8.821 String Theory
Fall 2008

8.821 F2008 Lecture 15:

2-Point Functions in P-space ; Introduction to 3-Point Functions

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1 Introduction

In today's lecture we will continue to carry out the calculation of $\langle\mathcal{O}\mathcal{O}\rangle$ in momentum space. Along the way, we will deal with some issues that arise when we work with a real time coordinate (*i.e.* with a Lorentzian metric), which shows up for modes with $k^2 < 0$. After a short discussion of which scaling dimensions of operators (namely Δ) are allowed from the bulk point of view, we will start our discussion on 3-point functions.

The following material will be relevant:

1. d'Hoker-Freedman, hep-th/0201253, §8.
2. Klebanov-Witten, hep-th/9905104.

2 Some Real-time Issues

We continue considering the equation,

$$0 = [z^{D+1}\partial_z(z^{-D+1}\partial_z) - m^2L^2 - z^2k^2]f_k(z) \qquad (*)$$

from last lecture. If $k^2 > 0$, that is spacelike (or Euclidean), the solution would be,

$$f_k(z) = A_k z^{D/2}K_\nu(kz) + A_I z^{D/2}I_\nu(kz)$$

where $\nu = \Delta - D/2 = \sqrt{(D/2)^2 + m^2L^2}$. Since each Bessel function shows definite behavior at the horizon as,

$$K_\nu(z) \sim e^{-kz} \qquad I_\nu(z) \sim e^{kz}$$

we see that the regularity in the interior uniquely fixes f_k and hence the bulk-to-boundary propagator. Actually there is a theorem (the Graham-Lee theorem) exactly addressing this issue for gravity fields which states that if you specify a Euclidean metric on the boundary of a Euclidean AdS (which in their formalism, would be a S^D) modulo conformal rescaling, the metric for the space inside of the S^D, whose topology would be that of AdS, would be uniquely determined.[1] A similar result holds for gauge fields.

In contrary to this, in Lorentzian signature with timelike k^2, i.e. on-shell states with $\omega^2 > \vec{k}^2$, there exist many normalizable solutions with the same leading z^{Δ_-} behavior. If we define $q = \sqrt{\omega^2 - \vec{k}^2}$,

$$K_{\pm\nu}(iqz) \sim e^{\pm iqz} \quad (z \to \infty)$$

so these modes behave like oscillating modes near the poincare horizon.

A better basis of writing the solution (one that is more convenient in dealing with this) is,

$$f_k(z) = A_1 z^{D/2} Y_\nu(qz) + A_2 z^{D/2} J_\nu(qz)$$

where the two terms behave at $z \to 0$ as $z^{\Delta_-}(\phi_0) + z^{\Delta_+}$ and z^{Δ_+} respectively. These are purely "normalizable." The ambiguity of the propagator for $k^2 < 0$, is equivalent to the ambiguity in the coefficient A_2. We see that now we can have different subleading terms in the propagator. But actually this is to be expected, as this corresponds to different choices of quantum states of the boundary QFT (Balasubramanian, Kraus, Lawrence and Trivedi, hep-th/9808017.) This is actually exactly the same statement as saying that adding different homogeneous solutions to propagators in a free QFT corresponds to the correlator of different states. As usual, the Wick rotation from the Euclidean answer $K_\nu(kz)$ gives the time-ordered Feynman propagator $H_\nu^{(1)}(qz)$.

One last thing we must deal with before proceeding is to define what we mean by a 'normalizable' mode, or solution, when we say that we have many normalizable solutions for $k^2 < 0$ with a given scaling behavior. In Euclidean space, ϕ is normalizable when $S[\phi] < \infty$. This is because when we are thinking about the partition function $Z[\phi] = \sum_\phi e^{-S[\phi]}$, modes with boundary conditions which force $S[\phi] = \infty$ would not contribute.

In real time, we say ϕ is normalizable if $E[\phi] < \infty$ where,

$$E[\phi] = \int_\Sigma d^{D-1}x dz \sqrt{h} T_{\mu\nu}[\phi] n^\mu \xi^\nu = \int_{x^0=\text{constant}} d^{D-1}x dz \sqrt{h} T^0{}_0[\phi]$$

where Σ is a given spatial slice, h is the induced metric on that slice, n^μ is a normal unit vector to Σ and ξ^μ is a timelike killing vector. T_{AB} is defined as,

$$T_{AB}[\phi] = \frac{2}{\sqrt{g}} \frac{\delta}{\delta g^{AB}} S_{Bulk}[\phi]$$

3 Bulk to Boundary Propagator in Position Space

We return to considering spacelike k in this section. The normalized solution at $z = \epsilon$ is given by the condition $f_k(z = \epsilon) = 1$. This means that the δ function in x space is given at $z = \epsilon$, so the

[1] There is a subtlety here as the specified metric on the boundary sphere needs to be in some finite region around the round metric. Witten provides some conjectural understanding of this region in terms of the conformal coupling of the $\mathcal{N} = 4$ scalars to the boundary metric on page 11 of hep-th/9802150.

bulk to boundary propagator would be different from the one we have obtained in the last lecture. And indeed the position space green's function obtained,

$$f_k(z) = \frac{z^{D/2} K_\nu(kz)}{\epsilon^{D/2} K_\nu(k\epsilon)}$$

is different from K_{Witten} obtained previously. The general position space solution can be obtained as,

$$\underline{\phi}^{[\phi_0]}(x) = \int d^D k e^{-ikx} f_k(z) \phi_0(k,\epsilon)$$

so the action would be,

$$\begin{aligned} S[\underline{\phi}] &= -\frac{\eta}{2}\int d^D x \sqrt{\gamma}\underline{\phi} n \cdot \partial \underline{\phi} \\ &= -\frac{\eta}{2}\int d^D x \int d^D k_1 \int d^D k_2 e^{-(k_1+k_2)x} \phi_0(k_1,\epsilon)\phi_0(k_2,\epsilon) z^{-D} f_{k_1}(z) z \partial_z f_{k_2}(z) \\ &= -\frac{\eta}{2}\int d^D k \phi_0(k,\epsilon)\phi_0(-k,\epsilon)\mathcal{F}_\epsilon(k) \end{aligned}$$

and therefore,

$$\langle \mathcal{O}(k_1)\mathcal{O}(k_2)\rangle_c^\epsilon = -\frac{\delta}{\delta\phi_0(k_1)}\frac{\delta}{\delta\phi_0(k_2)} S = (2\pi)^D \delta^D(k_1+k_2)\eta \mathcal{F}_\epsilon(k_1)$$

We note that if you don't like functional derivatives, you may see this by calculating,

$$\langle \mathcal{O}(k_1)\mathcal{O}(k_2)\rangle_c^\epsilon = \left(\frac{\partial^2}{\partial\lambda_1\partial\lambda_2} W[\phi_0(x) = \lambda_1 e^{ik_1 x} + \lambda_2 e^{ik_2 x}]\right)|_{\lambda_1=\lambda_2=0}$$

Now $\mathcal{F}_\epsilon(k)$ could be calculated to be,

$$\mathcal{F}_\epsilon(k) = z^{-D} f_{-k}(z) z \partial_z f_k(z)|_{z=\epsilon} + (k \leftrightarrow -k) = 2\epsilon^{-D+1}\partial_z \left(\frac{z^{D/2}K_\nu(kz)}{\epsilon^{D/2}K_\nu(\epsilon z)}\right)|_{z=\epsilon}$$

The near boundary behavior of K_ν for integer ν is,

$$\begin{aligned} K_\nu(u) =& u^{-\nu}(a_0 + a_1 u^2 + a_2 u^4 + \cdots) \\ &+ u^\nu \ln u (b_0 + b_1 u^2 + b_2 u^4 + \cdots) \end{aligned}$$

where the coefficients of the series $a_i,\ b_i$ depend on ν. For non-integer ν, there would be no $(\ln u)$ multiplied in the second line.

$$\begin{aligned} \mathcal{F}_\epsilon(k) &= 2\epsilon^{-D+1}\partial_z \left(\frac{(kz)^{-\nu+D/2}(a_0+\cdots) + (kz)^{\nu+D/2}\ln kz(b_0+\cdots)}{(k\epsilon)^{-\nu+D/2}(a_0+\cdots) + (k\epsilon)^{\nu+D/2}\ln k\epsilon(b_0+\cdots)}\right)|_{z=epsilon} \\ &= 2\epsilon^{-D}\left[\left\{\frac{D}{2} - \nu(1 + c_2(\epsilon^2k^2) + c_4(\epsilon^4k^4) + \cdots)\right\} + \left\{\nu\frac{2b_0}{a_0}(\epsilon k)^{2\nu}\ln(\epsilon k)(1 + d_2(\epsilon k)^2 + \cdots)\right\}\right] \\ &= \text{(I)} + \text{(II)} \end{aligned}$$

where (I) and (II) denote the first and second group of terms of the previous line.

(I) is a Laurent series in ϵ with coefficients which are positive powers of k (i.e. analytic in k at $k = 0$.) These are contact terms, i.e. short distance 'goo' that we can subtract off. We can see this by writing,

$$\int d^D k e^{-ikx} (\epsilon k)^{2m} \epsilon^{-D} = \epsilon^{2m-D} \Box_x^m \delta^D(x)$$

for $m > 0$. The ϵ^{2m-D} factor reinforces the notion that ϵ, which is an IR cutoff in AdS is a UV cutoff for the QFT.

The interesting bit of $\mathcal{F}(k)$ which gives the $x_1 \neq x_2$ behavior is non-analytic at $k = 0$.

$$(\text{II}) = -\eta \cdot 2\nu \cdot \frac{b_0}{a_0} k^{2\nu} \ln(k\epsilon) \cdot \epsilon^{2\nu - D}(1 + \mathcal{O}(\epsilon^2)), \qquad \frac{b_0}{a_0} = \frac{(-1)^{\nu-1}}{2^{2\nu}\nu\Gamma(\nu)^2} \text{ for } \nu \in \mathbb{Z}$$

The claim is that the fourier transformation of the leading term of (II) is given by,

$$\int d^D k e^{-ikx} (\text{II}) = \frac{2\nu\Gamma(\Delta_+)}{\pi^{D/2}\Gamma(\Delta_+ - D/2)} \frac{1}{x^{2\Delta_+}} \epsilon^{2\Delta_-}$$

We note that for $\nu \notin \mathbb{Z}$ obtaining the previous result is a lot more transparent as $\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle \sim k^{2\nu}$ and hence in position space this would be $\sim \frac{1}{x^{2\Delta}}$ since there are no logs concerned.

Also, since $\epsilon^{2\nu - D} = \epsilon^{\Delta_-}$ if we let $\phi_0(k, \epsilon) = \phi_0^{\text{Ren}}(k)\epsilon^{\Delta_-}$ as before the operation,

$$\frac{\delta}{\delta\phi_0(k,\epsilon)} = \epsilon^{-\Delta_-} \frac{\delta}{\delta\phi_0^{\text{Ren}}(k)}$$

removes this factor. We also see that for $\epsilon \to 0$, the $\mathcal{O}(\epsilon^2)$ terms vanish.

One last thing we must touch upon is the difference between the prefactors obtained in this lecture and the last, namely,

$$\langle \mathcal{O}\mathcal{O}\rangle = \frac{\nu}{\Delta} \langle \mathcal{O}\mathcal{O}\rangle_{Witten}$$

We claim that the l.h.s. is the correct correlator to obtain. This is discussed in d'Hoker-Freedman, hep-th/0201253; we'll come back to this when we talk about three-point functions.

4 Allowed Scaling Behavior at the Boundary

Now let's think about which Δ are attainable in our setting. We've seen that in Euclidean space, ϕ is normalizable if $S[\phi] < \infty$. This depends on the $z \to 0$ behavior of ϕ. For

$$S_{Bulk} = \int_\epsilon d^{D+1}x \sqrt{g}(g^{AB}\partial_A\phi\partial_B\phi + m^2\phi^2)$$

with $\sqrt{g} = z^{-D-1}$, if we have some $\phi \sim z^\Delta(1 + \mathcal{O}(z^2))$ with $\Delta = \Delta_+$ or Δ_-,

$$g^{zz}(\partial_z\phi)^2 = (z\partial_z\phi)^2 \sim \Delta^2 z^{2\Delta}$$

and hence,

$$g^{AB}\partial_A\phi\partial_B\phi + m^2\phi^2 \simeq \Delta^2 z^{2\Delta} + k^2 z^{2\Delta+2} + m^2 z^2 = (\Delta^2+m^2)z^{2\Delta}(1+\mathcal{O}(z^2))$$

in the limit $z \to 0$. Since for $\Delta = \Delta_\pm$, $\Delta^2 + m^2 = -D\Delta \neq 0$,

$$S_{Bulk}[z^\Delta] \sim \int_\epsilon dz z^{-D-1}(-D\Delta)z^{2\Delta}(1+\mathcal{O}(z^2)) \propto \frac{1}{2\Delta - D}e^{2\Delta-D}$$

We emphasize that only the boundary behavior of ϕ is defined, and it is not assumed that it satisfies the equation of motion. We see that

$$S_{Bulk}[z^\Delta] < \infty \Leftrightarrow \Delta > D/2$$

This does not saturate the unitary bound.

We note that the calculation of $S_{Bulk}[z^\Delta]$ can be done by looking at the boundary term, in which case we get,

$$\int_{z=\epsilon}\sqrt{\gamma}\phi n\cdot\partial_z\phi \sim \Delta e^{2\Delta-D}$$

which is off by the same factor in the coefficient mentioned at the end of the last section, namely Δ/ν.

Now consider the alternative action by Klebanov-Witten which is,

$$S^{KW}_{Bulk} = \int_\epsilon d^{D+1}x\sqrt{g}\phi(-\Box + m^2)\phi = S_{Bulk} - \int_{\partial AdS}\sqrt{\gamma}\phi n\cdot\partial_z\phi$$

For this action we see that,

$$\begin{aligned} S^{KW}_{Bulk}[\phi \sim z^\Delta(1+\mathcal{O}(z^2))] &= \int_\epsilon dz z^{-D-1}z^\Delta(1+\mathcal{O}(z^2))[(-\Delta(\Delta-D)+m^2)z^\Delta(1+\mathcal{O}(z^2)) + k^2 z^{2\Delta+2}] \\ &\sim \int_\epsilon dz z^{-D-1+2\Delta+2} \sim \epsilon^{2\Delta-D+2} < \infty \end{aligned}$$

is equivalent to

$$\Delta \geq \frac{D-2}{2}$$

which is exactly the unitary bound. We see that in this case both $\Delta_\pm$ give normalizable modes for $\nu \leq 1$. Note that it is actually Δ_- that gives the lowest value in the unitray bound, that is when,

$$\Delta_- = \left(\frac{D}{2} - \sqrt{\left(\frac{D}{2}\right)^2 + m^2L^2}\right)_{m^2 = 1-\frac{D^2}{4}} = \frac{D-2}{2}$$

The coefficient of z^{Δ_+} would be the source in this case.

What we have here is a different boundary CFT with the same bulk action, which we have obtained by adding a boundary term to the action.

5 3-Point Functions

Next we will talk about connected correlation functions of three or more operators. Unlike two-point functions, such observables are sensitive to the details of the bulk interactions, and we need to make a choice. We will consider the three point functions of the scalar fields coming from the action

$$S = \frac{1}{2}\int d^{D+1}x\sqrt{g}(\sum_{i=1}^{3}(\partial\phi_i)^2 + m_i^2\phi_i^2 + b\phi_1\phi_2\phi_3)$$

The arguments presented could be easily extended to n-point functions with $n > 3$.

The equations of motion we get is,

$$(\Box - m_i^2)\phi_i(z,x) = b\phi_j\phi_k\epsilon^{ijk}$$

We solve this perturbatively to obtain,

$$\begin{aligned}\underline{\phi}^i(z,x) &= \int d^Dx_1 K^{\Delta_i}(z,x;x_1)\phi_0^i(x_1)\\ &+b\epsilon^{ijk}\int d^Dx'dz'\sqrt{g}G^{\Delta_i}(z,x;z',x')\int d^Dx_1\int d^Dx_2 K^{\Delta_j}(z',x';x_1)K^{\Delta_k}(z',x';x_2)\phi_0^j(x_1)\phi_0^k(x_1)\\ &+\mathcal{O}(b^2\phi_0^3)\end{aligned}$$

where $G^{\Delta_i}(z,x;z'x')$ is the bulk-to-bulk propagator given by the condition,

$$(\Box - m_i^2)G^{\Delta_i}(z,x;z',x') = \frac{1}{\sqrt{g}}\delta(z-z')\delta^D(x-x')$$

so that

$$(\Box - m_i^2)\int\sqrt{g}GJ = J$$

for a source J.

The first and second terms would be obtained from the Witten diagrams, figure 1 and 2. A typical higher-order diagram would look something like figure 3.

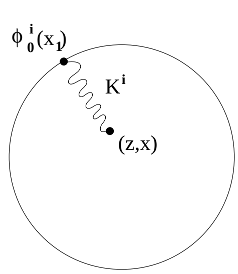

Figure 1: Witten diagram 1

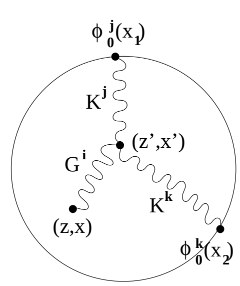

Figure 2: Witten diagram 2

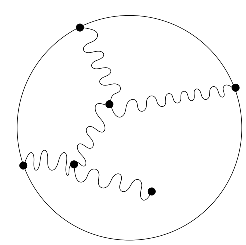

Figure 3: Witten diagram 3