8.851 Homeworks 8 & 9

Iain Stewart, May 6, 2006

This will be the last problem set. It includes more problems than usual and will count as two problem sets.

Problem 1) Non-Relativistic QCD at the ILC

Consider $e^+e^- \rightarrow t\bar{t}$ near the threshold for $t\bar{t}$ production ($\sqrt{s} \sim 2m_t \sim 350\,\mathrm{GeV}$) at the International Linear Collider. In this region the top quarks are non-relativistic with small residual energy, with $2E = \sqrt{s} - 2m \sim mv^2 \sim 5\,\mathrm{GeV}$, and non-relativistic momenta $\vec{p} \sim 30\,\mathrm{GeV}$. The top quarks have a width for $t \rightarrow bW$ that is numerically similar to mv^2, so we will include their total width as part of our LO potential Lagrangian:

$$\mathcal{L}_p = \sum_{\mathbf{p}} \psi_{\mathbf{p}}^\dagger \left(i\partial^0 - \frac{\mathbf{p}^2}{2m} + \frac{i\Gamma}{2} + \dots \right) \psi_{\mathbf{p}} + (\psi \rightarrow \chi) + \dots \tag{1}$$

where the ... indicate higher order relativistic corrections as well as the potential interactions. The Coulomb interaction is relevant so we will include it too. The cross section can be obtained from the imaginary part of a Green's function computed from this action, and in the $\overline{\mathrm{MS}}$ scheme is

$$G^0(a, v) = \frac{m^2}{4\pi} \left\{ iv - a \left[\ln\left(\frac{-iv}{\nu} \right) - \frac{1}{2} + \ln 2 + \gamma_E + \psi\left(1 - \frac{ia}{2v} \right) \right] \right\}. \tag{2}$$

Here $a = C_F \alpha_s(m\nu)$, $v^2 \equiv (\sqrt{s} - 2m + i\Gamma)/m$, $m = m_{\mathrm{top}}$. The polylog function $\psi(x) = d/dx \ln[\Gamma(x)]$. I have written the renormalization scale $\mu = m\nu$ in terms of a velocity renormalization parameter ν.

a) Lets see which diagrams generate $G^0(a, v)$. Consider the forward scattering diagrams from the time-ordered product of a production and annihilation current, $T J_i^\dagger(0) J_j(x)$. For $e^+e^- \rightarrow t\bar{t}$ through a virtual photon we have $J_i^\dagger = \psi_{\mathbf{p}}^\dagger \sigma_i (i\sigma_2) \chi_{-\mathbf{p}}^*$, but feel free to ignore the spin for your calculation. Compute the one-loop diagram and show that it reproduces the iv term in Eq. (2). Of all the diagrams that generate $G^0(a, v)$ which have UV divergences? Which diagram gives the term with a logarithm?

b) Using mathematica plot the normalized cross section $R = 8\pi \mathrm{Im} G^0 / m^2$ as a function of $\sqrt{s}$. Taking $\alpha_s(m_Z) = 0.118$, $m = 175\,\mathrm{GeV}$, renormalization velocity $\nu = 0.15$, and using one-loop running for $\alpha_s(m\nu)$:

i) Plot R for the values $\Gamma = 2\,\text{GeV}$, $\Gamma = 0.7\,\text{GeV}$, $\Gamma = 0.1\,\text{GeV}$, $\Gamma = 0.05\,\text{GeV}$. Describe the physics behind what you observe for $\sqrt{s} > 2m$ and $\sqrt{s} < 2m$.

ii) Take $\Gamma = 0.05\,\text{GeV}$. Analyically determine the location in $\sqrt{s}$ where you expect the first and second peaks (without using Eq. (2)) and see if it agrees with your plot of R for this Γ. (Hint: Remember that we're doing QCD with a Coulomb interaction. In QCD the coefficient of the Coulomb potential in the color singlet channel is $\mathcal{V}_c = -4\pi\alpha_s(m\nu)C_F$ with $C_F = 4/3$.)

iii) Taking the realistic value $\Gamma = 1.5\,\text{GeV}$ explain how this cross section could be used to measure the top mass m.

Problem 2) SCET Operators with Collinear Quarks

a) Start with the QCD Lagrangian for a massive quark and decompose $\slashed{D}$ in terms of n, $\bar{n}$, and $\perp$ components. As in lecture, write $\psi = \xi_n + \xi_{\bar{n}}$ and determine which terms are non-zero. Keeping all the terms integrate out the field $\xi_{\bar{n}}$ to generate an effective action for ξ_n.
[With power counting $m \sim p_\perp \sim Q\lambda$ this is the starting point to derive the action for a massive collinear quark, ie. prior to decomposing the gluon field into collinear and usoft pieces and prior to distinguishing between large and small momenta. The steps here are similar to those outlined in lecture except that you are keeping the mass.]

b) Consider the current for a $b \to u$ transition. In QCD $J = \bar{u}\Gamma b$. For SCET we did a matching calculation to find the leading order current

$$J^{(0)} = \bar{\xi}_n W \Gamma h_v \,, \tag{3}$$

where W included terms involving the order λ^0 collinear gluon field $\bar{n} \cdot A_n$. In lecture we explicitly computed the term in W with one $\bar{n} \cdot A_n$ field and wrote down the result for any number of $\bar{n} \cdot A_n$ fields. Do the matching computation for two $\bar{n} \cdot A_n$ fields (by expanding QCD diagrams with offshell propagators). Bonus: Verify that the result for one and two $\bar{n} \cdot A_n$ fields agree with the momentum space Feynman rules for the position space Wilson line

$$W(y^+) = P \exp\left(ig \int_{-\infty}^{0} ds\, \bar{n} \cdot A_n(s\bar{n} + y) \right),$$

where P is path-ordering.

Problem 3) Decoupling of Ultrasoft gluons in SCET_I

Consider a Wilson line built from usoft gluons

$$Y_n(x) = P \exp\left(ig \int_{-\infty}^{0} ds\, n \cdot A_{us}(sn^\mu + x^\mu) \right). \tag{4}$$

It satisfies $Y_n^\dagger Y_n = 1$ and has the equation of motion $n \cdot D_{us} Y_n = 0$. Start with the leading order Lagrangian, $\mathcal{L}^{(0)}$ for a collinear quark in SCET_I. This action gives eikonal couplings to ultrasoft gluons. Make the field redefinitions $\xi_n = Y_n \xi_n^{(0)}$ and $A_n = Y_n A_n^{(0)} Y_n^\dagger$ to obtain a Lagrangian $\mathcal{L}^{(0)}(\xi_n^{(0)}, A_n^{(0)})$. Show explicitly that this new Lagrangian has no coupling to $n \cdot A_{us}$ gluons.

Problem 4) Decoupling of Soft and Collinear Gluons in SCET_II

Consider an operator with one collinear quark and one soft quark in SCET_II. This case differs from SCET_I in that soft gluons knock collinear quarks off-shell and collinear gluons knock soft quarks offshell. To be definite lets make the soft quark a heavy quark from HQET, h_v, and the energetic quark a massless collinear quark, ξ_n. The SCET operator will be

$$(\bar{\xi}_n W_{\bar{n}}) \Gamma (S_n^\dagger h_v) \tag{5}$$

where W is the Wilson line we saw in problem 2b, and S_n is an exact analog of Eq. (4) but with A_s gluons rather than usoft gluons. To find Eq. (5) requires tree level matching (to determine the direction and gluon components appearing in the Wilson lines), and gauge invariance (to ensure that loops do not spoil the structure so obtained). The first non-trivial term in perturbation theory have one soft and one collinear gluon. By computing the graphs in Figure 1 expanded to LO, verify that the three-gluon interactions are responsible for putting the gluons in the right order in Eq. (5).
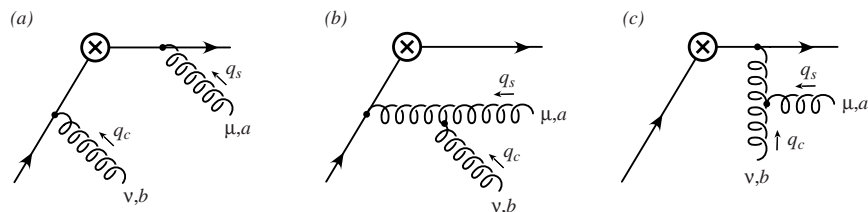

Figure 1: QCD graphs with collinear, q_c, and soft, q_s, momenta.

Problem 5) Pion Light-Cone Distribution Function

Factorization of degrees of freedom allows us to describe processes in terms of simpler (universal) objects. One such object is the pion light-cone distribution function. In QCD we can define

$$\langle \pi^a(p)| \, \bar{\psi}(y)\gamma^\mu\gamma^5 \frac{\tau^b}{\sqrt{2}} Y(y,x)\,\psi(x)\,|0\rangle \;=\; -if_\pi \delta^{ab} p^\mu \int_0^1 dz\, e^{i[zp\cdot y+(1-z)p\cdot x]}\,\phi_\pi(z)$$

$$+ \dots, \tag{6}$$

Here $f_\pi \simeq 131\,\text{MeV}$ and the field ψ denotes the isospin doublet (u,d). Consider the path from y^μ to x^μ to be light-like, $(y-x)^2 = 0$, with $y^\mu = y\bar{n}^\mu$, $x^\mu = x\bar{n}^\mu$. $Y(y,x)$ is a Wilson line along this path. For the leading order collinear operator in SCET

$$\langle \pi_{n,p}^a| \bar{\xi}_{n,p_1} W\, \slashed{\bar{n}}\gamma_5 \frac{\tau^b}{\sqrt{2}}\, \delta(\omega-\bar{\mathcal{P}}_+)\, W^\dagger \xi_{n,p_2}|0\rangle$$

$$= -if_\pi\, \bar{n}\cdot p\, \delta^{ab}\int_0^1 dz\, \delta[\omega-(2z-1)\bar{n}\cdot p]\,\phi_\pi(z)\,, \tag{7}$$

where $\bar{\mathcal{P}}_+ = \bar{\mathcal{P}}^\dagger + \bar{\mathcal{P}}$. Eq. (7) can be obtained from Eq. (6) by projecting onto the LO term with collinear quarks and taking a Fourier transform with $x = -y$.

a) Using the power counting for the collinear fields in $\lambda = \Lambda_{\text{QCD}}/Q$, together with mass dimensions, count the powers of Λ_{QCD} and Q on the LHS and RHS of Eq. (7) and verify that $\phi_\pi(z)$ is $\mathcal{O}(\lambda^0)$ and is dimensionless.

b) Under charge conjugation, $C^{-1}\xi_{n,p}(x)C = -[\bar{\xi}_{n,-p}(x)\mathcal{C}]^T$ where $\mathcal{C}$ is the usual charge conjugation matrix. For a π^0 state use charge conjugation together with Eq. (7) to prove that $\phi_{\pi^0}(z) = \phi_{\pi^0}(1-z)$. Physically what does this mean? What do you have to assume to prove that this is true for the π^+ and π^-?

c) Bonus: Carry out the steps to go from Eq. (6) to Eq. (7) ignoring the ellipses.