Effective Route Guidance in Traffic Networks

Lectures developed by
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Outline

- **Lecture 1**

  Route Guidance; User Equilibrium; System Optimum; User Equilibria in Networks with Capacities.

- Lecture 2

  Constrained System Optimum; Dantzig-Wolfe Decomposition; Constrained Shortest Paths; Computational Results.

The Context

- Olaf Jahn (Research Assistant).
- Rolf H. Möhring (Principal Investigator).

  Collaboration with and support by DaimlerChrysler, Berlin.

- Nicolas Stier (Research Assistant).
- Andreas S. Schulz (Principal Investigator).

  Supported by General Motors Innovation Grant and SMA.

Traffic

Forecast: Number of cars will increase further

Fact: Infrastructure will not be enhanced to the same extent

Remedy: Improve the efficiency of traffic by other means

2002 Urban Mobility Study

(http://mobility.tamu.edu/ums)

> "The bad news is that even if transportation officials do all the right things, the likely effect is that congestion will continue to grow. . . "

- Total congestion "bill" in 2000 was \$67.5 billion

  (= 3.6 billion hours delay + 5.7 billion gallons gas)

	1982	2000
time penalty for peak period travelers	16 hours	62 hours

Problem

People travel (between 6% and 19%) too much because of an unfavorable selection of their route.

(Beccaria & Bolelli 1992, Lösch 1995)

Potential Remedies

- Toll systems
- Dynamic traffic signal control
- Park and Ride
- Traveller information systems

Route Guidance

In-Car Navigation Systems

Shortest Path Routing

Improved network performance, but . . .

(Kaufman et al. 1991, Lee 1994)

Shortest Path Routing II

. . . the same simulations show the performance decreases as soon as many cars use the system.

Proposed Solutions

- Multiple path routing:
  - k shortest paths
  - random perturbation
- Feedback control:
  - iterative computation of shortest paths
- Traffic assignment:
  - user equilibrium
  - system optimum
  - a new approach

selfish users	central planner	the goal
optimize own travel time	optimize system welfare	
fair, not efficient	efficient, not fair	**fair, efficient**

How much can one gain?

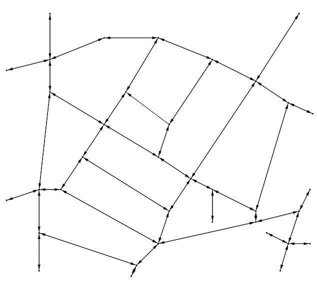
- Study worst-case ratios between guided / unguided traffic
- Without guidance: use game theory to predict traffic (Wardrop 1952)
- Users' behavior modeled as user equilibrium (Nash eq.)
- Price of anarchy is a measure of user equilibrium performance (Papadimitriou 2001)

The Traffic Model

Modeling Assumptions

Reality	Our Model
• microscopic → individual vehicles → exact position, speed	• macroscopic → one abstract measure → traffic flow
• dynamic → consider time → on a single point at any time	• static → time independent → simultaneously at any point of the path
• on-line → additional input over time	• off-line → all data known in advance

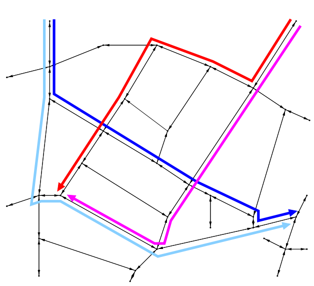
Representation of the Road Network



Flows

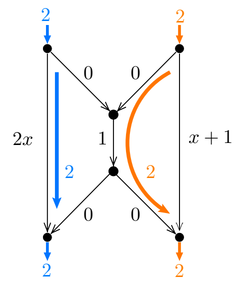
- Different drivers have different origins and destinations
- Flows on paths: f_P is the traffic along path P
- Flow on arcs:

$$f_a = \sum_{P:a \in P} f_P$$



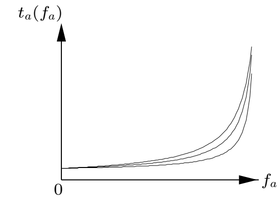
The Traffic Model

- Directed graph $G = (V, A)$,
  k demands (o_i, d_i) with rate r_i
- Flows on paths f_P. Can be non-integral.
- Traversal times: *latency functions* $t_a(\cdot)$
  $\rightarrow$ *continuous* and *nondecreasing*
  $\rightarrow$ belong to a given set $\mathcal{L}$ (e.g. linear)
- The total travel time of a flow is:
  $C(f) := \sum_{a \in A} t_a(f_a) f_a$



$2 \cdot 4 + 2 \cdot 1 = 10$

Traversal Time Functions



- Traversal time of an arc a depends on the flow f_a on a
- Dependence captured by the function $t_a(f_a)$
- Travel time along path P is denoted by $t_P(f) = \sum_{a \in P} t_a(f_a)$

System Optimum

Convex Multicommodity Min-Cost Flow Problem

$$
\begin{aligned}
\text{min} \quad & \sum_{a \in A} t_a(f_a) f_a \\
\text{s.t.} \quad & \sum_{P \ni a} f_P = f_a \qquad \text{for all } a \in A \\
& \sum_{P \in \mathcal{P}_i} f_P = r_i \qquad \text{for all } i = 1, \ldots, k \\
& f_P \geq 0 \qquad \text{for all } P \in \mathcal{P}
\end{aligned}
$$

where $\mathcal{P}_i$: set of paths from o_i to d_i
$\mathcal{P} = \cup \mathcal{P}_i$

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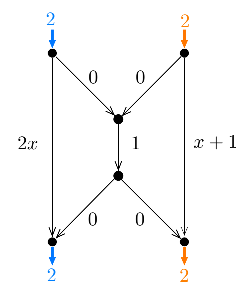
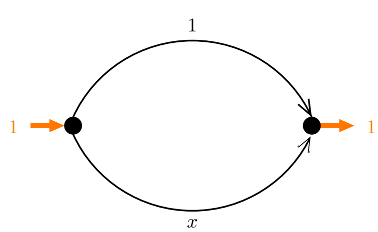


Example of SO

(Pigou 1920)



SO =min $f_a + f_b^2$
s.t. $f_a + f_b = 1$
$f_a, f_b \geq 0$

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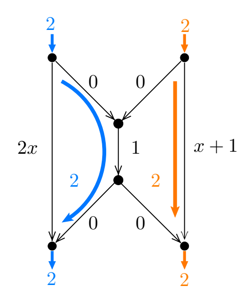
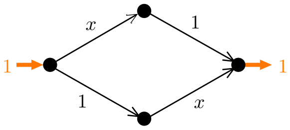


$2 \cdot 1 + 2 \cdot 3 = 8$

Braess Paradox

- **UE** non-monotone with network improvements (Braess 1968)



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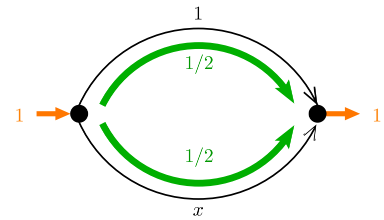

SO $=\min\ f_a + f_b^2$ $\quad = \min\ f_b^2 + 1 - f_b = 3/4$ and $f_a = \frac{1}{2}$

s.t. $f_a + f_b = 1$ $\quad$ s.t. $0 \le f_b \le 1$ $\quad f_b = \frac{1}{2}$

$f_a, f_b \ge 0$

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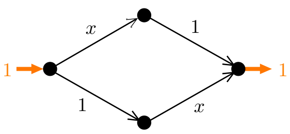
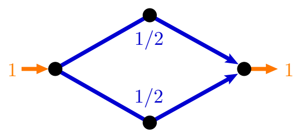


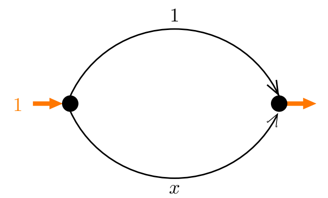


$C(f) = \frac{3}{2}$

User Equilibrium

Definition : A flow is a **UE** iff nobody can switch to a path with smaller travel time.

- Travel times of users between the same OD-pair are equal
- **UE** always exists and is unique (Beckmann et al. 1956)



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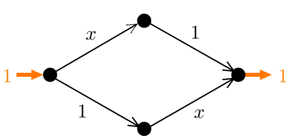

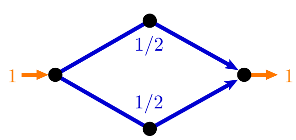

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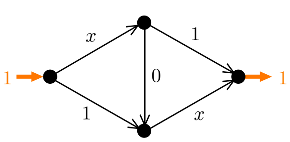

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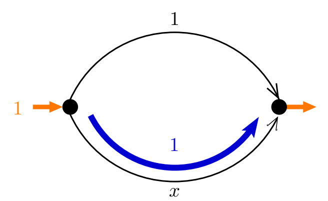
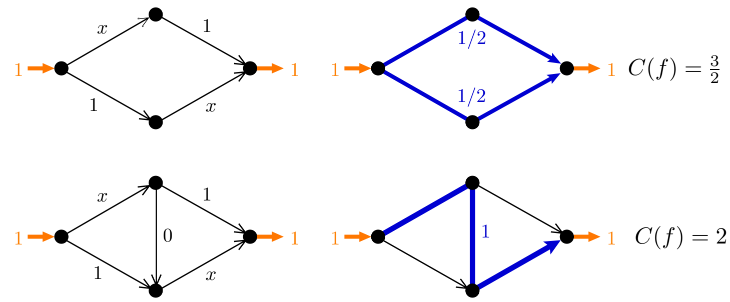


Braess Paradox

- **UE** non-monotone with network improvements (Braess 1968)



Price of Anarchy measures impact of lack of Central Coordination

(Papadimitriou 2001)

Price of Anarchy $\gamma := \max_{\text{inst.}} \frac{C(\textbf{UE})}{C(\textbf{SO})}$

- In general, γ unbounded (Roughgarden & Tardos 2000)
- If latencies are in $\mathcal{L}$, $\gamma \le \alpha(\mathcal{L})$, where $\alpha(\mathcal{L})$ depends only on $\mathcal{L}$
  In particular, $\alpha(\{\text{linear latencies}\}) = 4/3$
  (Roughgarden & Tardos 2000)
  (Roughgarden 2002)
- Pigou's and Braess' examples are worst possible

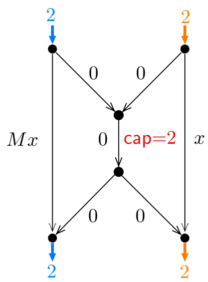
Networks with Capacities

Networks with Capacities

- Latencies model capacity only implicitly

What is the impact of having explicit capacities on arcs?

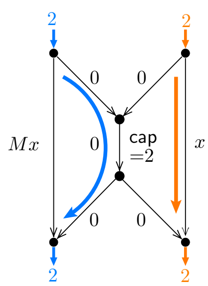
- Introduce capacities as hard constraints
- Straightforward to define **SO**
- What is now a **UE**?



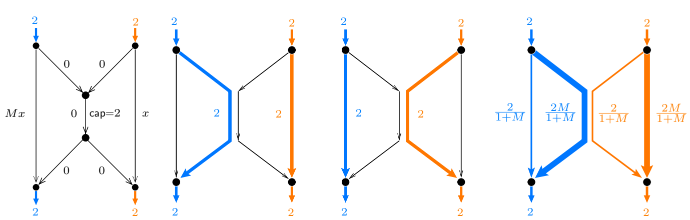
Equilibria in Networks with Capacities

Definition: A flow is a capacitated **UE** iff nobody can switch to a path with smaller travel time and residual capacity

- Travel times for users of same demand may differ (were constant w/o cap.)
- There may be multiple equilibria (**UE** w/o cap. was unique)
- How good is the best / worst eq.?



Multiple Equilibria



with costs of: 4 $4\,M$ $4\,\frac{M}{1+M}$

Worst UE can be unbounded!

Network with Capacities: Guarantees

Theorem. *For any instance of a network with capacities with latencies in $\mathcal{L}$, we have*

$$C(\textbf{BUE}) \leq \alpha(\mathcal{L})\, C(\textbf{SO})$$

In particular, if latencies are linear, $C(\textbf{BUE}) \leq \frac{4}{3} C(\textbf{SO})$

Guarantee does not change with introduction of capacities

Convex Optimization Review

- Let z be a continuously differentiable and convex function on a convex set.
- Then x^* is a global minimum of z iff

the gradient along all feasible directions is non-negative

Proof (for the Linear Case)

- Assume $t_a(f_a) = q_a f_a + r_a$ for all a and let $f =$ **BUE**

Beckmann UE

- Space of **UE** non-convex: Difficult to get Best **UE**
- Instead, Beckmann **UE** (**BUE**) is

$$\min \sum_{a \in A} \int_0^{f_a} t_a(x)dx$$

subject to f feasible flow
capacity constraints

- Opt. Cond. **BUE**: g feasible direction $\Rightarrow \sum_a g_a t_a(f_a) \geq 0$

Proof (for the Linear Case)

- Assume $t_a(f_a) = q_a f_a + r_a$ for all a and let $f =$ **BUE**
- Let $C^f(x) = \sum_a x_a t_a(f_a)$

Beckmann UE is an Equilibrium

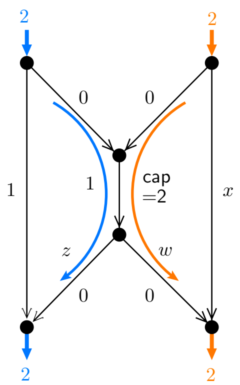
Lemma. *f is a* **BUE** $\Rightarrow$ *f is a* **UE**

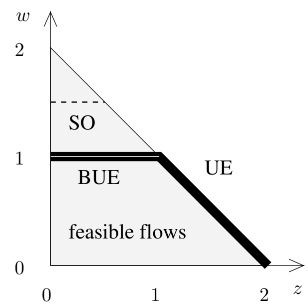
Proof:

- Suppose f is not a **UE** $\Rightarrow \exists$ two paths P, Q s.t. flow can be re-routed from P to Q and $t_P(f) > t_Q(f)$
- P and Q define a circulation g which is a feasible direction
- $\sum_a g_a t_a(f_a) < 0 \quad \Rightarrow \quad f$ is not a **BUE**

But BUE is not necessarily the best equilibrium

Non-convexity of UE



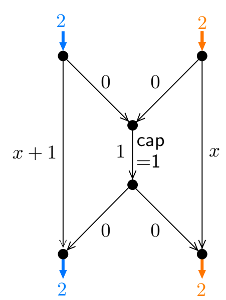


Proof (for the Linear Case)

- Assume $t_a(f_a) = q_a f_a + r_a$ for all a and let $f =$ **BUE**
- Let $C^f(x) = \sum_a x_a t_a(f_a)$
- For all flows x: $C(f) \leq C^f(x)$
  (same as $\sum_a (x_a - f_a) t_a(f_a) \geq 0$, the condition for **BUE**)

Best vs. Beckmann

- The **BUE** does not need to be best **UE**:



$C($**BUE**$) = 7$ and $C($best **UE**$) = C($**SO**$) = 6.875$

Proof (for the Linear Case)

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- For all flows x: $C(f) \leq C^f(x)$
  (same as $\sum_a (x_a - f_a) t_a(f_a) \geq 0$, the condition for **BUE**)
- $C^f(x) = \sum_a x_a(q_a f_a + r_a) \leq \sum_a x_a(q_a x_a + r_a) + \frac{1}{4}\sum_a f_a^2 q_a$
  because $(x - f/2)^2 \geq 0$

Review

No capacities	With capacities
UE unique	Set of **UE** may be non-convex
UE/**SO** $\geq \alpha(\mathcal{L})$	**UE**/**SO** unbounded
UE/**SO** $\leq \alpha(\mathcal{L})$	**BUE**/**SO** $\leq \alpha(\mathcal{L})$

Proof (for the Linear Case)

- Assume $t_a(f_a) = q_a f_a + r_a$ for all a and let $f =$ **BUE**
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  because $(x - f/2)^2 \geq 0$
- Last, $\sum_a x_a(q_a x_a + r_a) + \frac{1}{4}\sum_a f_a^2 q_a \leq C(x) + \frac{1}{4}C(f)$

$$\Rightarrow \quad \frac{3}{4}C(f) \leq C(x)$$