Massachusetts Institute of Technology
Department of Physics
8.962 Spring 2006

Problem Set 6

1. Constraint and evolution equations

   (Note: A long setup to a fairly short problem.)

   Maxwell's equations, written in terms of electric and magnetic fields and using cgs units with the speed of light $c = 1$, take the form

$$\partial_i E^i = 4\pi\rho \,, \qquad \partial_i B^i = 0 \,;$$
$$\partial_t E^i = \epsilon^{ij}{}_k \partial_j B^k - 4\pi J^i \,, \qquad \partial_t B^i = -\epsilon^{ij}{}_k \partial_j E^k \,.$$

   The first two equations — those relating the divergence of the fields — are called "constraint equations". This is because they involve no time derivatives — they are differential relations among the fields (and sources) at a single moment in time.

   The second two equations are called "evolution equations". Because they involve the time derivative of fields, they relate the fields and sources from one moment to the next. We can thus use the second pair of Maxwell equations to evolve "data" (which must satisfy the constraints) from some initial moment to an arbitrary late time.

   We will now prove that the Einstein field equations have a similar structure. Because the Einstein field equations are second order, we expect our "initial data" to consist of fields (the metric) and *first* time derivatives. Constraint equations should thus be those components of the Einstein equation which contain no more than a single time derivative; evolution equations are those components which contain two derivatives. (This is analogous to the kinematics of a particle: the "initial data" is the starting position and velocity; we use the acceleration as our "evolution equation" to find the particle's motion from then on.)

   Suppose we have chosen a timelike direction, so that $x^0 = t$; we do not specify the spatial coordinates x^i other than to say that they are coordinates covering the $x^0 =$ constant hypersurface.

   Show that the Einstein tensor components G_{00} and G_{0i} contain no more than one time derivative. Thus, the equations $G_{00} = 8\pi G T_{00}$ and $G_{0i} = 8\pi G T_{0i}$ can be considered constraints which relate the metric, its first time derivative, and sources at a single moment of time; the equations $G_{ij} = 8\pi G T_{ij}$ are evolution equations.

   (Hint: A brute force construction of the curvature tensor $G_{\alpha\beta}$ in terms of the metric and its derivatives will give you the correct solution to this problem. This is *not* a recommended procedure, though. A much quicker solution can be deduced by considering the Bianchi identity applied to the Einstein tensor.)

2. Action for a cosmological constant

   Show that varying the action

$$S = \int d^4x \sqrt{-g}\,(R + a) \ ,$$

   (where R is the Ricci scalar and a is a constant) yields the Einstein equation with a cosmological constant. How does a relate to the cosmological constant Λ?

3. Nordstrøm's gravity theory

   A metric theory devised by G. Nordstrøm in 1913 (before general relativity was finalized) relates $g_{\mu\nu}$ to $T_{\mu\nu}$ by the equations

$$C_{\alpha\beta\gamma\delta} = 0\,, \qquad R = \kappa g_{\mu\nu} T^{\mu\nu}$$

   where $C_{\alpha\beta\gamma\delta} = 0$ is the Weyl curvature tensor.

   The vanishing of Weyl tells us that the metric is conformally flat; this follows from the fact that vanishing Riemann implies that spacetime is truly flat (not just conformally flat), and that the Weyl tensor is invariant under conformal transformations. Conformal flatness means that

$$g_{\mu\nu} = e^{2\phi}\eta_{\mu\nu} \ ,$$

   where $\phi = \phi(x^\mu)$ is a function of the spacetime coordinates. (To relate this to the notation used in lecture, $e^\phi = \Omega$; the exponential form is convenient for the calculations here.)

   (a) Show that in the limits $\phi^2 \ll 1$ and $|\partial_t\phi| \ll |\partial_i\phi|$, the geodesic equation for a test body moving slowly ($u^i \ll 1$) in this spacetime reproduces the kinematics of Newtonian gravity. We'll call this the "Newtonian limit" from now on.

   (b) Show that in the Newtonian limit the Ricci scalar R is just a second order differential operator acting on ϕ. Compute that operator.

   (c) Show that Nordstrøm's field equation reduces in the Newtonian limit to the gravitational field equations, and determine the value of κ.

   (d) Is this theory consistent with the Pound-Rebka gravitational redshift experiment? (This is the experiment which established that light in fact redshifts as the equivalence principle predicts.)

   (e) Show that there is no deflection of light by the sun in this theory of gravity.

4. Weighing a relativistic body

An object of mass m is at rest on a bathroom scale in a weak, uniform, static gravitational field. That is, the object has fixed spatial coordinates (x, y, z) and the spacetime metric has the standard weak-field form $g_{\mu\nu} = \eta_{\mu\nu} + 2\phi\,\mathrm{diag}(1, 1, 1, 1)$, with ϕ the normal Newtonian potential. We take $\phi^2 \ll 1$, $\partial_z\phi = \text{constant} = -g$, and $\partial_\mu\phi = 0$ for $\mu \neq z$. Neglect terms of order ϕ^2 and ϕg in what follows.

In this problem, we will see that if one wants to interpret gravity as a force rather than as the effect of spacetime curvature, then it must be a velocity-dependent force. This is not a fundamental insight; the main purpose of the problem is to practice relating the metric to measurable quantities in curved spacetime.

(a) What force does the bathroom scale apply on the body? Compute both the components and the scalar magnitude of the 4-force. The principle to apply here is that the body does *not* follow a geodesic: The equation of motion for the body is

$$m\frac{D^2x^\mu}{d\tau^2} = mu^\beta\nabla_\beta u^\alpha = F^\mu \ .$$

This relation may be taken to *define* the 4-force F^μ.

(b) Now suppose that the object moves with constant, relativistic coordinate 3-velocity $v = dx/dt = (dx/d\tau)(dt/d\tau)^{-1}$ in the x-direction:

$$V^x = vV^t \ ; \qquad V^y = V^z = 0 \ .$$

What is V^t? (Don't just use a special relativity formula!) While the mass is on the bathroom scale, what force (components and magnitude) does the scale apply to the mass?

(c) Now transform coordinates by applying a naive Lorentz transformation: $\bar{t} = \gamma(t - vx)$, $\bar{x} = \gamma(x - vt)$, $\bar{y} = y$, $\bar{z} = z$. Evaluate the components of the metric in the new coordinate system, $g_{\bar{\mu}\bar{\nu}}$. To first order in ϕ, what are the force components in this new coordinate basis?

(d) Show that the barred coordinate basis can be transformed to an orthonormal basis, $\vec{e}_{\hat{\mu}} = E^{\bar{\mu}}{}_{\hat{\mu}}\vec{e}_{\bar{\mu}}$ with a tetrad matrix

$$E^{\bar{\mu}}{}_{\hat{\mu}} = \delta^{\bar{\mu}}{}_{\hat{\mu}} + \phi A^{\bar{\mu}}{}_{\hat{\mu}} \ .$$

Find the matrix $A^{\bar{\mu}}{}_{\hat{\mu}}$. To first order in ϕ, do the force components $F^{\hat{\mu}}$ differ from $F^{\bar{\mu}}$?

5. Converting between geometrized and "normal" units

Especially as we discuss astrophysical applications, we will find it useful to work in "geometrized units", in which the gravitational constant and the speed of light are both set to unity. When this is done, mass, length and time are measured in the same units.

We convert among different units by multiplying by powers of G and c; since such a factor is just 1 in geometrized units, we can include as many such factors as is necessary. For example, to express the solar mass as a time, we write $M_\odot^{\rm geom} = GM_\odot/c^3$. Using

$$\begin{aligned} M_\odot &= 1.99 \times 10^{33}\,\mathrm{gm}\ , \\ G &= 6.67 \times 10^{-8}\,\mathrm{cm^3 gm^{-1} sec^{-2}}\ , \\ c &= 3.00 \times 10^{10}\,\mathrm{cm/sec}\ , \end{aligned}$$

we find $M_\odot^{\rm geom} = 4.93 \times 10^{-6}$ seconds.

Do the following conversions:

(a) Mass of the earth ($M_\oplus = 5.98 \times 10^{27}$ gm) in centimeters.

(b) Characteristic mean density of neutron stars ($\bar\rho = 10^{15}$ gm/cm^3) in inverse square centimeters.

(c) Characteristic mean pressure at core of a neutron star ($\bar P = 10^{34}$ gm sec^{-2} cm^{-1}) in inverse square centimeters.

(d) Acceleration of gravity at the surface of the earth ($g = 9.8$ m/s^2) in inverse seconds and inverse years.

(e) The typical (isotropic) luminosity of a gamma ray burst: $L = 10^{53}$ erg/sec. (Reminder: 1 erg = 1 gm cm^2/sec^2.)

(f) Planck's constant ($\hbar = 1.05 \times 10^{-27}$ erg sec) in square centimeters.

The square root of this quantity is called the "Planck length", and is denoted l_p. Since it involves the constants which set quantum effects ($\hbar$), gravitational effects (G), and relativistic effects (c), it is thought that quantum gravitational effects (i.e., the "quantization of spacetime", whatever that actually means) must be important at lengthscales $L \sim l_p$.

(g) Convert l_p to a mass (gm); this is known as the "Planck mass". Convert that to an energy; express this energy in electron volts. Comment on the likelihood of observing Planck mass scale effects at a particle collider.