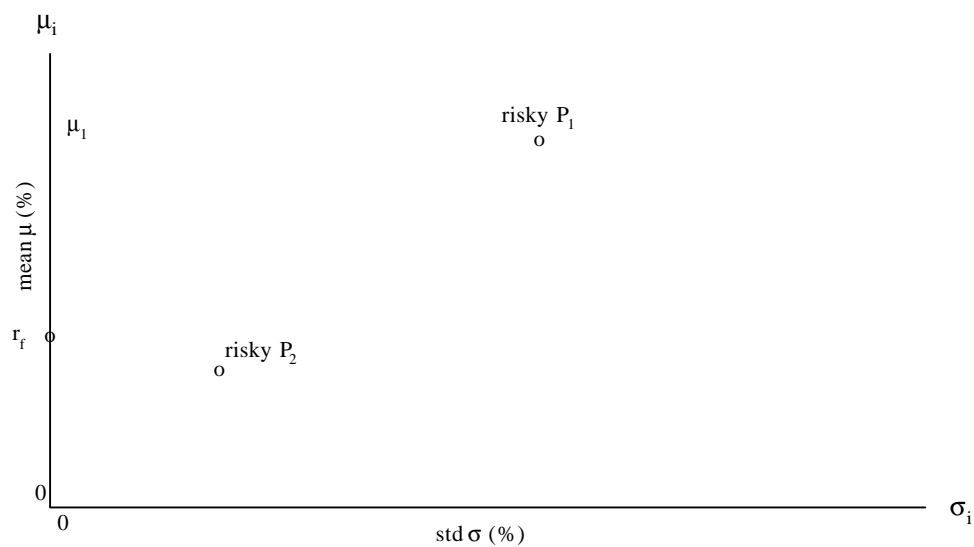
A Brief Review of the Sharpe Ratio

Figure 7: Efficient frontier.

One measure of the attractiveness of a portfolio r is its Sharpe Ratio:

$$S = \frac{E\left(r\right) - r_f}{std\left(r\right)} \tag{9}$$

What is its graphical interpretation?
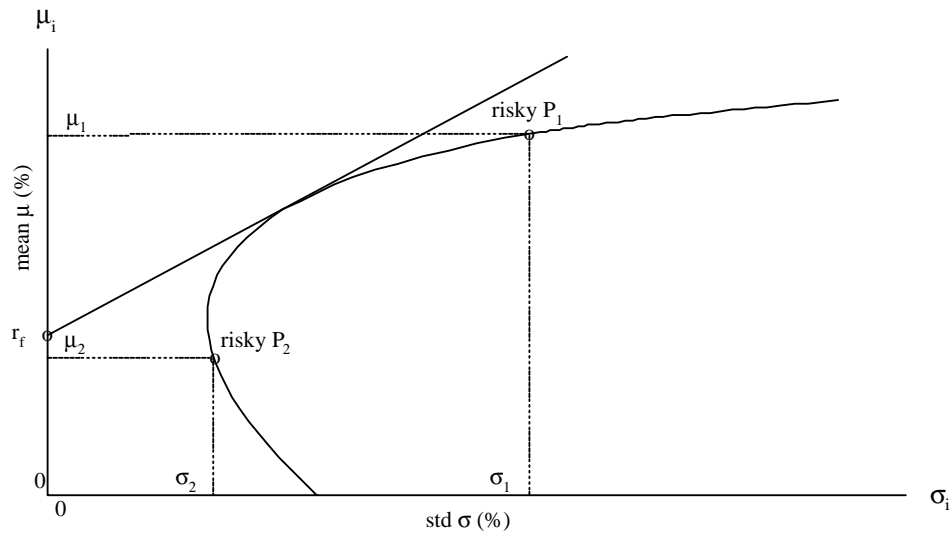
The Optimal Risky Portfolio

Figure 8: Efficient frontier.

Each CAL is uniquely identified by its slope.

The best CAL is the one with the steepest slope, or the highest Sharpe Ratio.

The Optimal Risky Portfolio is the Tangency Portfolio: the unique portfolio with the highest Sharpe-Ratio.

Conceptually, it is very important to notice that the definition of the Optimal Risky Portfolio does not involve the degree of risk aversion of any individual investor.

In such an ideal world, every investor, regardless of his level of risk aversion, will agree on the best CAL, and allocate his wealth between r_f and the optimal risky portfolio.

The portion y invested in the optimal risky portfolio, however, will depend on each investor's degree of risk aversion, as discussed in Class 3.

In practice, however, different investors might have very different ideas about their Optimal Risky Portfolio. Why?

Multiple Risky Assets

The opportunity set generated by the two risky assets is relatively simple. As we generalize to the case of multiple risky assets, the opportunity set becomes considerably more complicated. Suppose that we have n securities, whose random returns are denoted by $r_1, r_2, \ldots, r_n$

As usual, we construct portfolios by mixing these n securities:

$$r_p = w_1 \cdot r_1 + w_2 \cdot r_2 + \cdots + w_n \cdot r_n \tag{10}$$

where the w's are the portfolio weights, adding up to 1.

Each chosen w gives rise to an investment opportunity with

$$E(r_p) = \sum_{i=1}^{n} w_i \cdot E(r_i) \tag{11}$$

$$var(r_p) = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i \cdot w_j \cdot cov(r_i, r_j) \tag{12}$$

$$cov_{i,j} = \sum_{i=1}^{n} \frac{1}{n} (r_i - \bar{r}_i) \cdot (r_j - \bar{r}_j) \qquad equal\ weighted \tag{13}$$

$$cov_{i,j} = \sum_{i=1}^{n} p_t \cdot (r_i - \bar{r}_i) \cdot (r_j - \bar{r}_j) \qquad probability\ weighted \tag{14}$$

This results in an enormous degree of freedom, and a very rich opportunity set. Of course, not all portfolios in the opportunity set are good deals.

Constructing the mean?variance frontier: an efficient set of portfolios that achieve the lowest possible risk for any given target rate of return.

The Optimal Portfolio in Practice

Implication of the Separation Principle: A portfolio manager will offer the same risky portfolio to all clients.

In practice, different managers focus on different subsets of the whole universe of financial assets, derive different efficient frontiers, and offer different "optimal" portfolios to their clients.Why?

The theory of portfolio selection builds on many simplifying assumptions:

- No Market Frictions (tax, transactions costs, limited divisibility of financial assets, market segmentation).

- No Heterogeneity in Investors (e.g. rich vs. poor, informed vs. uninformed, young vs. old).

- Static expected returns and variance - no forecastability in returns or volatility (e.g. financial analysts, accounting information, macro?economic variables do not play any role in making an investment decision).

Our next step toward reality (in Classes 19-20): link the following two bodies of ideas,

- Security Analysis: subjective, judgmental

- Portfolio Selection: objective, statistical.

Some important questions to think about:

- Can security analysis improve portfolio performance?

- How do analysts' opinions enter in security selection?

A Digression to Diversification

Diversification is a universal concept. Put in simple terms, one should not put all eggs in one basket.

In investments, diversifying means holding similar amounts of many risky assets instead of concentrating all of your investment in only one.
In social science, it is believed that individuals are of "bounded rationality". One way to mitigate such a problem is through decision-making by multiple agents.

For example, in gymnastics or figure skating competitions, the score is averaged over multiple judges, after taking away extremes on either end.

The mathematical foundation of diversification - the strong law of large numbers!

$$\sigma_{\varepsilon_p}^2 = \sum_{i=1}^{n} \left(\frac{1}{n}\right)^2 \sigma_{\varepsilon_i}^2 = \frac{1}{n}\sigma_{\varepsilon}^2 \tag{15}$$

$$\beta_p = \frac{1}{n}\sum_{i=1}^{n} \cdot \beta_i \tag{16}$$

$$\alpha_p = \frac{1}{n}\sum_{i=1}^{n} \cdot \alpha_i \tag{17}$$

$$\varepsilon_p = \frac{1}{n}\sum_{i=1}^{n} \cdot \varepsilon_i \tag{18}$$

$$\sigma_p^2 = \beta_p^2 \sigma_M^2 + \sigma_{\varepsilon_p}^2 \tag{19}$$

Leisure Readings

"The Interior Decorator Fallacy" Chapter 3 of Capital Ideas by Peter Bernstein.

Focus:

BKM Chapter 8

- p. 210-213 (eq. 8.2, eq. 8.4, table 8.2)

- p. 217 middle to 229 (Markowitz, efficient frontier, mean variance)

- p. 234-239 (separation property, CAL)

Reader: Kritzman (1994)

type of potential questions: concept check question 1, 2, 3, p. 286 ff. questions 5, 11, 18

Questions for the Next Class

Please read BKM Chapters 9-11, Roll and Ross (1995) and Kritzman (1991), and think about the following questions:

The separation principle implies that every investor, regardless of his degree of risk aversion, will hold the same optimal risky portfolio. What implication does this result have on the entire market? Suppose there are I investors in the market. Each investor i has? a risk aversion coefficient A_j, with optimal exposure to the optimal risky portfolio.

$$y_i^* = \frac{E(r_p) - r_f}{var(r_p) \cdot A_i} \tag{20}$$

In equilibrium, adding y_i^* across all investors, what do we get?, (Hint: In equilibrium, supply equals demand, e.g., the amount of borrowing equals that of lending.)