Chapter 2: Deriving AdS/CFT

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Lecture 11

Important equations for this lecture from the previous ones:

1. The Virasoro constraints in the light-cone gauge, equation (32) and (33) in lecture 10:

$$\begin{aligned} 2v^+\partial_\tau X^- &= (\partial_\tau X^i)^2 + (\partial_\sigma X^i)^2 \quad , && (1)\\ v^+\partial_\sigma X^- &= \partial_\tau X^i \partial_\sigma X^i \quad , && (2) \end{aligned}$$

2.1.2: LIGHT-CONE QUANTIZATION (cont.)

Since the general classical solution strings (for closed strings X_R^μ and X_L^μ are independent periodic functions with period 2π, while for open strings $X_R^\mu = X_L^\mu$):

$$X^\mu(\sigma,\tau) = x^\mu + v^\mu\tau + X_R^\mu(\tau-\sigma) + X_L^\mu(\tau+\sigma) \quad . \tag{3}$$

Then it can be rewritten in terms of Fourier expansion. For closed string:

$$X^\mu(\sigma,\tau) = X^\mu + v^\mu\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^\mu e^{-in(\tau+\sigma)} + \tilde{\alpha}_n^\mu e^{-in(\tau-\sigma)}\right) \quad . \tag{4}$$

It's similar for open string, but from $X_R^\mu = X_L^\mu$ one arrives at $\alpha_n^\mu = \tilde{\alpha}_n^\mu$:

$$X^\mu(\sigma,\tau) = X^\mu + v^\mu\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos n\sigma \quad . \tag{5}$$

(The physical meaning of α_n^μ, $\tilde{\alpha}_n^\mu$ will be studied in Pset 3, so please do the homework.)

The center off mass motion can be found by averaging the position of the strings at a given timeslice ($l = 2\pi$ for closed strings and $l = \pi$ for open strings):

$$\frac{1}{l}\int_0^l d\sigma X^\mu(\sigma,\tau) = x^\mu + v^\mu\tau \quad . \tag{6}$$

The constant v^μ is identified with the strings' center of mass velocity.

The classical coefficients α_n^μ and $\tilde{\alpha}_n^\mu$ keep track of the oscillation modes of the strings. While the closed strings have independent left-moving and right-moving contributions, open string can be described as standing waves so that left-moving and right-moving are the same.

In the light-cone gauge, $X^+ = v^+\tau$ and X^- can be obtained by writing X^- in Fourier expansion, plugging equations (4) and (5) into equations (1) and (2) then equating the coefficients of different Fourier modes. The 0th (non-oscillating) mode gives the relations between the strings' center of mass velocity and the strings' oscillation modes. For closed string, from equation (1):

$$2v^+v^- = v_i^2 + \alpha' \sum_{n\neq 0} \left(\alpha^i_{-n}\alpha^i_n + \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n \right) \quad . \tag{7}$$

For open string, from equation (1):

$$2v^+v^- = v_i^2 + 2\alpha' \sum_{n\neq 0} \alpha^i_{-n}\alpha^i_n \quad . \tag{8}$$

For closed string, from equation (2):

$$\sum_{n\neq 0} \alpha^i_{-n}\alpha^i_n = \sum_{n\neq 0} \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n \tag{9}$$

This is known as the level matching condition for closed strings. Equation (2) satisfies trivially for open strings.

Poincare global symmetries of the action corresponds to the conserved currents on the worldsheet. For the moment, let's look at translation and apply the standard Noether procedure:

$$\Pi^\mu_a = \partial_{\partial_a X^\mu} \mathcal{L}_P \Big|_{\gamma^{ab}=\eta^{ab}} = \frac{1}{2\pi\alpha'} \partial_a X^\mu \quad . \tag{10}$$

Also note that $\partial^a \Pi^\mu_a = 0$, from the equation of motion for X^μ.
Π^μ_τ is the momentum density along the string, and the corresponded conserved current is the string momentum in spacetime:

$$p^\mu = \int_0^l d\sigma \Pi^\mu_\tau = \frac{1}{2\pi\alpha'} \int_0^l d\sigma \partial_\tau X^\mu = \frac{l}{2\pi} \frac{v^\mu}{\alpha'} \quad . \tag{11}$$

The mass-squared is related to the spacetime momentum of the strings (the mass shell condition):

$$M^2 = -p^\mu p_\mu = 2p^+p^- - p_i^2 \quad . \tag{12}$$

For closed string, from equation (7):

$$M^2 = \frac{1}{\alpha'} \sum_{n\neq 0} \left(\alpha^i_{-n}\alpha^i_n + \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n \right) = \frac{2}{\alpha'} \sum_{n\neq 0} \alpha^i_{-n}\alpha^i_n \quad . \tag{13}$$

For open string, from equation (8):

$$M^2 = \frac{1}{2\alpha'} \sum_{n\neq 0} \alpha^i_{-n}\alpha^i_n \quad . \tag{14}$$

One can concluded that the mass of a string can be determined from its oscillations.

After understanding the strings at classical level, the next step is to quantization − quantize independent degrees of freedom $X^i(\sigma,\tau)$ (with canonical momentum density Π^i) in the action:

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \partial^a X^i \partial_a X^i \quad , \quad \Pi^i = \frac{1}{2\pi\alpha'} \partial_\tau X^i \quad . \tag{15}$$

Nominate X^i to be a quantum operator, with the canonical commutation relation at a given timeslice:

$$[X^i(\sigma,\tau), X^j(\sigma',\tau)] = [\Pi^i(\sigma,\tau), \Pi^j(\sigma',\tau)] = 0 \quad , \quad [X^i(\sigma,\tau), \Pi^j(\sigma',\tau)] = i\delta^{ij}\delta(\sigma-\sigma') \quad . \tag{16}$$

The results are 0th mode x^i, p^i and oscillation modes α^i_n, $\tilde{\alpha}^i_n$ all become operators:

$$[x^i, p^j] = i\delta^{ij} \quad , \quad [\alpha^i_n, \alpha^j_m] = [\tilde{\alpha}^i_n, \tilde{\alpha}^j_m] = n\delta^{ij}\delta_{n+m,0} \quad . \tag{17}$$

Note that α^i_n, $\tilde{\alpha}^i_n$ can be related to the creation and annihilation operators (similar to canonical quantization of QFT, creation operators are associated with positive branch while annihilation operators are associated with negative branch so that the Hamiltonian is bounded from below):

$$\frac{1}{\sqrt{n}}\alpha^i_n = a^i_n \quad , \quad \frac{1}{\sqrt{n}}\alpha^i_{-n} = (a^i_{-n})^\dagger \quad , \quad \frac{1}{\sqrt{n}}\tilde{\alpha}^i_n = \tilde{a}^i_n \quad , \quad \frac{1}{\sqrt{n}}\tilde{\alpha}^i_{-n} = (\tilde{a}^i_{-n})^\dagger \quad , \quad n > 0 \quad . \tag{18}$$

Therefore, the oscillator vacuum state (labelled by string spacetime momentum p^μ) satisfies:

$$\alpha^i_n|0, p^\mu\rangle = \tilde{\alpha}^i_n|0, p^\mu\rangle = 0 \quad , \quad n > 0 \quad . \tag{19}$$

Excited states can be built from creation operators (α^i_{-n}, $\tilde{\alpha}^i_{-n}$ with $n > 0$):

$$\alpha^{i_1}_{-n_1}\alpha^{i_2}_{-n_2}\alpha^{i_3}_{-n_3}...\tilde{\alpha}^{j_1}_{-m_1}\tilde{\alpha}^{j_2}_{-m_2}\tilde{\alpha}^{j_3}_{-m_3}...|0, p^\mu\rangle \quad . \tag{20}$$

For closed string, define the oscillation number operator (no summation in i index, and the order of operators is very important):

$$N^i_n = \frac{1}{n}\alpha^i_{-n}\alpha^i_n \quad , \quad \tilde{N}^i_n = \frac{1}{n}\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n \quad . \tag{21}$$

Hence the level matching condition can be rewritten as:

$$\sum_{n\neq 0} nN^i_n = \sum_{n\neq 0} n\tilde{N}^i_n \tag{22}$$

For open string, only one set of oscillation is needed (let's pick α^i_{-n}).

The quantum version of the mass shell condition for closed strings, from equation (13):

$$M^2 = \frac{2}{\alpha'}\sum_{i=2}^{D-1}\sum_{n\neq 0} n(N^i_n + \tilde{N}^i_n) + a_0 = \frac{4}{\alpha'}\sum_{i=2}^{D-1}\sum_{n\neq 0} nN^i_n + a_c \quad . \tag{23}$$

The constant a_c is the zero-point energy for closed string, comes from rearranging the operator to normal ordered (for α^i_n and $\tilde{\alpha}^i_n$, negative n to the left and positive n to the right):

$$a_c = \frac{2(D-2)}{\alpha'}\sum_{n=1}^{\infty} n = \frac{2(D-2)}{\alpha'}\zeta(-1) = -\frac{D-2}{6\alpha'} \quad . \tag{24}$$

For open string, from equation (14):

$$M^2 = \frac{1}{\alpha'}\sum_{i=2}^{D-1}\sum_{n\neq 0} nN^i_n + a_o \quad . \tag{25}$$

The constant a_o is found to be:

$$a_o = \frac{(D-2)}{2\alpha'}\sum_{n=1}^{\infty} n = \frac{(D-2)}{2\alpha'}\zeta(-1) = -\frac{D-2}{24\alpha'} \quad . \tag{26}$$

The trick used here to find the zero-point energy is the ζ-function regularization:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad , \quad \zeta(-1) = -\frac{1}{12} \quad . \tag{27}$$

The sum is only convergant for $s > 1$, but by using analytical continuation (around the pole at $s = 1$) one can get a well-define finite result for $s = -1$. In path integral approach, the zero-point energy can also be found (on the general worldsheet) after mapping an asymptotic state to a point insertion, and the consistency of the CFT requires $-\frac{1}{12}$ (in that language, it's the central charge). Indeed, the divergent in equation (26) is just the artifact of a perturbative description (looking only at an asymptotic part instead of the worldsheet as a whole), and the analytical continuation trick (which is arise a lot in quantum field theories and statistical field theories) is used to read-off nonperturbative description from perturbative results.
For example of a fake divergent, consider a Taylor (perturbative) expansion of the following function:

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad . \tag{28}$$

$f(x)$ is well-define everywhere except for $x = 1$ (pole). However, the Taylor expansion description gives a summation series with divergent (ill-defined) whenever $|x| > 1$, which is not the true property of $f(x)$.

8.821 / 8.871 String Theory and Holographic Duality
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