[SQUEAKING] [RUSTLING] [CLICKING]

SCOTT HUGHES: Good afternoon. So we spent our last lecture laying out some of the basic foundations, making a couple of definitions. I want to quickly recap the most important concepts and definitions. And then, let me be blunt, I kind of want to get through these definitions, which I think it's important to do them precisely, but there's nothing significantly challenging about them. We just need to make sure they are defined very precisely. So now that you've kind of seen the style of these things, I would like to sort of move through the next batch of these definitions quickly enough that we can start to move into more interesting material.

So a quick recap-- and my apologies, my daughter has some kind of a virus that I am desperately trying to make sure I do not catch. And so I will be hydrating during this lecture. So I just want to recap some of the most important concepts we went over.

So this whole class is essentially a study in spacetime. Later, we're going to connect spacetime to gravity. And general relativity is going to become the relativistic theory of gravity.

So we began with a fairly mathematical definition. Spacetime is a manifold of events that is endowed with a metric. Manifold, for our purposes, is essentially just a set in which we understand how different members of the set are connected to each other. Events are really just when and where something happens. We haven't precisely defined metric yet. We will soon. But intuitively, just regard it as some kind of a mathematical object that gives me a notion of distance between these events.

I tried and I will continue to try to be very careful to make a distinction between geometric objects that live in the manifold, the events themselves with things like a displacement vector between two events, other vectors, which I will recap the definition of in just a moment. They have an existence and sort of a reality to them that is deeper and more fundamental than the representation of that object. So when I say that the displacement vector is delta t, delta x, delta y, delta z, that is according to observer O. And when I write that down-- I will be very careful as much as possible. I will occasionally screw this up-- but I will try to write this down without using an equal sign. Equal sign implies a degree of reality that I do not want to impart to that representation. So an equal dot with a little o here, that's my personal notation for delta x is represented by these components according to observer O. And for shorthand, I was sometimes just write this by the collection of indices delta x alpha.

A different observer O bar, they will represent this this factor using the exact same geometric object. They all agree that it's this displacement between two physical events in spacetime. But they assign different coordinates to it. They give a different representation to it. And we find that representation using a Lorenz transformation. And I'm not going to write out explicitly the Lorenz transformation matrix lambda. I gave it in the last lecture. And I'm assuming you're all experts in special relativity, and I don't need to go over that.

So I will then using this as sort of the prototype, the general notion of a vector in spacetime, which we'll often call a 4-vector for the obvious reason that it has 4 components, I'm going to treat that as any quartet of numbers that has transformation properties just like the displacement factor I just went over. So if there's some quantity A that has time like an x or y and z component, as long as a different observer, for reasons having to do with the underlying physics or whatever the heck this A is, as long as that different observer relates their components to observer O's components via the Lorenz transformation, you got yourself a 4-vector.

Any random set of four numbers, that ain't a 4-vector. You need to have some physics associated with it. And the physics has to tell it that it's a thing that's related by a Lorenz transformation.

So I want to pick up this discussion by introducing four particularly important and special vectors, which to be honest, we're not going to use too much beyond some of the first couple of weeks or so of the course, but they're very useful. And one should often bear in mind, even later in the course when they've sort of disappeared, that they're coming along for the ride secretly. And these are basis vectors.

So if I go into frame O-- I'm just going to pick some particular reference frame-- I can immediately write down four special vectors. So remember, this is a Cartesian type of coordinate system. So I'm going to introduce e0. I'm going to represent this by just the number 1 in the time slot and 0 everywhere else. e1, or ex if you prefer, I'm going to write that down like so. And you can kind of see where I'm going with this.

The analogy, if you've all seen unit vectors in other classes, hopefully this is fairly obvious what I'm doing. I'm just picking out a set of, you know, little dimensionless simple quantities that point along the preferred directions that I've set up in this inertial reference frame. A compact way of writing this-- so notice, I have four of these vectors. And these vectors each have four components.

And so what I can say is that the beta component of unit vector e alpha is representative, according to observer O, by delta alpha beta, the Kronecker delta. If you're not familiar with this one, I'm usually reluctant to send you to Wikipedia. But in this case, I'm going to send you to Wikipedia, Wiki via Kronecker delta.

Yeah, so what that does is it just emphasizes that at least in-- by the way, I should put little o's underneath all of these things, because I have chosen this according to observer o's representation. So this is just saying that a coin to observer O, these are four very special vectors. The utility of these things-- up high for everyone in back to be able to see-- the utility of doing this is that if I now want to write the vector A as a geometric object, I can combine the components that observer O uses with the basis vectors that observer O uses. And I can sum them all together. I can put them together.

And then I've got the-- it's not a representation. That's a damn vector. OK? I put it all together using sort of an internally consistent set of numbers and basis vectors. And so I am free to say this, where I actually use an actual equal sign.

You might stop and think, well, but those aren't the components of the observer O bar would use. And you're right. Observer O bar would not use those components. They would also not use those basis factors. We're going to talk about how those things change. But observer O bar does agree that if they were handed O's components and O's basis vectors, this would give me a complete representation of what the vector is. OK?

So again, I'm really harping on this sort of distinction between the geometric object and the representation. This is the geometric object. And we can take advantage of this. The fact that that combination of things is the geometric object is a tool that I'm going to now-- is a fact that I'm going to exploit in order to figure out how my basis vectors transform when I change reference frames.

So let me just repeat what I wrote of there. But-- oops-- good point for just a slight editorial comment. When you're talking and writing and there are millions of little sub scripts and indices, sometimes the brain and the appendages get out of sync with one another. I caught that one. I don't always. OK? So if you see something like that and you kind of go, um, why did that alpha magically turn into a beta, it's probably a mistake. Please call it out. OK?

All right, so this is how I build this geometric object, using the components and the basis vectors as measured by O, as used by observer O. Let's now write out what they would be if they were measured by a different observer, an O bar observer. I know how to get components, the barred components from the unbarred components. I don't yet know how to get the barred basis vectors from the unbarred basis vectors. But I know that they exist. And that once I know what they are, this equation is true. OK? These are just two different ways of writing this geometric object, which every observer agrees has its existence that transcends the representation.

So let's rewrite what I've got on the right-hand side of the rightmost equal sign here. I'm going to write this as lambda mu bar beta A beta e mu bar. And then now I'm going to use a trick, which is what the-- it's not even so much a trick, but I'm just going to use a fact that is great when you're working in index notation. So often when students first encounter this kind of notation, your instinct is to try to write everything out using matrices and things like row vectors and column vectors. It's a natural thing to do.

I urge you to get over that. If I get some bandwidth for that, I'm going to write up a set of notes this weekend showing how one can translate at least 2 by 2 objects-- two index objects and one index objects in a consistent way between matrices and row vectors and column vectors. But we're rapidly going to start getting into objects that are bigger than that, for which trying to represent them in matrix-like form gets untenable. In a little while we're going to have a three index object. And since we don't have three-dimensional chalkboards, making this sort of matrix representation of that is challenging. Soon after that, we'll have four index objects. And we'll occasionally need to take derivatives of that four index object, giving us a five index object. At that point, the ability to sort of treat them like matrices is hopeless.

So really, you should just be thinking of this as ordinary multiplication of the numbers that are represented by these components as written out here. And an ordinary multiplication like this, I can just go ahead and swap the order of multiplication very easily here. So what I'm going to do is move the-- hang on a second. Yeah, yeah, yeah, now I see what I'm doing. Sorry about that. I misread my notes.

I'm just going to move the A onto the other side of my lambda. And then I'm going to use the fact that beta on my right-hand side is a dummy index. So in that final expression I wrote down over there, beta is a dummy index. And I'm free to adjust it to put it into a form that's more convenient for me.

So let's begin. Let me just write where I've got this equation right now over there. So A alpha E alpha equals-- and what I've got then is component A beta lambda mu bar beta-- keep mu bar. Sorry about that. So I'm going to remap my dummy index. The reason I did that is I can now move this to the other side and factor out the A alpha.

OK, everyone can see that I hope. So see what I did was I arranged this so that I've isolated essentially only the transformation of the basis vector. So this equation has to hold no matter what vector I am working with. It's got to hold for an arbitrary A alpha. So the only way that can happen, this means that my transformation of basis vectors obeys a law that looks like this.

Now, on first inspection, you're going to go, ah, that's exactly what we got for the components of the 4-vector. Caution-- I'm going to remind you, it's actually on the board over there. OK? The barred component is playing a different role than the barred basis vector here. If you want to get the barred basis vector from the unbarred basis vector, you need to work with the inverse of this matrix. That's what this is telling us here. OK?

All that being said, if you're just working through this and you've got your components set up and you're sort of hacking through it, the algorithm for you to follow is actually simple. Really, all we're doing is lining up the indices. We're summing over the ones that are repeated and requiring that those that are on both the left-hand side and the right-hand side appear and equal one another. Or as an old professor of mine liked to say about 12 times a lecture, line up the indices. So that's essentially what we're doing.

In this case, I line up-- if I have my Lorenz transformation matrix with the barred index up top and the unbarred down below. Boom, I line up the indices. And that tells me what the unbarred basis components are from the barred ones, and vice versa for the components.

So what basically, as I just said, that tells me if I actually want to get this guy given this guy, I need to work with the inverse. Put this up high so that everyone in the back can see it. So this, again, is one of those places where you might be tempted to sort of write out a matrix and do a matrix inversion.

But before you do that, remember this is physics. OK? The inverse is going to be-- the inverse matrix is going to be the one that does, at least for certain quantities-- what the Lorenz transformation does is that it relates objects, if I'm at rest, if I consider myself at rest, it tells me about things according to a frame that it's moving with respect to me. The inverse matrix does the opposite. It would say to that person being at rest, what is the matrix that tells them about things according to me, and they see me moving with the exact same velocity, but in the opposite direction.

So to get the inverse matrix, to get the inverse Lorenz transformation, what we end up doing is we just reverse the velocity. So I've just wrote down for you-- and it's worth bearing in mind, every one of these lambdas that is there, they are really functions. And so my e alpha that I wrote down over there, I'm going to use an under tilde to denote a 3 vector. In one of your textbooks, it's written with a boldface. But that's hard to do on a blackboard.

So if I want to go the other direction, well, I just need to have the inverse transformation. And I have that. Bear in mind, I mean, those are really the exact same matrices. OK? In terms of the function that I'm working with here, I'm just flipping around the direction in order to get these things out. OK?

So, as I said, you might be tempted just to go ahead and do the matrix inversion. Let we just do a quick calculation to show you that that would work. And the reason I'm doing this is I just want to quickly step through a particular step, which is, again, sort of in the spirit of swatting mosquitoes with sledgehammers, it's the kind of analysis that you're going to sort of have to do off to the side now and again.

So given that I know e alpha is-- let's see, let's use beta-- e beta bar be e beta bar. But I now know that I could write this guy as lambda-- let's use a gamma-- beta bar minus v in the gamma direction. Now, you look at that, and you go, ooh, look, I'm actually summing over the betas there. Let's gather my terms a little bit differently.

So notice, I have unbarred basis vectors over here on the left, unbarred ones over here on the right, a bunch of junk in these braces here in the middle. The only way for that to work is if after summing over beta, that bunch of junk is, in matrix language, we'd say, it's the identity matrix. In component notation, we're going to call it the Kronecker delta.

Just as little aside, if I started with the barred on the left side and the unbarred in the right-hand side and did a similar analysis, it would take you at this point now that you are all fully expert in this kind of index manipulation, it should take you no more than about a minute to demonstrate to yourself that you can get a Kronecker delta on the barred indices in a similar way, just by using them in a slightly different order. OK?

All right, so far all still basically just formulas. So I'm going to start now doing a little bit of formalism that will very quickly segue into physics. We can all take a deep sigh of relief and go, ah, OK, something you can imagine measuring. So that's nice.

So I have introduced 4-vectors. I haven't introduced the basis vectors and their components. I haven't really done anything with them yet. So before we start doing things with them, let's think about some operations that we can do with 4-vectors.

So the first one which I'd like to introduce is a scalar product. To motivate the scalar product that I'm going to define in the same way that I defined 4-vectors as a quartet of numbers whose transformation properties are based on the transformation properties of the displacement, I'm going to motivate a general scalar product between 4-vectors by a similar kind of quantity that is constructed from the displacement.

So let me recall a result that I hope is familiar from special relativity. So working in units where the speed of light is 1, you are hopefully all familiar with the fact that this quantity is something that is in variance to all observers. I did not use a represented by symbol there, because no matter whose delta t, delta x, delta y, delta z I use there, they will all agree on the delta s squared that comes out of this. If I want to-- actually, let me write of a few things done before I say anything more.

So this is an invariant. This is the same-- let me actually write this in a slightly different way-- it is the same in all Lorenz frames. So pick some observer, get their delta t, delta x, et cetera, assemble delta s squared, pick a different observer, do the Lorenz transformation, assemble their delta t prime, delta x prime, et cetera, make that, boom, they all agree.

So we're going to use this to say, you know what, I'm going to call that the inner product of the displacement vector with itself. So I'm going to call this delta x dotted into delta x. And so what this means is that built into my scalar product-- so if I write this as a particular observer would compute it-- this is the scalar product that I'm going to define with respect to the displacement vector. And this is usually the point where somebody in the class is thinking, why is there a minus sign in front of the timelike piece?

I can't answer that. All I can say is that appears to be-- more than appears to be. There's a whole frickin' butt load of evidence-- that's not how nature is assembled. OK, it's connected to the fact-- so the fact that all of my spatial directions are sort of entered with the same sign, but my timelike direction, it has a different sign. It's connected to the fact that I can easily move left and right, front and back. It's a little bit of effort. I can move up and down. But I cannot say, oh, crap, I left my phone at home I'll go back 15 minutes and pick it up. You cannot move back and forth in time.

Time actually, which is the timelike component of this thing, it enters into the geometry in a fundamentally different way from the spatial things. And that's reflected in that minus sign. Anything deeper than that, let's just say that we're probably not likely to get very far with that conversation. Depending on what kind of muscle relaxants you enjoy using on the weekend, you might have some fun conversations about it. But it is not something that you're really going to get very far with. You just kind of have to accept that it's part of the built-in geometry of nature.

OK, so this I am defining as the inner products of the displacement vector with itself. I define vectors as having the same transformation properties as the displacement vector. We can similarly define an inner product of a 4-vector with itself. OK, now, I'll put this on another board.

So A dot A, I'm going to define this-- or rather I will say it is represented according to observer O as minus A0 squared plus A1 squared plus A2 squared plus A3 squared. I realize there can be a little bit of ambiguity in the way I'm writing it here. You just have to-- if you're ever confused, just ask for clarification about whether I'm writing something to a power or whether it's an index label. Context usually makes it clear. Handwriting sometimes obscures context though.

The reason why I'm doing this and a real benefit of this is that whatever this quantity is, because A has the same transformation properties of the displacement, this must be a Lorenz invariant as well. The underlying mathematics of Lorenz transformation doesn't care that I wrote a zero instead of delta x here. It doesn't care that I wrote A1 instead of delta x1, et cetera. It just knows that it's a thing that goes into that slot of the Lorenz transformation.

So all observers-- this is how I represent it according to observer O. But all observers agree on that form. And as a consequence, this is going to be something that we exploit a lot. Even when we move beyond the simple geometry of special relativity, a generalization of this will be extremely important for, not an exaggeration to say, everything.

So a little bit of terminology-- so if A dot A is negative, and depending upon which is bigger, A0 squared or the sum of the other ones-- it could very well be negative-- we say that A is a time like vector. This traces back to the fact that if A were the displacement vector, if the displacement, the invariant interval, were negative that would tell me that the two events, which are the beginning and the end of the interval, I could find a frame at which they're at the exact same location and are only separated in time. In the same way, this is basically saying that I can find a frame when this vector points parallel to some observer's time axis. If this is positive, A is spacelike. Everything I just said about timelike, lather, rinse, repeat, but replace time with space. OK?

And if A dot A equals 0, we say A is-- there's two words that are commonly used-- lightlike or null. Null just traces back obviously to the zero. Lightlike is because this is a vector that could lie tangent to the trajectory that a light beam follows in spacetime. OK?

OK, let me get some clean chalk. So, so far, I've only talked about this inner product, this scalar product. Oh, and, by the way, I'll use inner product and scalar products somewhat interchangeably. But this allows me to reiterate a point I made in Tuesday's lecture. When I say scalar, scalar refers to a quantity which is-- you know, it doesn't have any components associated with it. So in that sense, it's familiar from your eager intuition of, you know, not to vector. But it has a deeper meaning in this course, because I also want it to be something that is invariant between reference frames. So A dot A is the scalar product. It gives me a quantity that all observers agree on.

Now, I've only done scalar products of vectors, A and the displacement vector, with themselves. So a more general notion, if I have vectors A and B, then I will define the inner product between them, as observed by O, as constructed by observer O, rather, like so. It's not hard to convince yourself, given everything we've done so far, that this quantity must also be invariant. I'll sketch a really quick proof.

Let's define-- let's say we have two 4-vectors, A and B. Their sum, by the linearity rules that apply to these vectors, must also be a vector. And so if I compute, C dot C, this is an invariant. With a little bit of labor, that basically boils down to middle school algebra. You can show that C dot C is A dot A plus B dot B plus twice A dot B. This is invariant. This is invariant. This is invariant. And so this must be invariant.

So this is really useful for us, because we now have a way-- I've introduced these objects, these geometric objects, these 4-vectors. We are going to use them in this class to describe quantities that are of interest to the physicist who wants to make measurements in spacetime. We've now learned one of the things when you're doing stuff in relativity is you have to be careful who is measuring what. What are the components that 4-vector as seen by this observer? What about their friend who's jogging through the room at 3/4 quarters speed of light? What about their friend who's driving 2/3 the speed of light in the other direction? You have all these really annoying calculations that you can and sometimes have to do. This gives us a way to get certain things that are invariant out of the situation that everyone is going to agree on. Invariants are our friends.

So earlier today, earlier in today's lecture, I talked about how I can write my 4-vectors using the basis factors. So another way of writing this-- so what's sort of annoying is every time I've actually written out the inner product, I have used the represented by symbol. I don't want that. I want to have equal symbols in there, dammit. So let's take advantage of the fact that A dot B, I know how to expand A and B using components and basis vectors. And again, using the index notation, I can just pull everything out and rearrange this a little bit.

Whenever you get down to a point like this, we now get to do what every mathematician loves to do-- give something a name. I'm going to define the inner product of basis vector A with basis vector B to be a two index tensor-- eta alpha beta. What's lovely about this, this is a totally frame invariant quantity. We know that. And so I've now found a way to write this using the components as something that gives me a result that is totally frame invariant.

Now, when you hack through a little bit the algebra of this, what you'll find is that the components of this metric-- oh, shoot, I didn't want to actually say it out loud-- the components of this tensor, which pretend you didn't hear me say that-- the components of this tensor has the following components-- I just said something circular-- has the following values. This is, as I unfortunately gave away the plot, this is, in fact, the metric that I said at the beginning is the quantity that I must associate with spacetime in order for there to be a notion of distance between events.

I haven't really said what a tensor is carefully yet. I'm going to make a more formal definition of this in just a moment. But this is your first example of one.

And so the way in which this actually gives me a notion of distance is through this that I wrote down right here. If I have two events in spacetime that are separated by a displacement delta x, the delta s squared, which I obtained from this thing, is fundamentally the notion of distance between those two events that I use. And notice, it's a little less normal of a distance than you're used to when you do sort of ordinary Euclidean geometry. This is a distance whose square can be negative.

What we like to say is that when you're working in special relativity, it's not necessarily positive-- the distance between two events is not necessarily-- the distance squared is not necessarily positive definite. If it's negative, though, that just means it's sort of dominated by the time interval between them. If it's positive, you know it's dominated by the space interval between them. If it's zero, well, you actually know-- it's actually a little bit confusing at that point. They could be, in fact, you know, very widely separated in both space and time, but in such a way that a light beam could connect them. So there's a lot of information encoded in that.

Now, as we move forward-- hang on a second-- as we move forward, we're going to upgrade this. So right now, our metric is just this simple matrix of minus 1s, 0s, and 1s. One of things that we're going to do is sort of a warm-up exercise to the more complicated things we're going to do later is we're going to move away from special relativity and Cartesian coordinates. We're going to look at it in polar coordinates. That's going to be kind of like a warm-up zone.

And so when we do that, we're always going to reserve eta for the metric of special relativity when I'm working in Cartesian coordinates. It's just a great symbol to have for that. And it's a useful thing to always have that definition in mind.

I can continue to do special relativity, but then working in coordinates that are, you know, spherical-like or polar-like or something like that, then this is going to become a function. And what that is going to mean is that things like my little basis vector is going to have more complicated behavior. A little later in the course, we will then show that when gravity enters into the picture, essentially the essence of gravity is going to be encoded in this thing as well in a way where, again, it's going to be a function. It's going to vary as a function of space and time. And the dynamics of gravity will be buried in that.

It's sort of funny that it really does just sort of start out-- I mean, if you take that thing and you set delta t equals 0, this is just the bloody Pythagorean theorem. That is all this is. Put time back in, and it sort of is the generalization of Pythagoras to spacetime. And, in fact, we're going to take advantage of that and sort of define a geometry that looks like this as being flat in the same way that a board is flat, and the Pythagorean theorem works perfectly on it.

Then, we're going to start think about what happens when it becomes curved. And you start thinking about things like, what is the geometry on the surface of the sphere look like? That's just sort of pointing ahead. So I just throw that at you, so that you get ready for some of the concepts that we'll be talking about soon.

So let me write this actually in terms of differentials. It's sort of useful for what I want to say next. So a little differential, if I have two events in spacetime that are very close to one another, I can write them like so. And what I've essentially done here is written dx as dx alpha e alpha.

Before I get into some more sort of a couple of important, fairly important 4-vectors, the reason I did this is I want to make an important point about some notation and terminology that is used. If it is the case that the displacement vector is related to the differentials of your coordinates like so, we say that e alpha is a coordinate basis vector. What it does is it transforms a differential of your coordinate into a differential vector in spacetime.

Now, you, may be thinking to yourself, OK, well, what other kind can there be? Well, this is where my little spiel there a second ago about how we're going to start looking at more complicated things, it's going to become important. So when we're working in a Cartesian-like coordinate system, the fact that this is what we call a coordinate basis vector isn't very interesting.

Suppose I was working in some kind of a curvilinear coordinate system, OK? Spherical coordinates. Now, let's just focus on 3-space for a second. So if I write a sort of analogous equation in curvilinear coordinates-- OK, so here's the 3-space version of that. Now let's imagine that i equals 1 corresponds to radius, i equals 2 is theta, i equals 3 is phi. Then, this would be dr er plus d theta e theta plus d phi e phi.

Does that disturb you at all? OK, this has dimensions of length. These have the dimensions of angle. In order for this to work, er must be dimensionless. e theta must have the dimensions of length. e phi must have the dimensions of length.

This is what a coordinate basis looks like when I am dealing with-- well, we're going to use is a lot in this thing. I introduce this right now because you are all probably looking at that, and some small part of you inside is weeping, because what you want me to write down is this. Ah, isn't that better? OK, this looks like something you're used to.

So I throw this out here right now just because I want to make sure you're aware that there are some equations and some foundational stuff you guys have been doing over the years, particularly, this shows up a lot when you've done E&M out of a textbook like Purcell or Griffiths or Jackson, because there's some derivative operators, which are assuming that your basis vectors are what we call orthonormal. So my e i hat here, it is an orthonormal basis. And orthonormal basis is defined such that the dot product of any two members of this thing gives me back the chronic or delta.

That is not necessarily the case when I work with a coordinate basis. Our basis has er dot er equals 1. Yay, that one's nice. But when I do e theta dot e theta, I get r squared. e phi dot e phi will be r squared sine squared theta.

And what I'm going to do when I start generalizing these things, I'm going to change my-- this thing which I defined up here-- do I still have it on the board? Yeah, yeah, right here. So when I set eta alpha beta is e alpha dot e beta and I made it this thing, I'm going to generalize this and say at the dot product of any two basis vectors it gives me a more general notion of a metric tensor. And the values in the metric tensor maybe functions like this.

Right now, throw that out there, you know, this might be sort of just like a peak of the horrors that lie ahead. OK, we're not going to worry about this too much just yet. But I want you to be prepared for this. In particular, it's really useful to have this notion of a coordinate basis versus an orthonormal basis in your head. We're going to start defining some derivative operations soon. In fact, probably won't get to them today, but they will be present when we start doing Lecture 3.

And there's a couple of results that come up where everyone's sort of like, wait, I knew that the divergence had a factor of r on that derivative there. Where'd it go? It's because we're not working in an orthonormal basis.

All right, I'm a little sick of math. So let's do a little physics. So, so far, the actual only physical 4-vector that I've introduced is the displacement vector.

From the displacement vector, it's really easy to make the probably the first and simplest important 4-vector, which is known as the 4-velocity. This tells me the rate of displacement of an observer as this person moves through spacetime per unit-- and we're going to be careful about this in this class-- d tau is the time interval as measured along the trajectory of the observer with 4-velocity u. In other words-- that's a very long winded way of saying it-- it's an interval of proper time.

In English, the word proper time sounds very like, woo, I don't want to use improper time. I better use that. But this actually I think it comes from French. It just refers to the fact that it's one's own time. Apparently in German people say eigenzeit. So, you know, there's a couple of different words for it. But proper time is what we use.

In special relativity, if we see someone going by with constant velocity, a particular observer who sees, you know-- we're here in the room. Someone comes through. Their 4-velocity is u. We would see their 4-velocity to have the components gamma gamma v, where gamma, I'll remind you, is the special relativistic Lorenz factor. And I'll remind you again we've set speed of light to 1.

A very useful thing, which we're actually going to take advantage of quite a bit, is that in the rest frame of u-- pardon me for a second-- in the rest frame of u, or I should say of the observer whose 4-velocity is u, they just have 1 in atomic direction, C, if you want to put your factors back into their. And that's basically just saying that the person is standing still, but moving through time, because you are always moving through time.

All right, from the 4-velocity for an observer who has-- or for an object, I should say, who has some mass, we can easily define the 4-momentum, where this m is known as the rest mass of this object. It's worth a bit of description here. You will often see, particularly in some older textbooks that discuss special relativity, people like to talk about the relativistic mass. And that comes from the fact that if I write out what this thing looks like according to some particular observer, you have this gamma m entering into both of the components. And so older textbooks often called gamma m the relativistic mass.

That's not really the way people have-- over the course of the past couple decades, they've moved away from that. And it's just more useful to focus on the rest mass as the only really meaningful mass, because, as we'll see in a moment, it's a Lorenz invariant. We'll see how that is in literally about 3 minutes. And so what we're instead going to say is that as seen by some particular observer, this has a timelike component that is the energy that that observer would measure and a set of spacelike components that are the momentum that that observer will measure.

So where we get a bit of important physics out of all this stuff is by coupling these two 4-vectors to the scalar products that we made up. So the first one, if you do u dot u, according to any observer, that's just going to be minus gamma squared plus gamma squared v squared. And with about 20 seconds worth of analysis, you can find that this is always equal to minus 1.

Actually, there's an even trickier way to do this. Suppose I evaluate this in the rest frame of the observer whose 4-velocity is u? Well, in the rest frame, v is 0, and gamma is 1, and I get minus 1. And this is an invariant. So whatever I get in that particular frame must be obtained in all frames.

That's a trick we're going to use over and over and over again. Sometimes you can identify-- you know, you get some kind of God awful expression that just makes you want to vomit. But then you go, wait a minute, what would this look like in frame blah, blah, blah? And you sort of think about some particular frame. And in that frame, it may simplify. And if it does and it's a frame invariant quantity, mazel tov, you have just basically won the lottery. You've got this all taken care of. Go on with your life.

So the 4-velocity has a scalar product of itself that is always minus 1. OK? How about the 4-momentum? Well, the 4-momentum is just 4-velocity times mass. So that's just minus m squared. But we also know it's related to these two other quantities, which are important in physics. This is related to the energy and to the momentum. So this is also equal to minus e squared plus-- so this is the ordinary 3-- the magnitude of the 3-vector part of this thing as measured by the observer who breaks up the 4-momentum in this way.

So what this means is I can manipulate this guy around a little bit here. Anybody who works in particle physics is presumably familiar with this equation. Sometimes it appears with the p squared moved onto the other side. If it looks a little bit unfamiliar to you, let me put some factors of C back in this. So remember, we have set C equal to 1. When you put it back in, that's what this is. So it drops out of this in a very, very simple way.

One of the uses of this-- and many of you have done exercises presumably in some previous study that does this. And there'll be one exercise on the p set that was just posted where you exploit this. So a key bit of physics, the reason why we care about 4-momentum is it's in one mathematical object allows us to combine conservation energy and conservation of momentum.

So conservation of 4-momentum puts both conservation of energy and conservation of momentum into one mathematical object. So if I have n particles that interact, then the total 4-momentum is conserved in the interaction. OK? So, yeah, we talk a little bit more about this and then just sort of quickly move on.

So combining this with the fact that we are free to change our reference frames often it gives us a trick that allows us to really simplify a lot of analysis. So if I have n particles that are sort of swarming around and doing some horrible bit of business that I need to study and I need to have a good understanding of, we can often vastly simplify our algebra by choosing a special and very convenient frame of reference in which to do our analysis-- choose the center of momentum frame.

So this is the time frame in which that that p tote-- so C-O-M, center of momentum-- has zero spatial momentum. In that frame, you have just as much momentum going to the left as going to the right, just as much going forward as going backwards, as much going up as going down. And so this turns out to be-- so the classic example of where this is really useful is when you are studying particle collisions and you're looking at things like the production of new particles.

So imagine you've got particle A with some 4-momentum PA coming in like this.

Particle B's got some 4-momentum coming in like this. These guys collide. And they do so-- I work in the center of momentum frame. I might want to just calculate the energy at which they just happen to produce some new pair of particles at rest. I would like to find the threshold for this particular creation process. OK? So you're going to play with one problem on the p-set that's kind of like that.

Let's see, what do I have time to do? I think I will do-- yeah, I think I can do two more things.

So all the dot products that I have been talking about so far have been a dot product of a 4-vector with itself. I did u dotted into u. I did p dotted into p. I invented a frame in which p has a particularly simple form. And then when you actually do some of analysis, you would probably take that p tote and dot it into itself.

I haven't done anything that looks at the crossing between these two things, dotting one into the other. So let me to go through a very useful result that follows by combining p with u. There's a very specific notion of p and a very specific notion of u.

Let's let p be the 4-momentum of a particle. I call it a. Let's let u be the 4-velocity not of a, but the 4-velocity of observer O. So particle a might be a muon that was created the upper atmosphere and is crashing through our room right now. Observer O might be your hyperactive friend who is jogging through the room at half the speed of light.

The question I want to ask is, what does O measure as the energy of particle a. So the naive way to do this, which I will emphasize is not wrong, what you might do is sort of go, OK, well, we're sitting here. This room is our laboratory. I've measure this thing in my lab. So I know p as I measure it. I can see O jogging by. So I know O's 4-velocity as I measure it. So what I should do is figure out the Lorenz transformation that takes me into the rest frame of O. Once I have that Lorenz transformation, I'll apply that Lorenz transformation to the 4-vector p. Boom, that will give me that energy. That will work. That will absolutely work. But there's an easier way to do it.

So one thing you should note is that everybody represents that 4-velocity as an energy-- excuse me, they represent the 4-momentum as an energy and a 3-momentum. In particular, though, they represent it as the energy that they would measure and the 3-momentum that they would measure. So that means p as seen by O is e according to O and p according to O. That are acquainted o is what we want. And I just told you a moment ago, you know, if you have p in your own reference frame, and you have u in your own reference frame, you can do this whole math with Lorenz transformations and get it out.

But you also know that in O's own reference frame, O represents their 4-velocity as 1 in the timelike direction, 0 in the spatial direction. So what this means is if I go into O's reference frame, if I go into their inertial reference frame, notice that if I take the dot product of p and u, I get e times 1 and p times 0. So that is just negative. It's exactly what I want-- modulo minus sign. And so you go, OK, well, I'll flip my minus sign around. And you think, OK, great, but I did that using those quantities as written down in O's reference frame. And then you go, holy crap, that's the invariant scalar product. I'm done. Mic drop. Leave the room.

What this means is you start with p as you measure it, u as you measure it. Take the scalar product between the two of them. Boom, the answer you want pops out. No nonsense of Lorenz transformations. None of that garbage needs to happen. You just take that inner product and you've got it. So that sort of says in words, no matter what representation you choose to write down p and u in, take the dot product between the two of them, throw in a minus sign, you've got the energy of the particle with p as measured by the observer with u.

It's sort of late in the hour, or the hour and a half I should say. So let me just sort of emphasize, there are occasional moments in this class where if you're dozing off a little bit, I suggest you pop up and tattoo this into a neuron somewhere. This is one of those moments. OK? This is a result that we're going to use over and over and over again, because this holds-- this isn't just in special relativity. When we start talking about the behavior of things near black holes, there's going to a place where I basically at that point to say, well, I'm going to use the fact that the observer measures an energy that is given by-- and I'm going to write down that. The dot product that's involved is a little bit more complicated, because my metric is hairier. But it's the exact same physical concept. OK?

Let me just do one more. And then I'll go talk, without getting into the math, about what I will start with on Tuesday. Last 4-velocity, which is probably useful for us to quickly talk about is-- so we've talked a lot about 4-velocity. That is just one piece-- when we're talking about sort of the kinematics of bodies moving in spacetime, you need more information than just velocity. Sometimes things are moving around. There's additional forces acting on them.

And so we also care about the 4-acceleration. And so this is what I get when I take the derivative with respect to proper time of the 4-velocity. So will there be some homework exercises that use this.

The main thing which I want to emphasize to sort of conclude our calculations for today is that when I talk about a 4-velocity of acceleration, this has an extremely important property. It is always the case that a dotted into u equals 0. If you're used to sort of 3-dimensional intuition, that may seem weird. OK? Anytime you see a car accelerate from a stop, that's a case in which its acceleration is clearly not orthogonal to its velocity.

But the issue here is, this is not a spatial dot product. This is a space time dot product. And some of your intuition has to go out the window because of that. It's very simple to prove this. Remember, u dot u equals minus 1. So d of u dot u, d tau, which is just 2, u dot a is the derivative of minus 1, which is 0. OK?

So this is something that we will exploit. If you want to describe the relativistic kinetics of an accelerating body, this is a great thing that we can use to exploit. You often need a little bit more information. We have to give you as a bit of additional information some knowledge about what the orientation of the acceleration is and things like that. So whenever you are given-- I'll get to you in just a sec-- whenever you're given any kind of differential quantity like this, it's not enough to know-- it's like the acceleration of loss, you have to also have boundary conditions. And that sort of tells you what the initial direction is. Question?

AUDIENCE: Is that time still the proper time?

SCOTT HUGHES: That time is the proper time, yes.

AUDIENCE: So it's not accelerating?

SCOTT HUGHES: That's right. So you can still define a proper time for an accelerating observer. It will not relate-- hold that thought. You're going to play this a little bit more in a future problem set. I mean, the key thing is that the way-- if you have an d observer, an interval of proper time as compared to an interval of time for, you know, someone in a rest frame that sees this person accelerate away, the conversion between the intervals of time, the two measure, it evolves.

So, you know, let's say you're in this room with me. In fact, it turns out that if you accelerate at 1G for a year, you get to very close to the speed of light. So let's say that you were in a rocket ship right now that launched with an acceleration of G. Initially, you and I synchronize our watches. And so an interval of a second to me is the same as a second to you. Half a year later, you're moving at something like half the speed of light. And I will see a noticeable time delay. An interval of a second as you measure it looks long compared to me. Six months later, you're actually quite close to the speed of light, and it gets dilated even more.

So last thing, which I'm going to say, and I'm not going to get into too much detail with this yet, is we're going to begin next time by making a little bit more formal some of the notions that go around. So we've increased some physics and some vectors. I've given you guys one tensor so far, the metric tensor. And so I'm going to give you-- in fact, I will write down a very precise definition of this right now, and we'll pick it up from there on Tuesday.

So the basic idea-- so you guys have see-- the one tensor you've seen so far is the metric tensor. And what the metric is is it's sort of a mathematical object that I put in a pair of 4-vectors. And it spits out a quantity that is a Lorenz invariant scalar that characterizes what we call the inner product of those two 4-vectors.

More generally, I'm going to define a tensor of type 0 N as a function or mapping of N 4-vectors into Lorenz invariant scalars, which is linear in each of its N arguments. So I will pick it up here on Tuesday. And let me just say in words, the metric is a 0 2 tensor. I put in two 4-vectors. It spits out a Lorenz invariant scalar.

We're going to before too long come up with a couple of things that involve three real vectors-- excuse me three 4-vectors, too many numbers here, a trio of 4-vectors, which it then maps to a Lorenz invariant scalar. Some of them will take in four 4-vectors and produce a Lorenz invariant scalar.

Notice I wrote this in sort of funny way. The 0 N sort of begs for there to be sort of an N 0. OK? To do that I have to introduce an object that is sort of dual to a vector. We're going to talk about that. Those are objects called one-forms, which actually happen to be a species of vector.

We're actually going to then learn that the vector is itself a tensor. And so we will make a very general classification of these things. And we'll see that vectors are just a subset of these tensors. And at last, we'll sort of have all the mathematics in place. We can sort of lose some of these distinctions and just life goes on, and we can start actually doing some physics with these.

All right, I will pick it up there on Tuesday.