Free fermions, e^- with $q = -e$

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

Let $\psi = u(\vec{p})\, e^{-i p_\mu x^\mu}$

$$(\gamma^\mu p_\mu - m)\, u(\vec{p}) = 0$$

To find energy eigenstates

$$H u = (\alpha\cdot p + \beta m)u = E u$$

If $\vec{p} = 0$, in γ^0 diagonal representation:

$$\begin{pmatrix} mI & 0 \\ 0 & -mI \end{pmatrix} u = E u$$

$\therefore E = m, m, -m, -m$

for $u = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

for anti-fermions $q(e^+) = +e$

$E < 0$

Replace $p_\mu \to p_\mu$, $\psi = v(\vec{p})e$

$$(\gamma^\mu p_\mu + m)\, v(\vec{p}) = 0$$

$$H u = (\alpha\cdot p - \beta m)v = E v$$

$m \to -m$

$E = -m, -m, m, m$

If $\vec{p} \neq 0$,

$$H u = \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = E \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

or
$$\vec{\sigma}\cdot\vec{p}\, u_B = (E - m)\, u_A$$
$$\vec{\sigma}\cdot\vec{p}\, u_A = (E + m)\, u_B$$

Let $u_A^{(i)} = \chi^{(i)}$, $\chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\chi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

fermions, e^- solutions $\qquad\qquad$ e^+ solutions

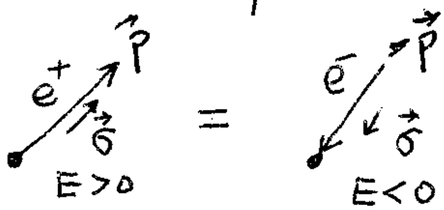
$E>0$

$$u^1 = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ p_z/(E+m) \\ \frac{p_x+ip_y}{E+m}\end{pmatrix}$$

$$u^2 = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix}$$

$E<0$

$$u^3 = \sqrt{|E|+m}\begin{pmatrix} \frac{-\vec{\sigma}\cdot\vec{p}}{|E|+m}\begin{pmatrix}1\\0\end{pmatrix} \\ 1 \\ 0\end{pmatrix} \qquad v^2(\vec{p}) = u^3(-\vec{p}) = \sqrt{E+m}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{(E+m)}\begin{pmatrix}1\\0\end{pmatrix} \\ 1 \\ 0\end{pmatrix}$$



$$u^4 = \sqrt{|E|+m}\begin{pmatrix} \frac{-\vec{\sigma}\cdot\vec{p}}{|E|+m}\begin{pmatrix}0\\1\end{pmatrix} \\ 0 \\ 1\end{pmatrix} \qquad v^1(\vec{p}) = u^4(-\vec{p}) = \sqrt{E+m}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\begin{pmatrix}0\\1\end{pmatrix} \\ 0 \\ 1\end{pmatrix}$$

$$\vec{\sigma}\cdot\vec{p} = (-\vec{\sigma})(-\vec{p})$$

$$\psi = u(\vec{p})\, e^{-ip_\mu x^\mu} \qquad\qquad \psi = v(\vec{p})\, e^{+ip_\mu x^\mu}$$

- Normalization: use $(\vec{\sigma}\cdot\vec{p})^2 = |\vec{p}|^2 I$

$$u^{(r)\dagger} u^{(s)} = 2E\,\delta^{rs} \qquad\qquad v^{(r)\dagger} v^{(s)} = 2E\,\delta^{rs}$$

$$\bar{u}^{(r)} u^{(s)} = 2m\,\delta^{rs} \qquad\qquad \bar{v}^{(r)} v^{(s)} = -2m\,\delta^{rs}$$

- Completeness:

$$\sum_{s=1,2} u^{(s)}\bar{u}^{(s)} = \not{p}+m \qquad\qquad \sum_{s=1,2} v^{(s)}\bar{v}^{(s)} = \not{p}-m$$

- Projection

$$\Lambda_\pm = \frac{\pm\not{p}+m}{2m} \qquad \Lambda_+ + \Lambda_- = I \qquad \Lambda_+^2 = \Lambda_+ \qquad \Lambda_-^2 = \Lambda_-$$

$$u^1(p) = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \dfrac{\sigma\cdot p}{E+m}\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ p_z/E+m \\ \dfrac{p_x+ip_y}{E+m}\end{pmatrix}$$

$$u^2(p) = \sqrt{\quad}\begin{pmatrix} 0 \\ 1 \\ \dfrac{\sigma\cdot p}{E+m}\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} = \sqrt{\quad}\begin{pmatrix} 0 \\ 1 \\ (p_x-ip_y)/E+m \\ -p_z\ /E+m\end{pmatrix}$$

$$u^3(p) = \sqrt{\quad}\begin{pmatrix} \dfrac{-\sigma\cdot p}{|E|+m}\begin{pmatrix}1\\0\end{pmatrix} \\ 1 \\ 0\end{pmatrix} = \sqrt{|E|+m}\begin{pmatrix} -(+p_x+ip_y)/(|E|+m) \\ -p_z/(|E|+m) \\ 1 \\ 0\end{pmatrix} \qquad v^2(p) = u^3(-p)$$

$$u^4(p) = \sqrt{\quad}\begin{pmatrix} \dfrac{-\sigma\cdot p}{|E|+m}\begin{pmatrix}0\\1\end{pmatrix} \\ 0 \\ 1\end{pmatrix} = \sqrt{\quad}\begin{pmatrix} -(p_x-ip_y)/(|E|+m) \\ +p_z/(|E|+m) \\ 0 \\ 1\end{pmatrix} \qquad v^1(p) = u^4(-p)$$

$$\beta = \gamma^0 = \begin{pmatrix} 1 & \\ 0 & -1 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \quad \text{or} \quad \beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \gamma^0; \quad \alpha = \begin{pmatrix} \cancel{\sigma} & 0 \\ 0 & \sigma \end{pmatrix} \quad \gamma_5 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\gamma_5 = \begin{bmatrix} 0 & 1 \\ +1 & 0 \end{bmatrix}$$

$$-i\,\sigma_2 = -i \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} & -1 \\ +1 & \end{bmatrix} \begin{bmatrix} c^* \\ d^* \end{bmatrix} = \begin{bmatrix} -d^* \\ c^* \end{bmatrix}$$

$$\gamma^\mu = [\beta, \beta\vec{\alpha}]$$

$$\gamma^2 = \beta\alpha^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}$$

$$i\gamma^2 = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix} = \begin{bmatrix} 0 & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & 0 & \end{bmatrix}$$

$$\psi_c^{(1)} = i\gamma^2 [u^{(1)}(p) e^{-i p_\mu x^\mu}]^* = \begin{bmatrix} 0 & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & 0 & \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \frac{\sigma \cdot p}{E+m} \\ 0 \end{bmatrix}^* e^{+ip \cdot x} = \begin{bmatrix} 0 \\ -\frac{\sigma \cdot p^*}{E+m} \\ 0 \\ 1 \end{bmatrix} e^{ip \cdot x} = u^4(p)\, e^{ipx}$$

$$\sigma \cdot p^* = \begin{bmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{bmatrix}^* = \begin{bmatrix} p_z & p_x + ip_y \\ p_x - ip_y & -p_z \end{bmatrix} = -\begin{bmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{bmatrix}$$

$$\psi_c^{(1)} = i\gamma^2 [u^1(p) e^{-ip_\mu x^\mu}]^* = \begin{bmatrix} & 0 & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ p_z/E+m \\ \frac{p_x + ip_x}{E+m} \end{bmatrix}^* e^{+ip_\mu x^\mu} = \begin{bmatrix} p_x - ip_x \\ -p_z/E+m \\ 0 \\ 1 \end{bmatrix} e^{+ip_\mu x^\mu}$$

$$= u^4(-p)\, e^{ipx} = v^1(p)\, e^{ipx}$$

$v^{(1)}(\vec{p})\, e^{ipx}$ $\qquad \xrightarrow[\Rightarrow \vec{\sigma}]{\vec{p}}$ e^+ $\qquad \frac{\vec{\sigma}\cdot\vec{p}}{E} \to +1$ as $\frac{m}{E} \to 0$

$$\equiv \left(\psi^{(1)}\right)^c = i\gamma^2 \left[u^{(1)}(\vec{p})\, e^{-ip\cdot x}\right]^* \qquad \gamma^2 = \begin{pmatrix} 0 & & & -i \\ & & i & \\ & i & & \\ -i & & & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix} u^{(1)}(\vec{p})\, e^{+ip\cdot x} = \sqrt{E+m} \begin{pmatrix} \frac{p_x + ip_x}{E+m} \\ -p_z/(E+m) \\ 0 \\ 1 \end{pmatrix} e^{ip\cdot x}$$

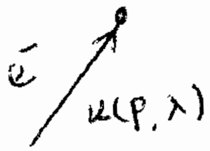
$$= u^{(4)}(-\vec{p})\, e^{ip\cdot x}$$

$\xleftarrow[\Leftarrow \vec{\sigma}]{-\vec{p}}$ e^-

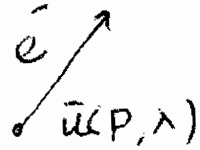
$$\frac{\vec{\sigma}\cdot\vec{p}}{E} = +1$$

$$u^{(1)}(\vec{p}) \quad u^{(2)}(\vec{p}) \quad u^{(3)}(-\vec{p}) \quad u^{(4)}(-\vec{p}) \quad e^{-ip\cdot x}$$

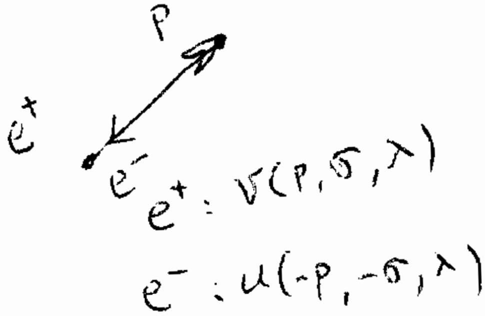
$$+ v^{(2)}(\vec{p}) \quad v^{(1)}(\vec{p}) \quad e^{+ip\cdot x}$$

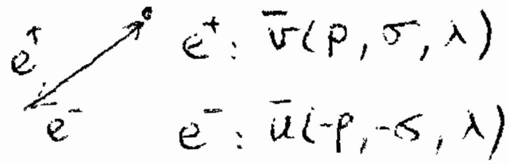


annihilation of e^-



creation of e^-





$\psi = u(\vec{p})\, e^{-ip_\mu x^\mu}$ $\quad (\gamma_\mu p^\mu - m)\psi = 0$ free fermion

Gauge invariance requires each fermion be accompanied by a field, e.g. A^μ, so $p^\mu \to p^\mu + eA^\mu$ $(q = -e)$

Dirac Eq $(\gamma^\mu p_\mu - m)\psi = -e\gamma^\mu A_\mu \psi = \gamma^0 V \psi$

(γ^0 since we multiplied γ^0 to $H\psi = (\vec{\alpha}\cdot\vec{p} + \beta m)\psi$ to get Dirac Eq)