1.9 The collisionless Boltzmann equation and Jeans' equations

We have emphasized Schwarzschild's view of a galaxy as a superposition of orbits. He prescribes a scheme that is straightforward in principle but difficult in practice. An alternate view of galaxies is as a system of particles in six dimensional *phase space.* The galaxy is then instantaneously described as a distribution function $f(\vec{\mathbf{x}}, \vec{\mathbf{v}})$ over the phase space.

Since Schwarschild's approach assumes a time independent potential, it guarantees that a set of orbits that reproduces that potential will do so for all time. By contrast a phase space density that reproduces the density and potential of a galaxy at one instant will not in general reproduce itself at later (or earlier times). The time evolution of the phase space density is governed by a a 6-dimensional equation of continuity that is analogous to the familiar 3-dimensional equation of continuity of fluid mechanics. Each point in phase space is described by a 6-D vector $\vec{\mathbf{w}} = (\vec{\mathbf{x}}, \vec{\mathbf{v}})$. The equation of continuity is then

$$\frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} \frac{\partial}{\partial w_\alpha}(f \dot{w}_\alpha) = 0. \tag{1.117}$$

We find that

$$\sum_{\alpha=1}^{6} \frac{\partial \dot{w}_\alpha}{\partial w_\alpha} = \sum_{i=1}^{3} \left(\frac{\partial v_i}{\partial x_i} + \frac{\partial \dot{v}_i}{\partial v_i} \right) = 0, \tag{1.118}$$

where the first part of the sum is zero because the velocities are necessarily not explicit functions of position (hence six dimensions in phase space and not fewer). The equation of motion tells us that $\dot{v}_i = -\frac{\partial}{\partial x_i}\Phi(\vec{\mathbf{x}})$ and thus $\dot{v}_i$ is a function of position only. Substituting back into our 6-dimensional equation of continuity we get the collisionless Boltzmann equation (CBE):

$$\frac{\partial f}{\partial t} + \vec{\mathbf{v}} \cdot \vec{\nabla} f - \vec{\nabla}\Phi \cdot \frac{\partial f}{\partial \vec{\mathbf{v}}} = 0. \tag{1.119}$$

It collisionless in that particles do not make instantaneous jumps in $\vec{\mathbf{x}}$ or $\vec{\mathbf{v}}$, a consequence of a potential Φ that is smooth in space and time. The CBE can be used simultaneously for many different species in a galaxy, each with its own distribution function. This is important in implementing the fourth step of galaxy construction: calculating the distribution of light produced by all the different types of stars in the galaxy.

The phase space density $f(\vec{\mathbf{x}}, \vec{\mathbf{v}})$ is far too unwieldy to produce a useful description of a galaxy. But if the dependence of the phase space density upon velocity is relatively smooth and free of singularities, one can collapse the 6-dimensional phase space density into set of functions of 3-dimensional position by taking moments of the velocities. The zeroth moment is just the number denisity, $\nu(\vec{\mathbf{x}})$, give by

$$\nu(\vec{\mathbf{x}}) \equiv \int f(\vec{\mathbf{x}}, \vec{\mathbf{v}}) d^3\vec{v} \quad . \tag{1.120}$$

For each of three velocity components the first moment gives a mean velocity,

$$\bar{v}_i(\vec{\mathbf{x}}) \equiv \frac{1}{\nu(\vec{\mathbf{x}})} \int v_i f(\vec{\mathbf{x}}, \vec{\mathbf{v}}) d^3\vec{v} \quad . \tag{1.121}$$

One can likewise define higher order moments with combinations of powers of the three velocity components. The second moments give a quantity related to the velocity dispersion tensor, σ_{ij}^2,

$$\overline{v_i v_j}(\vec{\mathbf{x}}) \equiv \frac{1}{\nu(\vec{\mathbf{x}})} \int v_i v_j f(\vec{\mathbf{x}}, \vec{\mathbf{v}}) d^3\vec{v} = \sigma_{ij}^2 + \bar{v}_i \bar{v}_j \quad . \tag{1.122}$$

Ideally the velocity distribution functions are not cuspy and so are reasonably well described by the low order moments. There is at least some observational support for this. A density and a set of low order moments may therefore give a reasonably complete description of a galaxy.

By multiplying the CBE by powers of the velocity components, and integrating over velocity space we obtain a series of differential equations for the various velocity moments. The zeroth moment of the CBE yields

$$\int \frac{\partial f}{\partial t} d^3\vec{v} + \int v_i \frac{\partial f}{\partial x_i} d^3\vec{v} - \frac{\partial \Phi}{\partial x_i} \int \frac{\partial f}{\partial v_i} d^3\vec{v} \quad . \tag{1.123}$$

Note that we have dropped the summation signs and adopted the implicit summation convention, in our case over i. We can eliminate the last term term by application of the divergence theorem,

$$\int \vec{g} \cdot \vec{\nabla}_v f d^3\vec{v} = \int f \vec{g} \cdot d\vec{S} - \int f \vec{\nabla}_v \cdot \vec{g} d^3\vec{v} \quad . \tag{1.124}$$

In the present case $g = \vec{\nabla}\Phi(\vec{x})$ and is not a function of $\vec{v}$ and so may move freely inside and outside the integral. We have $\vec{\nabla}_v \cdot \vec{g} = 0$ and the surface integral goes to zero if the phase space density goes to zero at infinity. Incorporating our definitions of number density and mean velocity we have

$$\frac{\partial}{\partial t}\nu + \frac{\partial}{\partial x_i}(\nu \bar{v}_i) = 0, \tag{1.125}$$

which looks just like a standard 3-D continuity equation. This should come as no surprise since to derive it we just collapsed the 6-D continuity equation by integrating over all velocity.

The first moment of the CBE is found by multiplying by v_j and integrating over velocity. Again taking the spatial and temporal derivatives outside the velocity integrals, we get

$$\frac{\partial}{\partial t} \int v_j f d^3\vec{v} + \frac{\partial}{\partial x_i} \int v_j v_i f d^3\vec{v} - \frac{\partial \Phi}{\partial x_i} \int v_j \frac{\partial f}{\partial v_i} d^3\vec{v} = 0 \quad . \tag{1.126}$$

Integrating by parts and expressing our results in terms of our average velocities, we have

$$\frac{\partial}{\partial t}(\nu\bar{v}_j) + \frac{\partial}{\partial x_i}(\nu\overline{v_i v_j}) + \frac{\partial\Phi}{\partial x_i}\int f\frac{\partial v_j}{\partial v_i}d^3\vec{v} = 0 \quad . \tag{1.127}$$

In an orthogonal coordinate system, $\frac{\partial v_j}{\partial v_i} = \delta_{ij}$ so the last term on the left hand side becomes $\nu\delta_{ij}$. Applying the product rule and our continuity equation to the first term we get

$$\nu\frac{\partial\bar{v}_j}{\partial t} - \bar{v}_j\frac{\partial}{\partial x_i}(\nu\bar{v}_j) + \frac{\partial}{\partial x_i}[\nu(\sigma_{ij}^2 + \bar{v}_i\bar{v}_j)] = -\nu\bar{v}_i\frac{\partial\Phi}{\partial x_j} \quad , \tag{1.128}$$

where we have made use of the relation between the second moments and the velocity dispersion. Differentiating the second term on the left hand side, part of the result cancels part of the third term and we arrive *Jeans' equations* for a collisionless fluid,

$$\begin{array}{ccccccccc} \nu\dfrac{\partial\bar{v}_j}{\partial t} & + & \bar{v}_i\nu\dfrac{\partial\bar{v}_j}{\partial x_i} & = & -\nu\dfrac{\partial\Phi}{\partial x_j} & - & \dfrac{\partial}{\partial x_i}(\nu\sigma_{ij}^2) & (j = 1, 2, 3). \\ \text{acceleration} & + & \begin{array}{c}\text{kinematic}\\ \text{viscosity}\end{array} & = & \text{gravity} & + & \text{pressure} & \end{array} \tag{1.129}$$

While in Cartesian coordinates all three Jeans equations have the same form, this is not true in other coordinate systems.

By setting the acceleration and shear terms to zero, we can recover the equation for hydrostatic equilibrium. Other uses include calculating the number density and potential self-consistently, assuming a given model for the velocity dispersion.

As an example, we will calculate the surface density of a sheet of fluid with plane parallel symmetry and reflection symmetry in the z-axis. The mass surface density, $\Sigma(z)$, is defined by

$$\Sigma(z) = \int_{-z}^{z}\rho(z')dz' \quad . \tag{1.130}$$

Taking σ_{zz}^2 to be constant (iosthermal), the $j = 3$ Jeans Equation gives

$$\frac{1}{\nu}\frac{\partial}{\partial z}(\nu\sigma_{zz}^2) = -\frac{\partial\Phi}{\partial z} = -2\pi G\Sigma(z), \tag{1.131}$$

where we have used Gauss' theorem to write the last part of the equality. The solution to (1.122) is of the form

$$\Sigma(z) = -\frac{\sigma_{zz}^2}{2\pi G}\left(\frac{d}{dz}\ln\nu\right). \tag{1.132}$$

If we consider the population of of nearby G and K dwarf stars, their measured velocity dispersion is found to be $\sigma_{zz}^2 = (20\text{ km/s})^2$ and their scale height h, defined as

$$h^{-1} \equiv \frac{d}{dz}\ln\nu \tag{1.133}$$

is found to be approximately 300 pc. This gives a mass surface density of $\Sigma \approx 50\ M_\odot\text{pc}^{-2}$ for the disk of the Milky Way.

Applying the same equation to the Mestel disk model, we should be able to calculate a surface density from

$$v_c^2 = 2\pi G \Sigma_{\rm Mestel} R_0, \tag{1.134}$$

where measured values of v_c^2 and R_0 for the Milky Way are (220 km/s)^2 and 8 kpc, respectively. Substituting, we find that the ratio of the predicted Mestel disk surface density and that of the perpendicular plane model is

$$\frac{\Sigma_{\rm Mestel}}{\Sigma_{\rm perp}} = \frac{v_c^2}{\sigma_{zz}^2} \cdot \frac{h}{R_0} = \left(\frac{220}{20}\right)^2 \cdot \left(\frac{0.3}{8}\right) \approx 4.5. \tag{1.135}$$

The assumptions going into the Jeans analysis are involve considerably less extrapolation than those going into the Mestel analysis. In particular we have assumed that *all* of the mass of the Milky Way lies in the disk. This might at first seem reasonable, since all but perhaps 15% of the light in the Milky Way lies in the disk. But if the disk were embedded in a more nearly spherical component, this would affect the circular velocity but not in the perpendicular velocity dispersion. The currently favored interpretation is that some large fraction of the mass interior to the Sun's orbit lies in an unseen, dark component. We will see that clusters of galaxies and the cosmic microwave background (CMB) provide additional evidence for such dark matter. A great many investigators have searched for Milky Way dark matter in various forms: ionized gas, atomic gas, molecular gas, compact stars and planet sized condensations. All have proved unsuccessful. Observations of the CMB in particular lead us to believe that the dark matter is non-baryonic – that it is not composed of protons, neutrons and electrons.