15.082J and 6.855J and ESD.78J

December 2, 2010

Multicommodity Flows 2

On the Multicommodity Flow Problem O-D version

K origin-destination pairs of nodes

$(s_1, t_1), (s_2, t_2), \ldots, (s_K, t_K)$

Network G = (N, A)

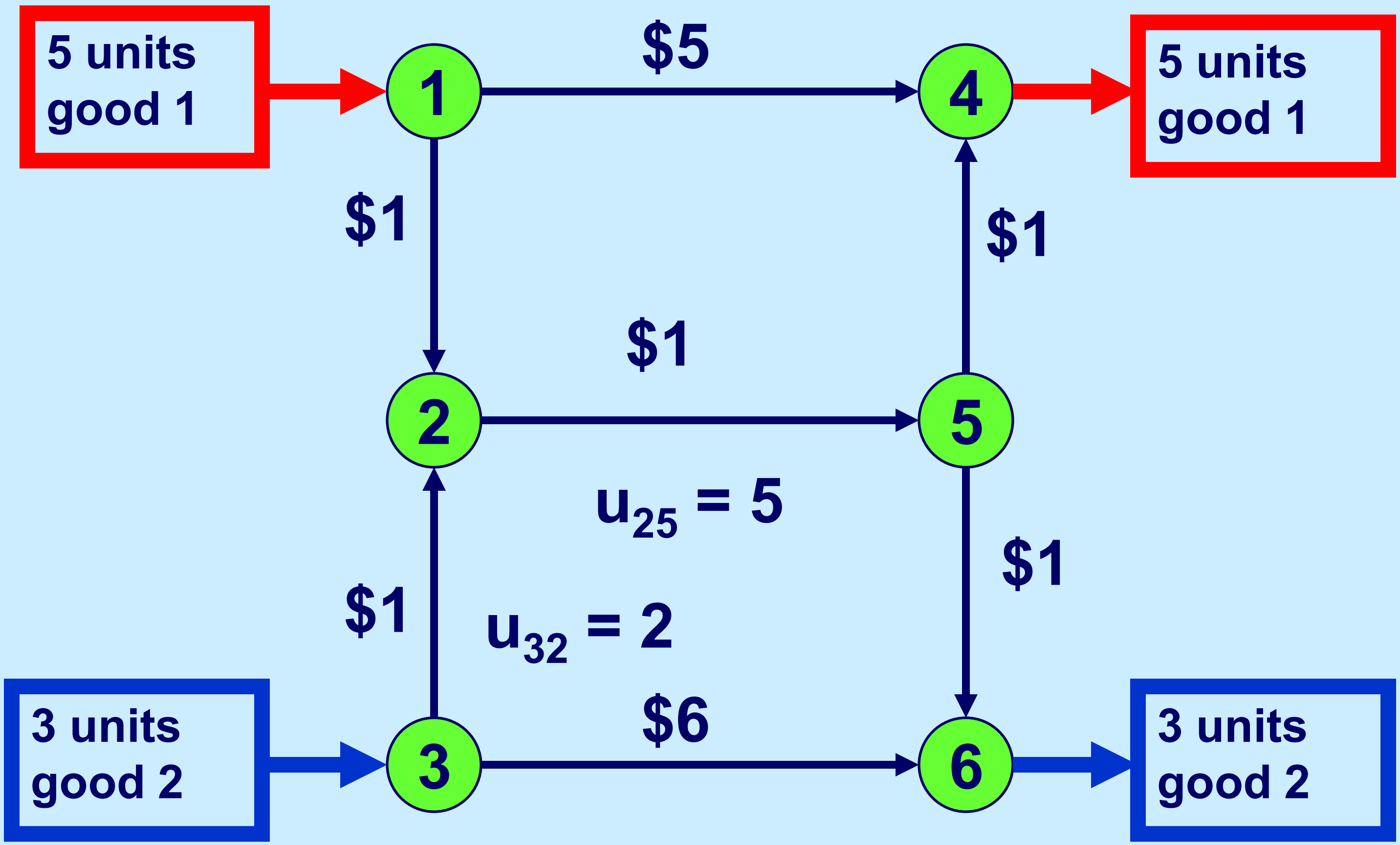
d_k = amount of flow that must be sent from s_k to t_k.

u_{ij} = capacity on (i,j) shared by all commodities

c_{ij}^k = cost of sending 1 unit of commodity k in (i,j)

x_{ij}^k = flow of commodity k in (i,j)

A Linear Multicommodity Flow Problem



The Multicommodity Flow LP

Min $\displaystyle\sum_{(i,j)\in A}\sum_{k} c_{ij}^{k} x_{ij}^{k}$

$$\sum_{j} x_{ij}^{k} - \sum_{j} x_{ji}^{k} = \begin{cases} d_k & \text{if } i = s_k \\ -d_k & \text{if } i \in t_k \\ 0 & \text{otherwise} \end{cases}$$

Supply/ demand constraints

$$\sum_{k} x_{ij}^{k} \le u_{ij} \quad \text{for all } (i,j) \in A$$

Bundle constraints

$$x_{ij}^{k} \ge 0 \quad \forall (i,j) \in A,\ k \in K$$

Assumptions

- **Homogeneous goods.** Each unit flow of commodity k on (i,j) uses up one unit of capacity on (i,j).
- **No congestion.** Cost is linear in the flow on (i,j) until capacity is totally used up.
- **Fractional flows.** Flows are permitted to be fractional.
- **OD pairs.** Usually a commodity has a single origin and single destination.

Optimality Conditions: Partial Dualization

Theorem. The multicommodity flow $x = (x^k)$ is an optimal multicommodity flow for (17) if there exists non-negative prices $w = (w_{ij})$ on the arcs so that the following is true

1. If $w_{ij} > 0$, then $\sum_k x_{ij}^k = u_{ij}$

2. The flow x^k is optimal for the k-th commodity if c^k is replaced by $c^{w,k}$, where

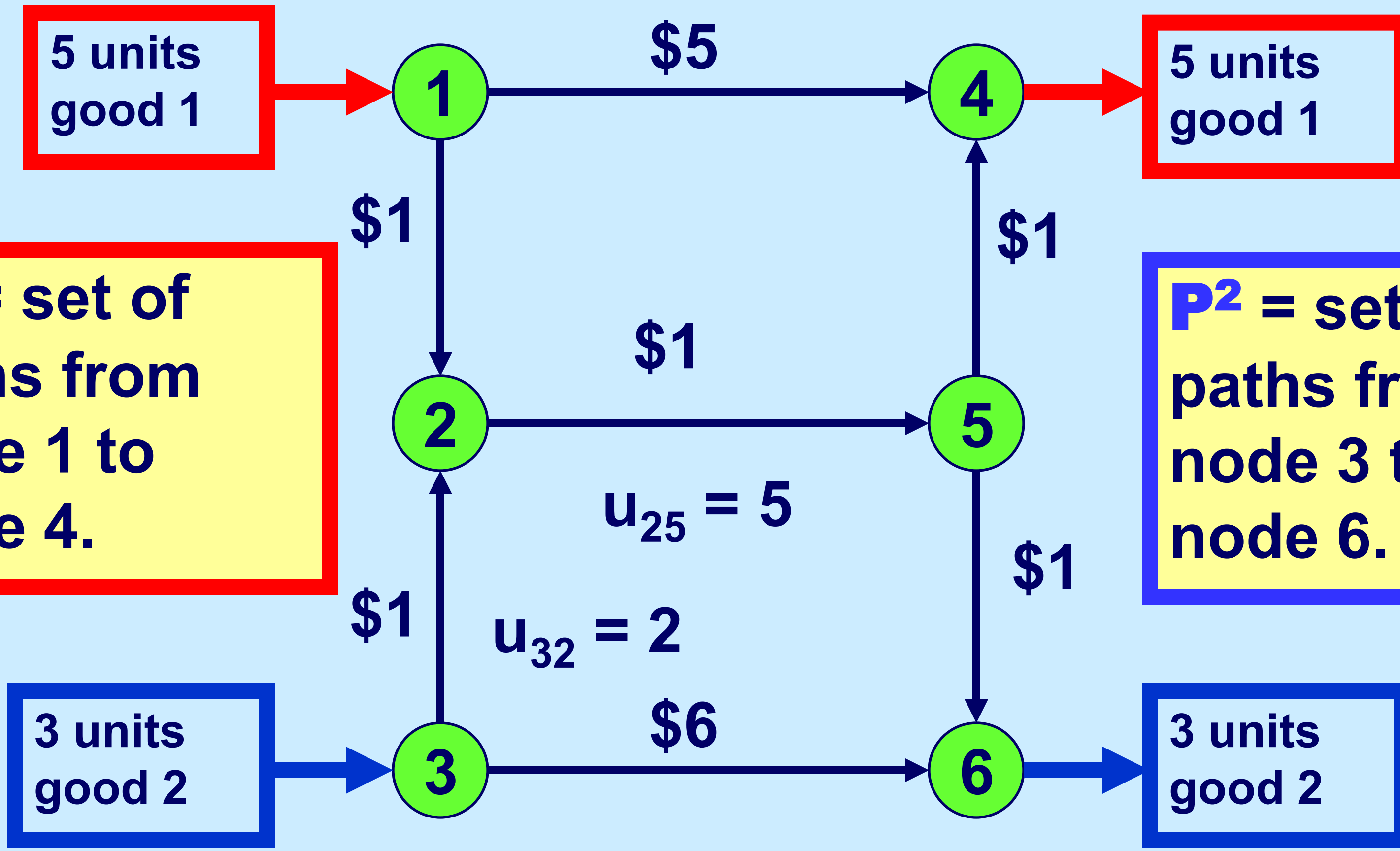
$$c_{ij}^{w,k} = c_{ij}^k + w_{ij}$$

Recall: x^k is optimal for the k-th commodity if there is no negative cost cycle in the kth residual network.

Another approach: path-based approach

- Represent flows from s_k to t_k as the sum of flows in paths.
- The resulting LP may have an exponential number of columns
- Use “column generation” to solve the LP.

A Linear Multicommodity Flow Problem



P^1 = set of paths from node 1 to node 4.

P^2 = set of paths from node 3 to node 6.

$P^1 = \{1\text{-}4,\ 1\text{-}2\text{-}5\text{-}4\}$

$P^2 = \{3\text{-}6,\ 3\text{-}2\text{-}5\text{-}6\}$

A path based formulation

$f(P)$ = flow in path P

$c(P)$ = cost of path P

c(1-4)	=	5
c(1-2-5-4)	=	3
c(3-6)	=	6
c(3-2-5-6)	=	3

Minimize $5\ f(1\text{-}4) + 3\ f(1\text{-}2\text{-}5\text{-}4) + 6\ f(3\text{-}6) + 3\ f(3\text{-}2\text{-}5\text{-}6)$

subject to

$$f(1\text{-}4) + f(1\text{-}2\text{-}5\text{-}4) = 5$$

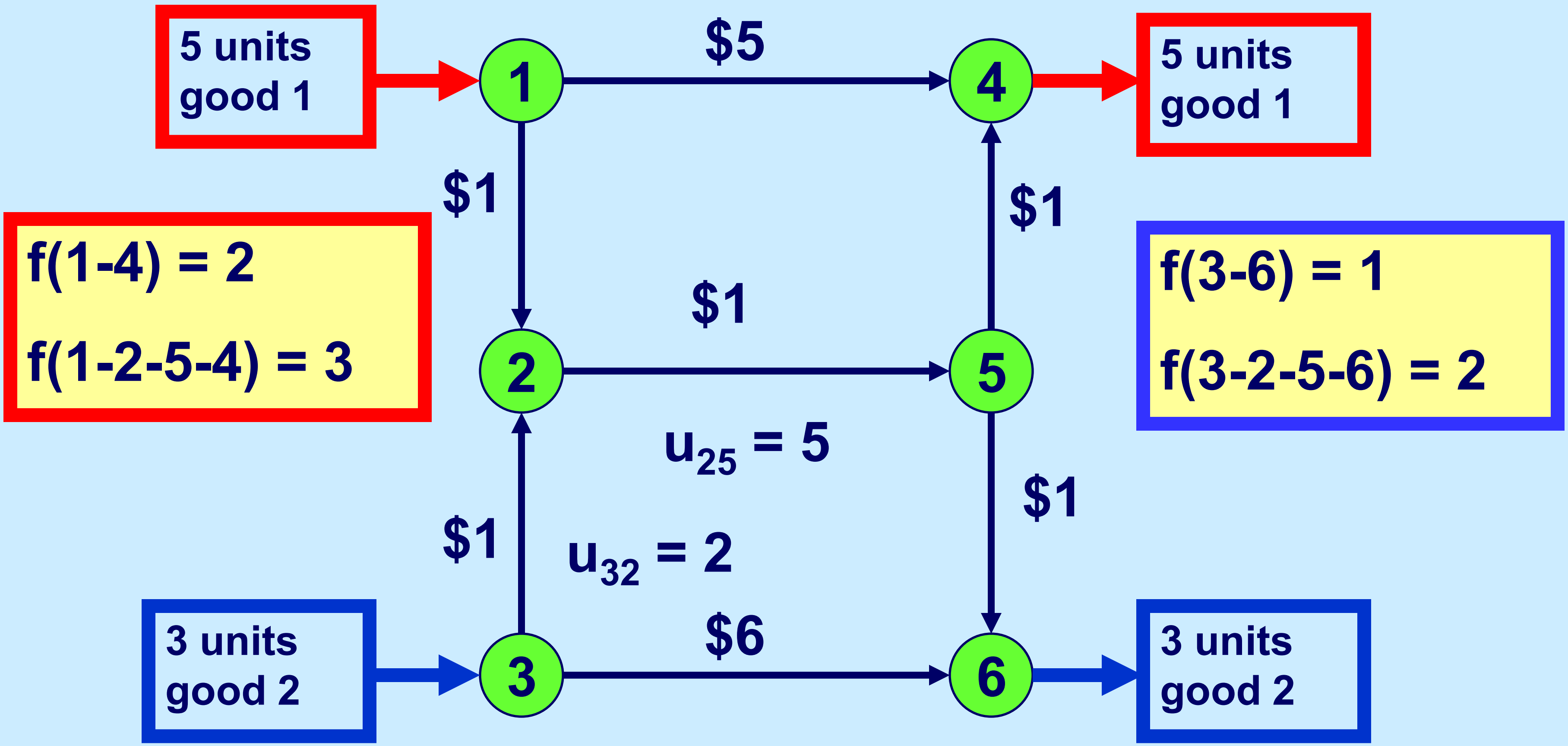
$$f(3\text{-}6) + f(3\text{-}2\text{-}5\text{-}6) = 3$$

$$f(1\text{-}2\text{-}5\text{-}4) + f(3\text{-}2\text{-}5\text{-}6) \leq u_{25} = 5$$

$$f(3\text{-}2\text{-}5\text{-}6) \leq u_{32} = 2$$

$$f(P) \geq 0 \text{ for all paths } P$$

Optimal solution for the path based version



The path based LP can be solved using the simplex method.

General formulation for the path based version

Let $\mathbf{P}^k$ = set of directed paths from s_k to t_k

Let $c^k(P)$ = cost of path $P \in \mathbf{P}^k$.

Let $f(P)$ = flow on path P.

$$\text{Let } \delta_{ij}(P) = \begin{cases} 1 & \text{if } (i,j) \in P \\ 0 & \text{otherwise} \end{cases}$$

Master Problem

$$\text{Minimize} \quad \sum_k \sum_{P \in \mathbf{P}^k} c^k(P) f(P)$$

$$\sum_k \sum_{P \in \mathbf{P}^k} \delta_{ij}(P) f(P) \le u_{ij} \qquad \text{for all } (i,j) \in A$$

$$\sum_{P \in \mathbf{P}^k} f(P) = d^k \qquad \text{for } k = 1 \text{ to } K$$

$$f(P) \ge 0 \qquad \text{for } P \in \bigcup_{k=1}^{K} \mathbf{P}^k$$

$$\text{Minimize} \quad \sum_{k} \sum_{P \in \boldsymbol{P}^k} c^k(P) f(P)$$

$$\sum_{k} \sum_{P \in \boldsymbol{P}^k} \delta_{ij}(P) f(P) \le u_{ij} \qquad \text{for all } (i, j) \in A$$

$$\sum_{P \in \boldsymbol{P}^k} f(P) = d^k$$

$$f(P) \ge 0 \qquad \text{for } P \in \bigcup_{k=1}^{K} \boldsymbol{P}^k$$

bundle constraints: one for each capacitated arc.

supply demand constraints: one for commodity.

variables: one for each path from origin to destination

On the path-based formulation

- m + K constraints
- exponentially many variables
- There is some optimum solution where at most m + K paths have positive flow?

Key questions

1. How can one recognize if a solution is optimal?
2. How can one deal with an LP with exponentially many variables?

FACT: One can use linear programming to optimize over the path based formulation if there are not too many paths?

Optimality Conditions: Partial Dualization

Theorem. The multicommodity flow $x = (x^k)$ is an optimal multicommodity flow for (17) if there exists non-negative prices $w = (w_{ij})$ on the arcs so that the following is true

1. If $w_{ij} > 0$, then $\sum_k x_{ij}^k = u_{ij}$

2. The flow x^k is optimal for the k-th commodity if c^k is replaced by $c^{w,k}$, where

$$c_{ij}^{w,k} = c_{ij}^k + w_{ij}$$

Recall: x^k is optimal for the k-th commodity if there is no negative cost cycle in the kth residual network.

The Restricted Master Problem

Let $\mathbf{S}^k$ = subset of $\mathbf{P}^k$ = directed paths from s_k to t_k

Let $c^k(P)$ = cost of path $P \in \mathbf{S}^k$.

$$\text{Let } \delta_{ij}(P) = \begin{cases} 1 & \text{if } (i,j) \in P \\ 0 & \text{otherwise} \end{cases}$$

Let $f(P)$ = flow on path P.

Restricted Master Problem

$$\text{Minimize} \quad \sum_k \sum_{P \in \mathbf{S}^k} c^k(P) f(P)$$

$$\sum_k \sum_{P \in \mathbf{S}^k} \delta_{ij}(P) f(P) \le u_{ij} \qquad \text{for all } (i,j) \in A$$

$$\sum_{P \in \mathbf{S}^k} f(P) = d^k \qquad \text{for } k = 1 \text{ to } K$$

$$f(P) \ge 0 \qquad P \in \mathbf{S} = \bigcup_{k=1}^{K} \mathbf{S}^k$$

Recognizing Optimality

Let f_S be the optimal set of flows for the restricted master and let $w = W_S$ the optimum tolls (prices) on arcs.

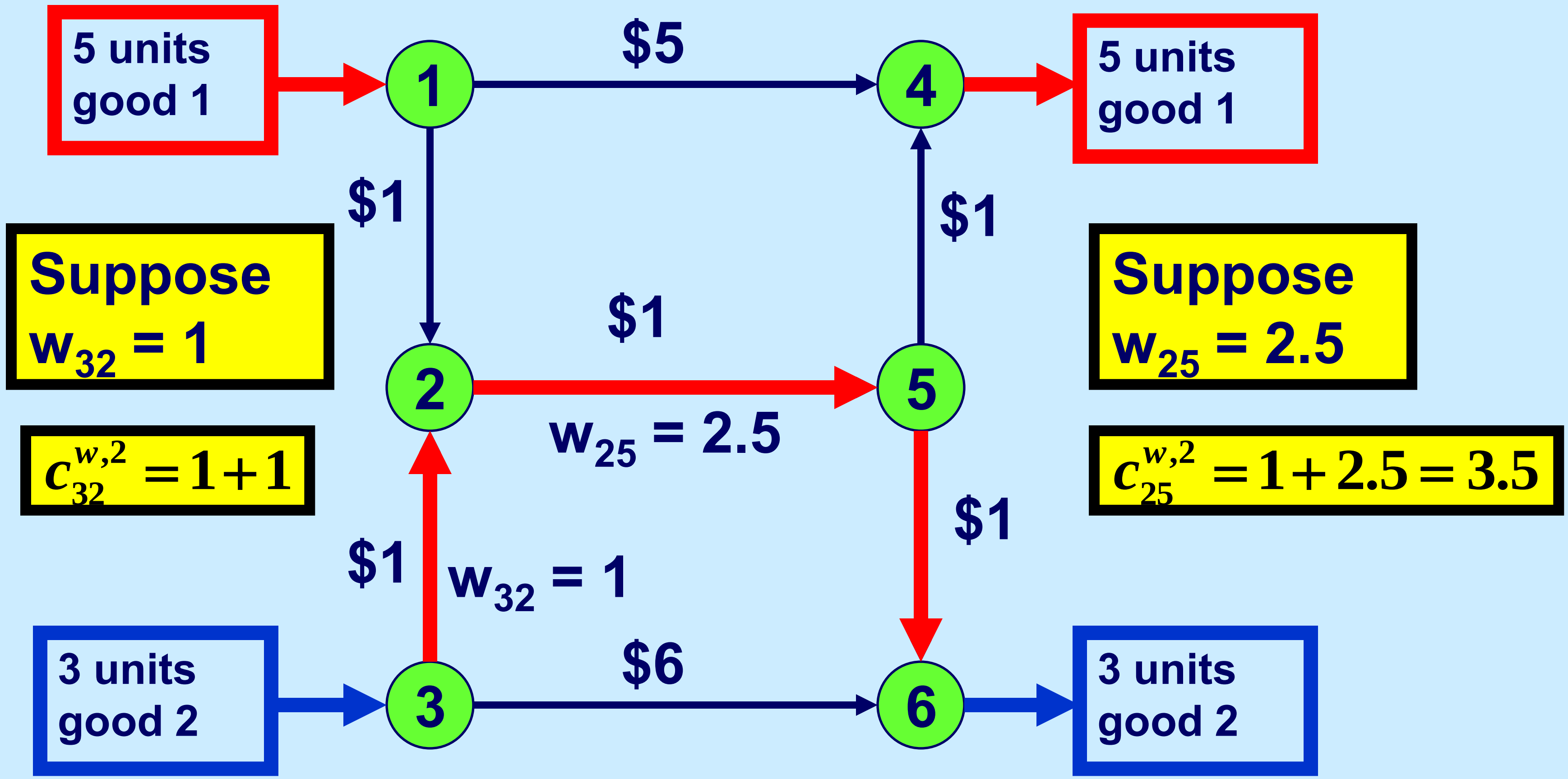
FACT: If $w_{ij} > 0$, then $\sum_P \delta_{ij} f_S(P) = u_{ij}$

Let $c_{ij}^{w,k} = c_{ij}^{k} + w_{ij}$ $\quad$ $c^{w,k}(P) = \sum_{(i,j)\in P} c_{ij}^{w,k}$

Theorem: f_S is optimum for the multicommodity flow problem if it is feasible and if the following is true:

If $f_S(P) > 0$ and $P \in \mathcal{S}^k$, then P is a shortest path in from s_k to t_k with respect to $\mathcal{P}^k$.

Illustration of definitions

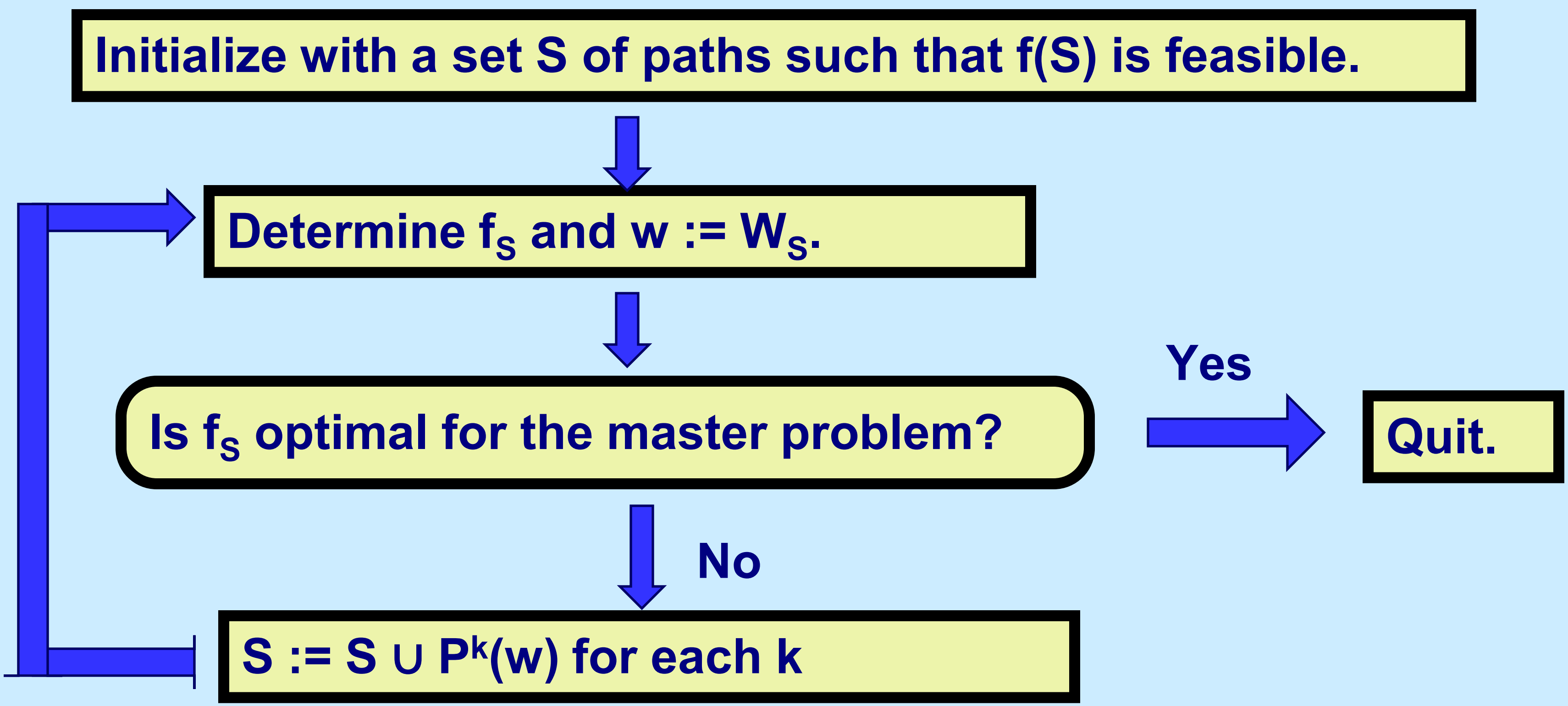


$$c^{w,2}(3-2-5-6) = 2+3.5+1 = 6.5$$

Constraint Generation for Solving the Master Problem

Let f_S be the optimal set of flows and W_S the optimum tolls (prices) on arcs for the restricted master over set S.

Let $P^k(w)$ be a shortest path from s_k to t_k using costs $c^{w,k}$



Solving the Master Problem

1. Initialize $\mathcal{S}^k$ for each k.
2. Solve the restricted master problem for paths in $\mathcal{S} = \cup_k \mathcal{S}^k$ obtaining solution $x = (x^k)$.
3. Check to see if x is optimal for the master problem. If not, find new paths to add to $\mathcal{S}$ and return to step 2.

Summary of Method

- Convert multicommodity flow problem to a problem on paths

- Solve the path problem over a set S. Check if the solution is optimal for the original problem; if not add one or more paths to S, and repeat.

Comments

- One can initialize with artificial paths with infinite capacity and very high costs.
- This approach was developed by Ford and Fulkerson, and generalized by Dantzig and Wolfe to LP's
- Is often very efficient for getting close to the optimum solution. It slows down and converges slowly as more paths are generated

Mental Break

The expression "second string" means "replacement", as in "he is a second string quarterback". Where does this expression come from?

Archers in the middle ages carried a second string in case the first string of their bow broke.

How many different times is the Red Sea mentioned in the Bible?

0 times.

What are the four horsemen of the Apocalypse?

Conquest, slaughter, famine, and death.

Mental Break

What do the following Popes have in common: Pope Boniface IX, and Pope Benedictine XIII, and Pope Alexander V?

They all served at the same time in 1400 and were all “infallible.” They were selected by vying factions of the Catholic Church during the great schism. Two of them are now referred to as “Anti-popes.”

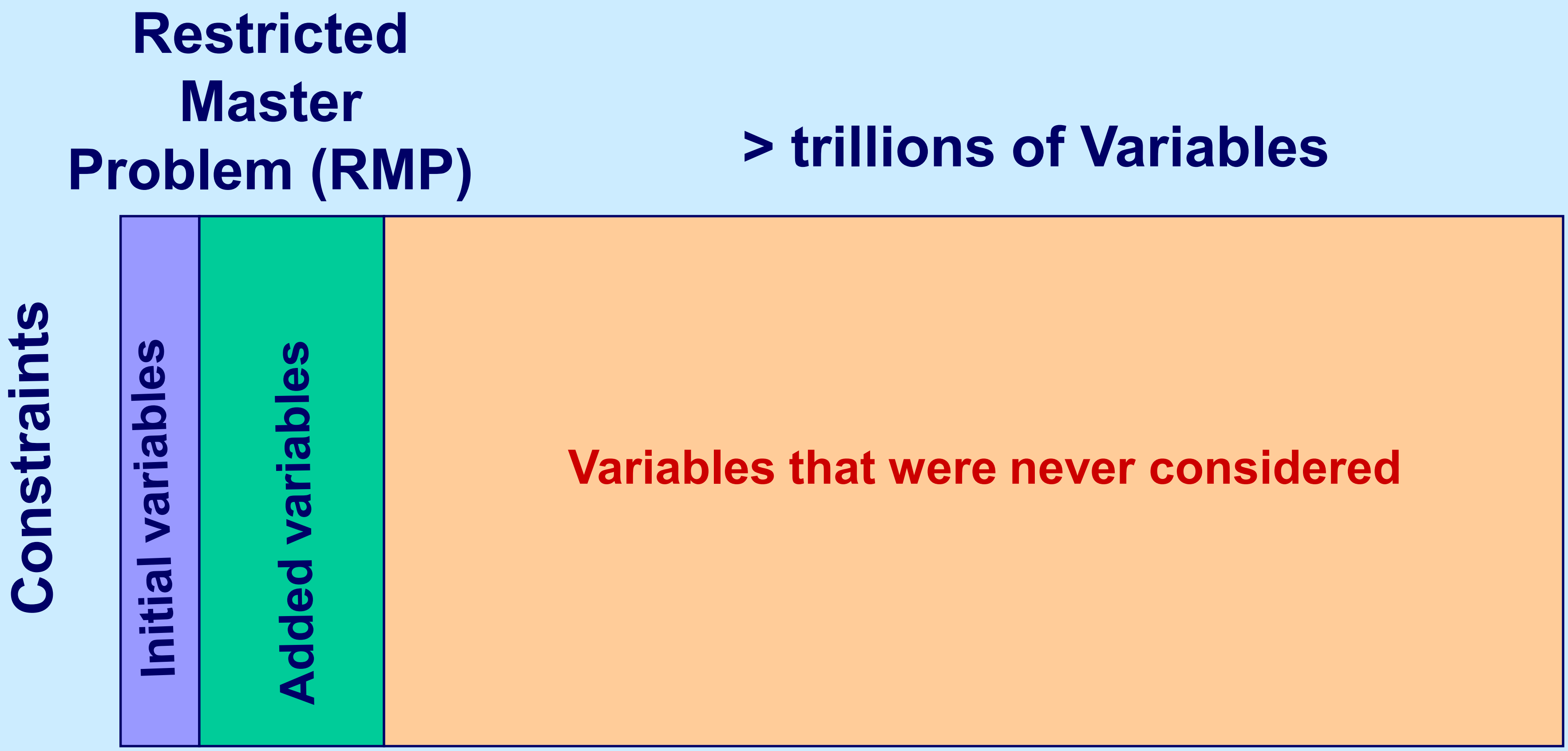
When was toilet paper first used?

In 1391 in China. They were sheets that were 2 ft. by 3 ft.

When was the first time that a magician sawed a woman in half?

1799. The magician was the Count de Grisley.

Column Generation



A story of shared resources.

Tina and Donald head separate divisions for XYZ industries. They have been asked to come up with monthly plans for their divisions.

If they knew all the resources that they had, their planning problem could be written as a linear program. But they have to share resources. The difficulty arises because they are really bad at negotiating with each other.

Their LPs

The LPs (ignoring the shared constraints) are:

$$\begin{array}{ll} \min & ax \\ \text{s.t.} & x \in X \end{array}$$

Tina's LP

$$\begin{array}{ll} \min & cy \\ \text{s.t.} & y \in Y \end{array}$$

Donald's LP

Shared resources:

$$Tx + Dy \le b$$

I know how to make good decisions. Let me decide what resources to use.

Donald Trump

I trust your decisions as much as I trust your choices of hair style.

Perhaps someone else could help mediate.

Tina Brown

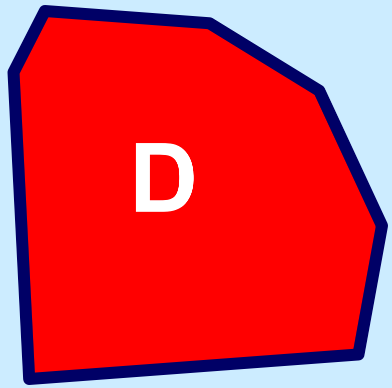
Perhaps I can help. I’ll use LP Dantzig-Wolfe decomposition.

George Dantzig

Your LPs are both bounded. Any solution can be represented as a convex combination of extreme points. Just send me extreme points, and I'll find the best solution.

George Dantzig

$$\begin{aligned} \min \quad & ax + cy \\ \text{s.t.} \quad & Tx + Dy \le b \end{aligned}$$

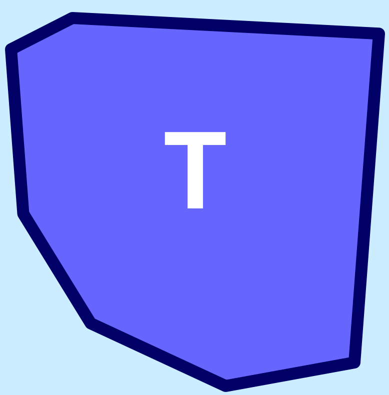


$$x = \sum_i \lambda_i X^i$$

$$y = \sum_j \mu_j Y^j$$

$$\sum_i \lambda_i = 1 \quad \sum_j \mu_j = 1$$

$$\lambda \ge 0, \quad \mu \ge 0$$



I have no idea what you're talking about.

Me neither. Besides that, the number of extreme points can be exponentially large. There is no way I'll send you that many points.

No problem. I'll give you prices for the shared resources. You send me your best solution, taking into account these prices. I'll do the rest.

Donald Trump

Tina Brown

George Dantzig

And so Dantzig helped them out.

Dantzig's first set of prices for the shared resources was very high. Tina and Donald came up with their best solutions X^1 and Y^1, taking into account the costs of the resources. Dantzig saw that the solution X^1 and Y^1 satisfied the shared resource constraint, and he was pleased.

He then set prices of 0 for the shared resources. Tina and Donald came up with solutions X^2 and Y^2 that used lots of shared resources. Dantzig saw that X^2 and Y^2 did not satisfy the resource constraints, but he was still pleased.

So all I need to do is solve my LP with prices p for shared resources.

$$\min \quad ax + (pT)x \quad \text{s.t.} \quad x \in X$$

$$\min \quad cy + (pD)y \quad \text{s.t.} \quad y \in Y$$

Not having to negotiate with Donald is such a pleasure.

Donald Trump

Tina Brown

I'll keep sending them prices and they'll keep sending me extreme points. I'll then take combinations of the extreme points to develop good solutions.

Eventually, I'll get the optimum solution. Or I'll quite when I'm close enough. But first things first.

George Dantzig

I'll solve the restricted master using these first four solutions. I'll then use the prices from the LP on the shared resources. They can then give me their best solutions again.

George Dantzig

The Restricted Master Problem.

min $ax + cy$

s.t. $Tx + Dy \leq b$

$$x = \lambda_1 X^1 + \lambda_2 X^2$$

$$y = \mu_1 Y^1 + \mu_2 Y^2$$

$$\lambda_i + \lambda_i = 1 \qquad \mu_1 + \mu_2 = 1$$

$$\lambda \geq 0, \quad \mu \geq 0$$

Dantzig-Wolfe Algorithm

Initialize with a set E_X of extreme points of X and a set E_Y of extreme points of Y.

Solve the restricted master problem, obtaining an optimal solution x', y' (for the restricted master) and a price vector p' on shared resources.

Ask Tina and Donald to solve their problems while using p' as prices on shared resources. They obtain X' and Y'.

Is X' in E_X and Y' in E_Y?

No

Replace E_X by $E_X \cup X'$

Replace E_Y by $E_Y \cup Y'$.

Yes

Quit with an optimal solution x' and y'

More on Dantzig-Wolfe Decomposition

- **Dantzig-Wolfe decomposition works when there are resources shared by two divisions or 1 million divisions, or even in the case of a single division.**

- **The multi-commodity flow algorithm decomposed into problems for each commodity. The single commodity problems were solved taking into account the arc prices.**

- **The technique works whenever solutions can be combined or mixed. It is used a lot for solving the hardest linear programs.**

Summary

- Multicommodity Flows

- Column generation approach (generate variables one at a time when needed).
  - Path-based generalization
  - Generalization to Dantzig-Wolfe Decomposition

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