Optimization Methods in Management Science

MIT 15.053, Spring 2013

Problem Set 6, Due: Thursday April 11th, 2013

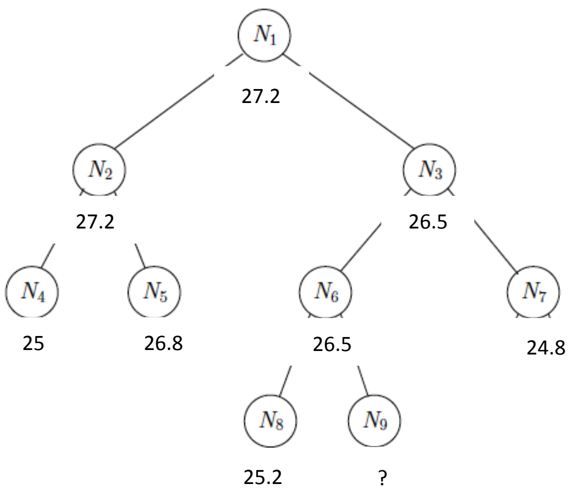
Problem Set Rules:

1. Each student should hand in an individual problem set.
2. Discussing problem sets with other students is permitted. Copying from another person or solution set is *not* permitted.
3. Late assignments will *not* be accepted. No exceptions.
4. The non-Excel solution should be handed in at the beginning of class on the day the problem set is due. The Excel solutions, if required, should be posted on the website by the beginning of class on the day the problem set is due. Questions that require an Excel submission are marked with Excel submission . For Excel submission questions, only the Excel spreadsheet will be graded.

Problem 1

(Total 16 points) We are using Branch-and-Bound to solve an Integer Program with an objective function in maximization form. All coefficients of the objective function are integer valued.

We currently have the following Branch-and-Bound tree, where nodes are labeled $N_1, \ldots, N_9$ and the numbers below each node indicate the value of its LP relaxation. The incumbent solution was obtained in solving the LP at N_4. The optimal LP solution was feasible for the IP and had objective value 25.



(a) (4 points) Let v_9 be the optimum value of the LP associated with node N_9. Choose the best answer. (It is the answer that is correct and provides the most information.)

i. $v_9 \leq 27.2$

ii. $v_9 \leq 26.5$

iii. $v_9 = 26.5$

iv. $v_9 \geq 25.2$

(b) (4 points) With the information that we currently have, what are the best upper and lower bounds that we can give on the value v^* of the optimal solution for the integer program?

(c) (8 points) For each of the following nodes of the tree, say whether it is active (A) or fathomed (F) or whether there is not enough information (NEI) to know. We recall that fathoming is the same as pruning.

(i) N_4

(ii) N_5

(iii) N_7

(iv) N_8

(v) N_9

Problem 2

(Total 24 points) Consider the following capital budgeting problem: We have a set of six possible investments with the following characteristics:

Investment	1	2	3	4	5	6
NPV Added	\$33	\$45	\$25	\$17	\$39	\$23
Cash Required	\$10	\$14	\$8	\$6	\$12	\$8

We want to find the optimal set of investments that maximizes the total Net Present Value (NPV) while limiting the amount of initial investment to \$28.

(a) (4 points) Write an integer program to determine the optimal set of investments that maximizes the net present value.

(b) <u>Excel submission</u> (15 points) Start with the incumbent $x_1 = 1, x_2 = 0, x_3 = 0, x_4 = 1, x_5 = 1$, and $x_6 = 0$. Its objective value is 89. Solve for the first five nodes of the Branch and Bound tree as given on the spreadsheet. (The spreadsheet is already set up to solve the linear program.) You should adjust upper or lower bounds in each case to solve the LP. Write the solutions in the spaces indicated on the spreadsheet. Also enter the objective values manually after the solutions to the nodes of the tree. (Don't copy cell K19 because it contains a formula.) Indicate in column M whether the nodes of the tree are fathomed or not.

(c) (5 points) Either node 2 or node 3 will repeat the solution from node 1. Explain why. Either node 4 or node 5 repeats the solution of node 2. Explain why.

Problem 3

(All parts are <u>Excel submission</u> .) (Total 18 points) Consider the same integer program from Problem 2. However, this time, we will solve the problem using cutting planes. We will start with the same incumbent as in Problem 2. This incumbent is the optimal integer solution.

(a) (4 points) Solve the linear program. Does the solution to this LP prove that the incumbent is optimal for the IP? Why or why not? If your answer to part (a) is "no", then continue to Part (b).

(b) (7 points) You obtained a solution for Part (a) in which two variables are 1 and one is a fraction less than 1. Add to the LP the "cut" $x_i + x_j + x_k \leq 2$, where these are the three decision variables that were positive in the LP solution. (And write the cut on the spreadsheet in the indicated place.) Solve the revised linear program. Does the solution to this LP prove that the incumbent is optimal for the IP? Why or why not? If your answer to Part (b) is "no", then continue to Part (c).

(c) (7 points) You obtained a solution for Part (b) in which two variables are 1 and one is a fraction less than 1. Add to the LP from Part (b) the "cut" $x_i + x_j + x_k \leq 2$, where these are the three decision variables that were positive in the LP solution. (And write the cut on the spreadsheet in the indicated place.) Solve the revised linear program. Does the solution to this LP prove that the incumbent is optimal for the IP? Why or why not?

Problem 4

(Total 12 points, 3 points each) We want to find valid inequalities for the following 0-1 knapsack problem:

$$\left.\begin{array}{lrcl} \max & 22x_1 + 10x_2 + 16x_3 + 11x_4 + 18x_5 + 6x_6 & & \\ \text{s.t.:} & 4x_1 + 3x_2 + 7x_3 + 6x_4 + 5x_5 + 8x_6 & \leq & 15 \\ & x_1, x_2, x_3, x_4, x_5, x_6 & \in & \{0,1\}. \end{array}\right\} \text{(KP)}$$

For each of the inequalities below, identify whether or not they are valid.

(a) $x_1 + x_3 + x_4 \leq 2$.

(b) $x_2 + x_3 + x_5 \leq 2$.

(c) $x_3 + x_5 \leq 1$.

(d) $x_1 + x_2 + x_4 + x_6 \leq 3$.

Problem 5

(Total 30 points)We want to solve the following integer program with two variables:

$$\left.\begin{array}{lrcl} \max & 4x_1 + 3x_2 & & \\ \text{s.t.:} & 2x_1 + x_2 & \leq & 11 \\ & -x_1 + 2x_2 & \leq & 6 \\ \forall j = 1,2 & x_j & \geq & 0 \\ \forall j = 1,2 & x_j & \in & \mathbb{Z}. \end{array}\right\}$$

Let s_1, s_2 be the slack variables for the first and second constraint respectively. Solving the LP relaxation for this problem yields the following optimal Simplex tableau:

Basic	x_1	x_2	s_1	s_2	Rhs
$(-z)$			-11/5	-2/5	-133/5
x_1	1		2/5	-1/5	16/5
x_2		1	1/5	2/5	23/5

(a) (4 points) Slack variables are usually allowed to be fractional. If x_1 and x_2 are both integers, will s_1 and s_2 also be integers? Briefly explain why or why not.

(b) (6 points) Derive a Gomory cut from each of the first two rows in the optimal Simplex tableau.

(c) (6 points) Express the cuts in terms of the original variables x_1 and x_2. Graph the feasible region for x_1 and x_2, and illustrate the cuts on the graph.

(d) (6 points) We now append the cuts (or the cut, if only one of them is needed) to the LP relaxation, and resolve. We provide the optimal Simplex tableau after resolving below:

Basic	x_1	x_2	s_1	s_2	s_3	Rhs
$(-z)$			-2		-1	-26
x_1	1		1/2		-1/2	7/2
s_2			1/2	1	-5/2	3/2
x_2		1			1	4

where s_3 is the slack variable corresponding to the appended cut. Which rows can be used to derive Gomory cuts? Compute the cuts. Rewrite them in terms of x_1 and x_2.

(e) (8 points) Draw the cuts on your sketch and find the new optimal solution graphically. Is this new solution optimal? (Hint: if you did everything correctly, the new solution is optimal with objective function value 25.)