15.083J/6.859J Integer Optimization

Lecture 5: Ideal formulations I

1 Outline

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- Total unimodularity
- Dual Methods

2 Total unimodularity

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- $S = \{\boldsymbol{x} \in \mathcal{Z}_+^n \mid \boldsymbol{Ax} \leq \boldsymbol{b}\}$, $\boldsymbol{A} \in \mathcal{Z}^{m \times n}$ and $\boldsymbol{b} \in \mathcal{Z}^m$.
- $P = \{\boldsymbol{x} \in \Re_+^n \mid \boldsymbol{Ax} \leq \boldsymbol{b}\}$.
- When $P = \text{conv}(S)$ for all integral vectors $\boldsymbol{b}$?

2.1 Cramer's rule

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- $\boldsymbol{A} \in \Re^{n \times n}$ nonsingular.
- $\boldsymbol{Ax} = \boldsymbol{b} \iff \boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b} \iff \forall\, i: \; x_i = \frac{\det(\boldsymbol{A}^i)}{\det(\boldsymbol{A})}$.
- $\boldsymbol{A}^i$: $\boldsymbol{A}_j^i = \boldsymbol{A}_j$ for all $j \in \{1, \dots n\} \setminus \{i\}$ and $\boldsymbol{A}_i^i = \boldsymbol{b}$.

2.2 Definition

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- $\boldsymbol{A} \in \mathcal{Z}^{m \times n}$ of full row rank is **unimodular** if the determinant of each basis of $\boldsymbol{A}$ is 1, or -1. A matrix $\boldsymbol{A} \in \mathcal{Z}^{m \times m}$ of full row rank is **unimodular** if $\det(\boldsymbol{A}) = \pm 1$.
- A matrix $\boldsymbol{A} \in \mathcal{Z}^{m \times n}$ is **totally unimodular** if the determinant of each square submatrix of $\boldsymbol{A}$ is 0, 1, or -1.

2.3 Examples

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- $\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$ is not TU: $\det\left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right) = -2$.
- $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ is TU.

2.4 Proposition

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- $\boldsymbol{A}$ is TU if and only if $[\boldsymbol{A}, \boldsymbol{I}]$ is unimodular.
- $\boldsymbol{A}$ is TU if and only if $\begin{bmatrix} \boldsymbol{A} \\ \text{-}\boldsymbol{A} \\ \boldsymbol{I} \\ \text{-}\boldsymbol{I} \end{bmatrix}$ is TU.
- $\boldsymbol{A}$ is TU if and only if $\boldsymbol{A}'$ is TU.

2.5 Theorem

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- $\boldsymbol{A}$ integer matrix of full row rank. $\boldsymbol{A}$ is unimodular if and only if $P(\boldsymbol{b}) = \{\boldsymbol{x} \in \Re^n_+ \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$ is integral for all $\boldsymbol{b} \in \mathcal{Z}^m$ for which $P(\boldsymbol{b}) \neq \emptyset$.
- $\boldsymbol{A}$ integer matrix. $\boldsymbol{A}$ is TU if and only if $P(\boldsymbol{b}) = \{\boldsymbol{x} \in \Re^n_+ \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}$ is integral for all $\boldsymbol{b} \in \mathcal{Z}^m$ for which $P(\boldsymbol{b}) \neq \emptyset$.

2.5.1 Proof

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- Assume that $\boldsymbol{A}$ is unimodular. $\boldsymbol{b} \in \mathcal{Z}^m$ and $P(\boldsymbol{b}) \neq \emptyset$.
- $\boldsymbol{x} = (\boldsymbol{x}_B, \boldsymbol{x}_N)$ extreme point of $P(\boldsymbol{b})$, $\boldsymbol{x}_B = \boldsymbol{A}_B^{-1}\boldsymbol{b}$ and $\boldsymbol{x}_N = \boldsymbol{0}$.
- Since $\boldsymbol{A}$ unimodular $\det(\boldsymbol{A}_B) = \pm 1$. By Cramer's rule and the integrality of $\boldsymbol{A}_B$ and $\boldsymbol{b}$, $\boldsymbol{x}_B$ is integral.
- $P(\boldsymbol{b})$ is integral.

- Conversely, $P(\boldsymbol{b})$ integral for all $\boldsymbol{b} \in \mathcal{Z}^m$.
- $B \subseteq \{1, \ldots, n\}$ with $\boldsymbol{A}_B$ nonsingular.
- $\boldsymbol{b} = \boldsymbol{A}_B\boldsymbol{z} + \boldsymbol{e}_i$, where $\boldsymbol{z}$ integral: $\boldsymbol{z} + \boldsymbol{A}_B^{-1}\boldsymbol{e}_i \geq \boldsymbol{0}$ for all i.
- $\boldsymbol{A}_B^{-1}\boldsymbol{b} = \boldsymbol{z} + \boldsymbol{A}_B^{-1}\boldsymbol{e}_i \in \mathcal{Z}^m$ for all i.
- ith column of $\boldsymbol{A}_B^{-1}$ is integral for all i.
- $\boldsymbol{A}_B^{-1}$ is an integer matrix, and thus, since $\boldsymbol{A}_B$ is also an integer matrix, and $\det(\boldsymbol{A}_B)\det(\boldsymbol{A}_B^{-1}) = 1$, we obtain that $\det(\boldsymbol{A}_B)$ =1 or -1.
- For second part: $\boldsymbol{A}$ is TU if and only if $[\boldsymbol{A}, \boldsymbol{I}]$ is unimodular. For any $\boldsymbol{b} \in \mathcal{Z}^m$ the extreme points of $\{\boldsymbol{x} \in \Re^n_+ \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}$ are integral if and only if the extreme points of $\{(\boldsymbol{x}, \boldsymbol{y}) \in \Re^{n+m}_+ \mid \boldsymbol{A}\boldsymbol{x} + \boldsymbol{I}\boldsymbol{y} = \boldsymbol{b}\}$ are integral.

2.6 Corollary

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Let $\boldsymbol{A}$ be an integral matrix.

- $\boldsymbol{A}$ is TU if and only if $\{\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{0} \leq \boldsymbol{x} \leq \boldsymbol{u}\}$ is integral for all integral vectors $\boldsymbol{b}$ and $\boldsymbol{u}$.
- $\boldsymbol{A}$ is TU if and only if $\{\boldsymbol{x} \mid \boldsymbol{a} \leq \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b},\ \boldsymbol{l} \leq \boldsymbol{x} \leq \boldsymbol{u}\}$ is integral for all integral vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}$.

2.7 Theorems

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- $\boldsymbol{A}$ is TU if and only if each collection J of columns of $\boldsymbol{A}$ can be partitioned into two parts so that the sum of the columns in one part minus the sum of the columns in the other part is a vector with entries 0, +1, and -1.
- $\boldsymbol{A}$ is TU if and only if each collection Q of rows of $\boldsymbol{A}$ can be partitioned into two parts so that the sum of the rows in one part minus the sum of the rows in the other part is a vector with entries only 0, +1, and -1.

2.8 Corollary

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The following matrices are TU:

- The node-arc incidence matrix of a directed graph.
- The node-edge incidence matrix of an undirected bipartite graph.
- A matrix of zero-one elements, in which each column has its ones consecutively.

2.9 Example

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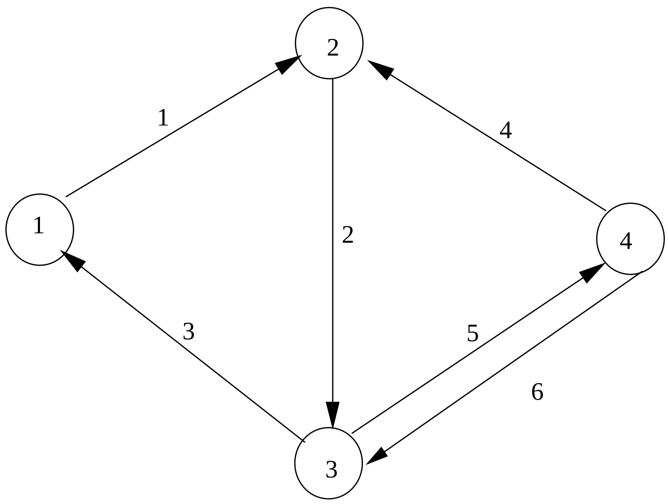
$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{bmatrix}$$



2.10 Implications

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Following problems can be solves as LOs:

- Network flows
- Matching in biparite graphs
- Stable set in biparite graphs.

3 Dual methods

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- $$\begin{aligned} Z_{\mathrm{LP}} = \min \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x} \in P \end{aligned}$$
- Let P be a nonempty polyhedron with at least one extreme point. The polyhedron P is integral if and only if Z_{LP} is integer for all $\boldsymbol{c} \in \mathcal{Z}^n$.
- For converse, assume $\boldsymbol{x}^* \in P$, extreme point with x_j^* fractional. $\boldsymbol{c} \in \mathcal{Z}^n$: $\boldsymbol{x}^*$ unique optimum.
- There exist $a \in \mathcal{Z}$: $\boldsymbol{x}^*$ optimum for $\overline{\boldsymbol{c}} = \boldsymbol{c} + (1/a)\boldsymbol{e}_j$. $a\overline{\boldsymbol{c}}'\boldsymbol{x}^* - a\boldsymbol{c}'\boldsymbol{x}^* = x_j^*$, either $a\overline{\boldsymbol{c}}'\boldsymbol{x}^*$ or $a\boldsymbol{c}'\boldsymbol{x}^*$ is fractional. Contradiction.

3.1 Key idea

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Construct a solution to the dual of the LP relaxation and an integer solution, feasible to IO with $Z_{\mathrm{H}} = Z_{\mathrm{D}}$. Since $Z_{\mathrm{D}} \leq Z_{\mathrm{LP}} \leq Z_{\mathrm{IP}} \leq Z_{\mathrm{H}}$, if $Z_{\mathrm{H}} = Z_{\mathrm{D}}$, $Z_{\mathrm{LP}} = Z_{\mathrm{IP}}$.

3.2 Submodular functions

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- $f : 2^N \mapsto \Re_+$ is **submodular** if

$$f(S) + f(T) \geq f(S \cap T) + f(S \cup T), \qquad \forall\ S, T \subset N.$$

- $f : 2^N \mapsto \Re_+$ is **supermodular** if

$$f(S) + f(T) \leq f(S \cap T) + f(S \cup T), \qquad \forall\ S, T \subset N.$$

- It is **nondecreasing**, if

$$f(S) \leq f(T), \qquad \forall\ S \subset T.$$

3.3 Polymatroids

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$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j\in S} x_j \le f(S), && S \subset N, \\ & x_j \in \mathcal{Z}_+, && j \in N. \end{aligned}$$

$$P(f) = \left\{ \boldsymbol{x} \in \Re_+^n \;\Big|\; \sum_{j\in S} x_j \le f(S),\ \forall\ S \subset N \right\}.$$

3.3.1 Theorem

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If the function f is submodular, nondecreasing, integer valued, and $f(\emptyset) = 0$, then $P(f) = \text{conv}(F)$, F set of feasible integer solutions.

3.4 Proof

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$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j\in S} x_j \le f(S), && S \subset N. \\ & x_j \ge 0, && j \in N, \end{aligned}$$

dual

$$\begin{aligned} \text{minimize} \quad & \sum_{S\subset N} f(S) y_S \\ \text{subject to} \quad & \sum_{\{S \mid j\in S\}} y_S \ge c_j, && j \in N, \\ & y_S \ge 0, && S \subset N. \end{aligned}$$

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- $c_1 \ge c_2 \ge \cdots \ge c_k > 0 \ge c_{k+1} \ge \ldots \ge c_n$. $S^j = \{1, \ldots, j\}$ for $j \in N$, and $S^0 = \emptyset$.
- $$x_j = \begin{cases} f(S^j) - f(S^{j-1}), & \text{for } 1 \le j \le k, \\ 0, & \text{for } j > k. \end{cases}$$
  $$y_S = \begin{cases} c_j - c_{j+1}, & \text{for } S = S^j,\ 1 \le j < k, \\ c_k, & \text{for } S = S^k, \\ 0, & \text{otherwise.} \end{cases}$$
- $\boldsymbol{x}$ is integer, $x_j \ge 0$

- $$\begin{aligned}\sum_{j\in T} x_j &= \sum_{\{j \mid j\in T,\, j\le k\}} \big(f(S^j) - f(S^{j-1})\big)\\ &\le \sum_{\{j \mid j\in T,\, j\le k\}} \big(f(S^j\cap T) - f(S^{j-1}\cap T)\big)\\ &= f(S^k\cap T) - f(\emptyset)\\ &\le f(T) - f(\emptyset)\\ &= f(T).\end{aligned}$$

- $\boldsymbol{y}$ is dual feasible because $y_S \ge 0$ and

$$\sum_{\{S \mid j\in S\}} y_S = y_{S^j} + \cdots + y_{S^k} = c_j,\ if\ j \le k,$$

$$\sum_{\{S \mid j\in S\}} y_S = 0 \ge c_j,\ if\ j > k.$$

- Primal objective value: $\sum_{j=1}^k c_j\big(f(S^j) - f(S^{j-1})\big)$
- Dual objective value:

$$\sum_{j=1}^{k-1}(c_j - c_{j+1})f(S^j) + c_k f(S^k) = \sum_{j=1}^{k} c_j\big(f(S^j) - f(S^{j-1})\big).$$

3.5 Matroids

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- $(N,\mathcal{I})$ independence system, $r(T) = \max\{|S| :\ S\in\mathcal{I},\ S\subset T\}$.
- $$\begin{aligned}\text{maximize} \quad & \sum_{j\in N} c_j x_j\\ \text{subject to} \quad & \sum_{j\in S} x_j \le r(S), \qquad \forall\ S\subset N,\\ & x_j \in \{0,1\}.\end{aligned}$$
- Theorem: $(N,\mathcal{I})$ independence system. It is a matroid if and only if its rank function $r(S) = \max\{|S| :\ S\in\mathcal{I},\ S\subset T\}$ is submodular.

3.6 Greedy algorithm

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1. Given a matroid $(N,\mathcal{I})$, and weights c_j for $j\in N$, sort all elements of N in decreasing order of c_j: $c_{j_1}\ge c_{j_2}\ge\cdots\ge c_{j_n}$. Let $J=\emptyset$; $k=1$.

2. For $k=1,\ldots,m$, if $J\cup\{j_k\}$ is an independent set, let $J = J\cup\{j_k\}$;

3. An optimum solution is given by the set J.