**15.053/8** **February 28, 2013**

# 2-person 0-sum (or constant sum) game theory

# Quotes of the Day

**"My work is a game, a very serious game."**

**-- M. C. Escher (1898 - 1972)**

**"Conceal a flaw, and the world will imagine the worst."**

**-- Marcus Valerius Martialis (40 AD - 103 AD)**

**"Reveal your strategy in a game, and your outcome will be the worst."**

**-- Professor Orlin (for the lecture on game theory)**

# Game theory is a very broad topic

| | |
|---|---|
| 6.254 | Game Theory with Engineering Applications |
| 14.12 | Economic Applications of Game Theory |
| 14.122 | Microeconomic Theory II |
| 14.126 | Game Theory |
| 14.13 | Economics and Psychology |
| 14.147 | Topics in Game Theory |
| 15.025 | Game Theory for Strategic Advantage |
| 17.881 | Game Theory and Political Theory |
| 17.882 | Game Theory and Political Theory |
| 24.222 | Decisions, Games and Rational Choice |

# From Marilyn Vos Savant's column.

*"Say you're in a public library, and a beautiful stranger strikes up a conversation with you. She says: 'Let's show pennies to each other, either heads or tails. If we both show heads, I pay you $3. If we both show tails, I pay you $1. If they don't match, you pay me $2.'*

*At this point, she is shushed. You think: 'With both heads 1/4 of the time, I get $3. And with both tails 1/4 of the time, I get $1. So 1/2 of the time, I get $4. And with no matches 1/2 of the time, she gets $4. So it's a fair game.' As the game is quiet, you can play in the library."*

*But should you? Should she?*

**submitted by Edward Spellman to Ask Marilyn on 3/31/02**

**Marilyn Vos Savant has a weekly column in Parade. She has the highest recorded IQ on record.**

# 2-person 0-sum (or constant sum) Game Theory

- **Two people make decisions at the same time.**
- **The payoff depends on the joint decisions.**
- **Whatever one person wins the other person loses (or the sum of their winnings is a constant).**
  - **Marilyn vos Savant answered the question incorrectly. http://www.siam.org/siamnews/06-03/gametheory.pdf**

# Payoff (Reward) Matrix for Vos Savant's Game

You (the Row Player)choose heads or tails

The beautiful stranger chooses heads or tails

**Beautiful Stranger**

| | $C_1$ Heads | $C_2$ Tails |
|---|---|---|
| $R_1$: Heads | 3 | -2 |
| $R_2$: Tails | -2 | 1 |

This matrix is the payoff matrix for you, and the beautiful stranger gets the negative.

# Payoff Matrix for Rock-Paper-Scissors

Row Player chooses a row: either $R_1$, $R_2$, or $R_3$. The three rows are referred to as strategies for the Row Player.

Column Player chooses a column: either $C_1$, $C_2$, or $C_3$, which are referred to as strategies for the Column Player.

| | Rock | Paper | Scissors |
|---|---|---|---|
| Rock | 0 | -1 | 1 |
| Paper | 1 | 0 | -1 |
| Scissors | -1 | 1 | 0 |

This matrix is the payoff matrix for the Row Player, and the column player gets the negative.)

# The Goal of today's lecture

- Introduce some useful concepts in game theory.
- Focus on "guaranteed payoffs".
- Determine optimal strategies for playing two-person constant-sum games.
- Show the connection with linear programming.

# A payoff matrix

Row Player chooses a row: either $R_1$, $R_2$, or $R_3$

Column Player chooses a column: either $C_1$, $C_2$, or $C_3$

| | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| $R_1$ | -2 | 1 | 2 |
| $R_2$ | 2 | -1 | 0 |
| $R_3$ | 1 | 0 | -2 |

e.g., Row Player chooses $R_3$; Column Player chooses $C_1$

Row Player gets 1; Column Player gets -1.

# The column player minimizes the payoff to the row player.

| | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| $R_1$ | -2 | 1 | 2 |
| $R_2$ | 2 | -1 | 0 |
| $R_3$ | 1 | 0 | -2 |

**Row Player gets 1; Column Player loses 1**

# Player strategies

- Player 1 (the row player) has three strategies: choose row 1, or row 2, or row 3.
- The column player has three strategies:  choose column 1, or column 2, or column 3.
- We will later refer to these as pure strategies, for reasons that will become apparent when we describe mixed strategies.

# A guaranteed payoff for the Row Player.

**Floor($R_i$) is the min payoff in the row $R_i$.**

| | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| $R_1$ | -2 | 1 | 2 |
| $R_2$ | 2 | -1 | 0 |
| $R_3$ | 1 | 0 | -2 |

Floor($R_1$) = min {-2,1, 2} = -2.

Floor($R_2$) = min{2, -1, 0} = -1.

Floor($R_3$) = min{1, 0, -2} =-2.

**If the Row Player selects Row j, her payoff will be at least Floor($R_j$).**

**The value of the game for the Row Player is at least Max {Floor($R_j$): j = 1, …, m}.** -1

# The Column Player's guarantee

**Ceiling($C_j$) is the max payoff in the column $C_j$.**

| | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| $R_1$ | -2 | 1 | 2 |
| $R_2$ | 2 | -1 | 0 |
| $R_3$ | 1 | 0 | -2 |

**Ceiling ($C_1$) = max {-2, 2, 1} = 2.**

**Ceiling ($C_2$) = max{ 1, -1, 0} = 1.**

**Ceiling ($C_3$) = max{ 2, 0, -2} = 2.**

**If the Column Player selects Column $C_j$, the Row Player's payoff will be at most Ceiling($C_j$).**

**The value of the game for the Row Player is at most Min {Ceiling($C_j$): j = 1, …,n}. (1)**

# The two guarantees

- The row player can guarantee a payoff of at least -1.
- The column player can guarantee that the payoff to the row player is at most 1.

## If you are the row player, what row (strategy) would you choose?

| | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| $R_1$ | 100 | 50 | 5 |
| $R_2$ | 50 | 30 | 10 |
| $R_3$ | 25 | 20 | 15 |

1. $R_1$
2. $R_2$
3. $R_3$

# Dominance

| | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| $R_1$ | 100 | 50 | 5 |
| $R_2$ | 50 | 30 | 10 |
| $R_3$ | 25 | 20 | 15 |

**We say that column i dominates column $C_j$ if $C_i \leq C_j$.**

**Column $C_3$ dominates $C_1$ and $C_2$. Since the column player is rational, she will choose $C_3$.**

**If the Column Player chooses $C_3$, then the row player chooses $R_3$.**

**It is called a saddle point. If either player chooses a different strategy, the payoff is worse for that player.**

# Saddle points

| | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| $R_1$ | 100 | 50 | 5 |
| $R_2$ | 50 | 30 | 10 |
| $R_3$ | 25 | 20 | 15 |

**Suppose that there is a row $R_i$ and a column $C_j$ such that Floor($R_i$) = Ceiling($C_j$).**

**Then the element $a_{ij}$ is a saddlepoint of the game, and the value of the game for the row player is $a_{ij}$.**

**The value of this game is 15 for the Row Player. The Row Player will Choose $R_3$, and the column Player will choose $C_3$. Any switching of strategy lowers the value of the game to the player.**

# Next: 2 volunteers

**Row Player puts out 1, 2 or 3 fingers.**

**Column Player simultaneously puts out 1, 2, or 3 fingers**

**We will run the game for 5 trials.**

| | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| $R_1$ | -2 | 1 | 2 |
| $R_2$ | 2 | -1 | 0 |
| $R_3$ | 1 | 0 | -2 |

| RP |
|---|
| |
| |
| |
| |
| |
| **Total** |
| |

**RP tries to maximize his or her total**

**CP tries to minimize R's total.**

# Who has the advantage, R or C?

| | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| $R_1$ | -2 | 1 | 2 |
| $R_2$ | 2 | -1 | 0 |
| $R_3$ | 1 | 0 | -2 |

1. R
2. C
3. neither

# Mental Break

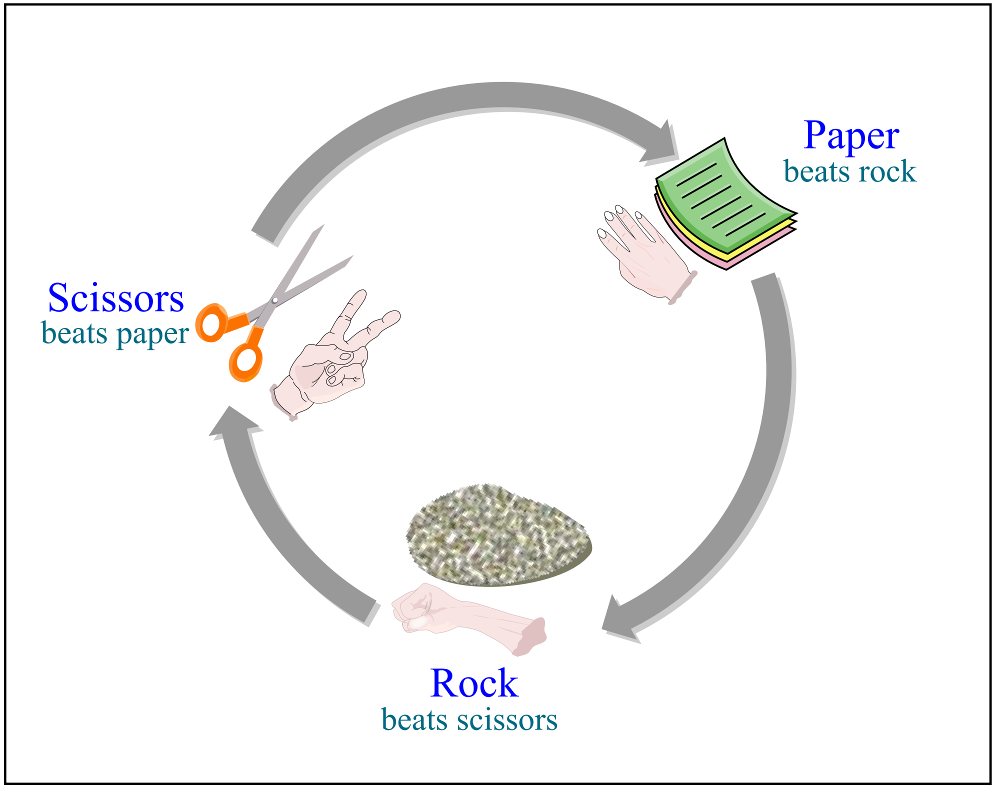


Image by MIT OpenCourseWare.

# Random (mixed) strategies

Suppose we permit the Row Player to choose a mixed strategy, that is, the strategy is one in which she chooses rows with probabilities. (Her strategy is randomized).

| | $C_1$ | $C_2$ | $C_3$ | Prob. |
|---|---|---|---|---|
| $R_1$ | -2 | 1 | 2 | 1/2 |
| $R_2$ | 2 | -1 | 0 | 0 |
| $R_3$ | 1 | 0 | -2 | 1/2 |

Example: Suppose that Strategy 1 consists of choosing $R_1$ with probability ½, and $R_3$ with probability ½.

The row player flips a coin. If it is heads, she chooses $R_1$. If tails, she chooses $R_3$.

| | $C_1$ | $C_2$ | $C_3$ | Prob. |
|---|---|---|---|---|
| $R_1$ | -2 | 1 | 2 | 1/2 |
| $R_2$ | 2 | -1 | 0 | 0 |
| $R_3$ | 1 | 0 | -2 | 1/2 |
| Expected Payoff | | | | |

Probabilities for the mixed strategy.

$Min(E(S_2))$ = min { , , } =

The Row Player can guarantee receiving a payoff of at least

| | $C_1$ | $C_2$ | $C_3$ | Prob. |
|---|---|---|---|---|
| $R_1$ | -2 | 1 | 2 | 1/3 |
| $R_2$ | 2 | -1 | 0 | 1/3 |
| $R_3$ | 1 | 0 | -2 | 1/3 |
| Expected Payoff | | | | |

**Probabilities for Strategy $S_2$**

# The Row Player's Problem

| | $C_1$ | $C_2$ | $C_3$ | Prob. |
|---|---|---|---|---|
| $R_1$ | -2 | 1 | 2 | $x_1$ |
| $R_2$ | 2 | -1 | 0 | $x_2$ |
| $R_3$ | 1 | 0 | -2 | $x_3$ |

**Choose x so that**

$x_1 + x_2 + x_3 = 1$

$x_1, x_2, x_3 \geq 0.$

**How can the Row Player choose probabilities so as to maximize the guaranteed expected payoff?**

**Let z be the guaranteed expected payoff.**

# Row Player's Problem

| | $C_1$ | $C_2$ | $C_3$ | Prob. |
|---|---|---|---|---|
| $R_1$ | -2 | 1 | 2 | $x_1$ |
| $R_2$ | 2 | -1 | 0 | $x_2$ |
| $R_3$ | 1 | 0 | -2 | $x_3$ |

Choose x so that

$x_1 + x_2 + x_3 = 1$

$x_1, x_2, x_3 \geq 0.$

$C_1$

$C_2$

$C_3$

# Row Player's Problem

$$\text{Max} \quad z$$

$$z \le -2x_1 + 2x_2 + x_3 \qquad E(C_1)$$

$$z \le x_1 - x_2 \qquad E(C_2)$$

$$z \le 2x_1 - 2x_3 \qquad E(C_3)$$

$$x_1 + x_2 + x_3 = 1$$

$$x_1, x_2, x_3 \ge 0.$$

$$\text{Max} \quad z = \text{Min}\,\{E(C_1), E(C_2), E(C_3)\}$$

s.t.

$$x_1 + x_2 + x_3 = 1$$

$$x_1, x_2, x_3 \ge 0$$

This is the called the ***maximin problem***

# An *optimal* mixed strategy for the Row Player.

| | $C_1$ | $C_2$ | $C_3$ | Prob. |
|---|---|---|---|---|
| $R_1$ | -2 | 1 | 2 | $x_1 = 7/18$ |
| $R_2$ | 2 | -1 | 0 | $x_2 = 5/18$ |
| $R_3$ | 1 | 0 | -2 | $x_3 = 1/3$ |
| Expected Payoff | 1/9 | 1/9 | 1/9 | |

Strategy $x_{opt}$

**The guaranteed (minimax) payoff is 1/9.**

# The Column Player

Suppose that the column player chooses a mixed strategy.

| | | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|---|
| $R_1$ | | -2 | 1 | 2 |
| $R_2$ | | 2 | -1 | 0 |
| $R_3$ | | 1 | 0 | -2 |
| Prob. | | 1/3 | 1/3 | 1/3 |

If v is the guaranteed payoff, then v is the minimum value s.t.,

$v \geq E(R_1) = 1/3$

$v \geq E(R_2) = 1/3$

$v \geq E(R_3) = -1/3$

The column player can guarantee that R is payed at most 1/3.

# The Column Player's Problem

**Choose a mixed (randomized) strategy y that minimizes the guaranteed payoff to the Row Player.**

This is the ***minimax payoff.***

| | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| $R_1$ | -2 | 1 | 2 |
| $R_2$ | 2 | -1 | 0 |
| $R_3$ | 1 | 0 | -2 |
| Prob. | $y_1$ | $y_2$ | $y_3$ |

$$\min \quad v$$

$$v \geq E_y(R_1)$$

$$v \geq E_y(R_2)$$

$$v \geq E_y(R_3)$$

# An *optimal* randomized strategy for the Column Player

| | $C_1$ | $C_2$ | $C_3$ | Exp. payoff |
|---|---|---|---|---|
| $R_1$ | -2 | 1 | 2 | 1/9 |
| $R_2$ | 2 | -1 | 0 | 1/9 |
| $R_3$ | 1 | 0 | -2 | 1/9 |
| Prob. | $y_1$=1/3 | $y_2$=5/9 | $y_3$=1/9 | |

$v = \max\{ E_y(R_1), E_y(R_2), E_y(R_3)\}$

$= \max\{ 1/9, 1/9, 1/9\} = 1/9.$

That is, CP can play to guarantee at most 1/9 for R.

# On the Optimal Expected Payoffs

**Version 1. You are the row Player and you choose the strategy given earlier. The most intelligent person on earth is playing against you with the aid of the most powerful computer.**

**Your expected payoff is 1/9$^{th}$ on average.**

**Version 2. You are the Row Player and have access to the most powerful computer on Earth and a brilliant game theorist. You are playing against column Player who is using the simple randomized strategy given earlier.**

**Your expected payoff is at most 1/9$^{th}$ on average.**

# Fundamental Theorem of 2-person 0-sum Game Theory.

**Let X be the set of feasible mixed strategies for the Row Player.**

$z = \max_{x \in X} \{\min\{E_x(C_1), E_x(C_2), \ldots E_x(C_m)\}$

$= \max_{x \in X} \{$ min of the expected column payoffs$\}$

**Let Y be the set of mixed strategies for the Column Player.**

$v = \min_{y \in Y} \{\max\{E_y(R_1), E_y(R_2), \ldots, E_y(R_n)\}$

$= \min_{y \in Y} \{$ max of the expected row payoffs$\}$

***Theorem.*** ***For any 2-person 0-sum game, the maximin value is equal to the minimax value.***

**The maximin value is the minimax value.**

# Developers of Game Theory

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**John Von Neumann**

**Oskar Morgenstern**