8.851 Homework 2

Iain Stewart, February 19, 2003

Problem 1) Field Redefinitions

In class we considered the following Lagrangian for a scalar field

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \lambda\phi^4 + \eta g_1 \phi^6 + \eta g_2 \phi^3 \partial^2 \phi \qquad (1)$$

where $\eta \ll 1$. We then made a field redefinition $\phi \to \phi'$ which eliminated the last term, giving a new Lagrangian $\mathcal{L}'$.

a) Show at $\mathcal{O}(\eta)$ that the field redefinition gives the same answer for $\mathcal{L}'$ as applying the $\eta = 0$ equations of motion to the last term. Determine the new λ' and g_1' parameters in terms of those in the original Lagrangian.

b) Demonstrate the equivalence of the 4-point and 6-point functions to $\mathcal{O}(\eta)$ by explicit computation of the tree level Feynman graphs with $\mathcal{L}$ and $\mathcal{L}'$. Use your result from a).

c) Draw the diagrams you would need to compute to demonstrate the equivalence of the 4-point functions at one-loop and $\mathcal{O}(g_2\,\eta)$. Roughly sketch how the equivalence would work by taking zero momenta on the external lines and looking at the form of the loop integrals (do not worrry about the symmetry factors needed to show it explicitly).

d) With the original field redefinition determine the terms induced at $\mathcal{O}(\eta^2)$ in $\mathcal{L}'$, and then show that a second field redefinition can be used to again eliminate the terms proportional to the equations of motion.

Problem 2) Quark Masses and Renormalons

Consider QCD with n_f flavors.

a) Compute the quark mass anomalous dimension at one-loop in $\overline{\text{MS}}$. Solve to get $\bar{m}(\mu)$. (For a bonus, also compute the relation between the pole mass and $\overline{\text{MS}}$ mass at one-loop.)

b) Suppose we measured the charm mass $\bar{m}_c(\mu = \bar{m}_c) = 1.4\,\text{GeV}$, and wanted to know what value to use for a process at $\mu = 500\,\text{GeV}$. By matching and running in effective theories determine the appropriate value.

In QCD the pole mass has an infrared renormalon which leads to an intrinsic ambiguity in its value. This ambiguity is related to the convergence of perturbative series with terms α_s^n where the coefficients may grow as $n!$. If a function $f(\alpha_s) = f(0) + \sum_{n\geq 0} f_n \alpha_s^{n+1}$ then the convergence is improved for the Borel transformed function $\tilde{f}(t)$ where

$$\tilde{f}(t) = f(0)\delta(t) + \sum_{n=0}^{\infty} \frac{f_n}{n!} t^n \,, \qquad\qquad f(\alpha_s) = \int_0^\infty dt e^{-t/\alpha_s}\, \tilde{f}(t)\,. \tag{2}$$

For the fermion self energy a unique set of diagrams, $\mathcal{G}$, are proportional to $\alpha_s[n_f\alpha_s]^k$ for $k \geq 1$ (draw them). They can be used to examine the renormalon by looking at their contribution to the pole mass.

c) Examine Eq.(4.73) and (4.74) of the text. Using the result, compute the Borel transform of the graphs in $\mathcal{G}$ for the pole mass to arrive at the form in Eq.(4.78) of the text. (Subtract by hand the pole at $u = 0$ which corresponds to an ultraviolet renormalization.)

d) The pole at $u = 1/2$ is called an infrared renormalon. It makes the inverse Borel transform in Eq.(2) ambiguous, and shows that the pole mass is sensitive to infrared momenta. (The $\overline{\mathrm{MS}}$ mass is a short distance mass and does not have a renormalon ambiguity.) Compute the size of the ambiguity, $\delta m_{\rm pole}$, from the ambiguity in deforming the contour in the integral in Eq.(2) to avoid the $u = 1/2$ pole. Express your answer in terms of $\Lambda_{\rm QCD}$.

Problem 3) $b \to s\gamma$ and QCD Equations of Motion

After integrating out the top, W, Z, and Higgs, local operators are induced which mediate the flavor changing charge neutral process $b \to s\gamma$. The operators that are induced include 4-quark operators and the following ten 2-quark operators:

$$\begin{aligned}
Q_1 &= \bar{s}_L \not{D} D_\mu D^\mu b_L \,, \\
Q_2 &= \bar{s}_L D_\mu \not{D} D^\mu b_L - \frac{1}{2}\bar{s}_L D_\mu D^\mu \not{D} b_L \,, \\
Q_3 &= \bar{s}_L D_\mu \not{D} D^\mu b_L \,, \\
Q_4 &= \bar{s}_L \not{D}\not{D}\not{D} b_L \,, \\
Q_5 &= g\bar{s}_L T^a \gamma^\mu b_L D^\nu G^a_{\mu\nu} \,, \\
Q_6 &= g G^a_{\mu\nu} \bar{s}_L T^a \gamma^\mu D^\nu b_L \,,
\end{aligned}$$

$$
\begin{aligned}
Q_7 &= g\tilde{G}^a_{\mu\nu}\bar{s}_L T^a \gamma^\mu D^\nu b_L\,, \\
Q_8 &= e m_b\,\bar{s}_L\sigma^{\mu\nu} b_R F_{\mu\nu}\,, \\
Q_9 &= g m_b\,\bar{s}_L\sigma^{\mu\nu} T^a b_R G^a_{\mu\nu}\,, \\
Q_{10} &= m_b\,\bar{s}_L \not{D}\not{D} b_R
\end{aligned} \tag{3}
$$

where $D_\mu = \partial_\mu + igT^a A^a_\mu + ieQA_\mu$ and Q is the charge operator.

a) Write down the QCD equations of motion for the gluons and for the quark fields b_L, b_R, and s_L.

b) Take $m_s = 0$, but keep the b-quark mass $m_b \neq 0$. Use the equations of motion to reduce all the operators to linear combinations of Q_8 and Q_9 plus 4-quark operators. (You do not need to write the 4-quark operators out explicitly, but do write "+ 4-quark" to indicate when your manipulations have induced them.) You may find identities like $2D^\mu = \{\gamma^\mu, \not{D}\}$ useful. For Q_7 you will need the identity for three gamma matrices, Eq.(1.119) in the text.

c) Write down the Feynman rule for Q_8.