1 Lecture 6. Vortices, superfluidity. Trapped gases. BEC at finite temperature.

To treat hydrodynamics and BEC in spatially varying background, need a more general approach that does not assume condensation into a particular plane wave state. One can formulate theory of BEC so that the wavefunction of the condensate is an arbitrary function in space-time that is determined self-consistently from a classical nonlinear field equation, known as the Gross-Pitaevskii equation.

1.1 Gross-Pitaevskii equation

Let us start again from the Hamiltonian of weakly interacting Bose gas with short-range interaction,

$$\mathcal{H} = \int \left(\hat{\varphi}^+(x) \left(-\frac{\hbar^2}{2m} \nabla_x^2 + U(x) \right) \hat{\varphi}(x) + \frac{\lambda}{2} \hat{\varphi}^+(x) \hat{\varphi}^+(x) \hat{\varphi}(x) \hat{\varphi}(x) \right) dx \tag{1}$$

To formulate the condensate dynamics, start with the Heisenberg evolution, $i\hbar\partial_t \hat{\varphi} = [\hat{\varphi}, \mathcal{H}] = \left(-\frac{\hbar^2}{2m} \nabla^2 + U(x) \right) \hat{\varphi} + \lambda \hat{\varphi}^+ \hat{\varphi}^2$, and replace $\hat{\varphi}$, $\hat{\varphi}^+$ by classical variables φ, $\bar{\varphi}$, which gives a classical field dynamics problem,

$$i\hbar\partial_t \varphi = \left(-\frac{\hbar^2}{2m} \nabla^2 + U(x) \right) \varphi + \lambda \bar{\varphi} \varphi^2 \tag{2}$$

$$-i\hbar\partial_t \bar{\varphi} = \left(-\frac{\hbar^2}{2m} \nabla^2 + U(x) \right) \bar{\varphi} + \lambda \bar{\varphi}^2 \varphi \tag{3}$$

called the Gross-Pitaevskii equations.

It is instructive to write eqs. separately for the modulus and phase $\varphi = |\varphi| e^{i\theta}$.

$$\partial_t n + \nabla \mathbf{j} = 0 \,, \quad n = |\varphi|^2 \,, \quad \mathbf{j} = \frac{\hbar}{2mi} \left(\bar{\varphi} \nabla \varphi - \varphi \nabla \bar{\varphi} \right) = \frac{\hbar}{m} |\varphi|^2 \nabla \theta \tag{4}$$

and

$$\hbar \partial_t \theta = - \left(U(\mathbf{x}) + \lambda n + \frac{\hbar^2}{2m} |\nabla \varphi|^2 \right) \tag{5}$$

Comparing the above expressions for the density and current, obtain the superflow velocity

$$\mathbf{v} = \frac{\hbar}{m} \nabla \theta \tag{6}$$

The flow is <u>irrotational</u>, $\nabla \times \mathbf{v} = 0$ (this is true away from singularities in θ), and

$$m\partial_t \mathbf{v} = -\nabla \left(\tilde{\mu} + m\mathbf{v}^2/2 \right) \,, \quad \tilde{\mu} = U(\mathbf{x}) + \lambda n + \frac{\hbar^2}{2m\sqrt{n}} \nabla^2 \sqrt{n} \tag{7}$$

(compare to the Euler equation).

1.2 Superfluidity. Vortices.

Let us consider the circulation of velocity in a superflow. It follows from the relation between the velocity and the phase, Eq. (6), that the circulation around any contour C obeys

$$\Gamma = \oint_C \mathbf{v} \cdot d\mathbf{r} = \frac{\hbar}{m} 2\pi l \tag{8}$$

with some integer l. We see that

- The circulation is quantized in multiples of h/m;
- The flows in multi-connected geometries, such as a ring-shape tube, are discrete;
- Quantum leaps are required to change a flow.

The fact that a superflow, due to the dicreteness of circulation, cannot be dissipated gradually, but only in dicrete steps, is the origin of superfluidity. The only way to eliminate a superflow is to produce excitations with discrete vorticity and then remove them (along with the vorticity) from the system.

Also mention the Landau criterion for superfluidity: The quasiparticle energy $\epsilon'(\mathbf{k}) = \epsilon(\mathbf{k}) - \mathbf{v} \cdot \mathbf{k}$, Doppler-shifted due to the flow, should be positive, to prevent massive production of quasiparticles. This criterion defines a critical velocity

$$v_c = \min_{\mathbf{k}} \epsilon(\mathbf{k})/|\mathbf{k}| \tag{9}$$

above which the flow without excitations is unstable, thus showing that superfluidity can be sustained only at the flow velocity below certain critical value.[1] The Landau criterion points at a neccessary condition for superfluidity. However, since it does not take into account vortices which can be generated in the flow even at velocities below v_c, one cannot use it to predict the actual value of critical velocity. The observed critical velocities are system-dependent (non-universal) and are typically several orders of magntude below v_c estimated from quasiparticle dispersion.

Now let us consider velocity field in a vortex with singularity on the z axis.The flow lines are concentric circles parallel to the (x, y) plane. Constant circulation requires that the velocity falls inversely with the distance ρ from the z axis:

$$\mathbf{v}(\mathbf{r}) = \frac{\hbar l}{m} \frac{\hat{\theta}}{2\pi\rho} \tag{10}$$

where $\hat{\theta}$ is the asimuthal unit vector of the cylindrical coordinate system.

Density variation is important only close to the vortex core, at distances where the kinetic energy per particle exceeds the interaction energy, $\frac{1}{2}mv^2 > \lambda n$, which gives

$$\rho \leq \xi = \frac{\hbar}{\sqrt{\lambda m}} \tag{11}$$

[1] For weakly interacting Bose gas, the critical velocity is equal to Bogoliubov sound velocity.

The so-called healing length ξ determines the size of the vortes core where density is depleted below its bulk value.

This qualitative picture can be confirmed by an analysis based on the Gross-Pitaevskii equation. One can look for a solution of the equation that describes a vortes. For a single-quantized vortex with unit circulation, $l = 1$, we expect $\varphi(\rho \to \infty) = \sqrt{n}e^{i\theta}$, and $\varphi(\rho \ll \xi) \propto \rho e^{i\theta}$. Thus one can take a trial function of the form

$$\varphi(\mathbf{r}) = \sqrt{n}\frac{\rho}{\sqrt{\rho^2 + r_0^2}}e^{i\theta} \tag{12}$$

and minimize the energy $\mathcal{H}(\varphi, \bar{\varphi})$, which gives $r_0 = \sqrt{2}\xi$, in agreement with the estimate above.

Let us consider The energy of the vortex, that can be estimated as the kinetic energy of the flow, is positive. Thus vortices do not appear unless the system is driven, or stirred. Let us consider a cylindrical jar rotating with angular velocity Ω, and find the critical rotation velocity at which vortices start to appear.

For a vortex located on the symmetry axis of the jar of radius b and height L, the kinetic energy of the flow is

$$E_v = \int_0^L \int_0^b n(\rho)\frac{1}{2}mv^2(\rho)2\pi\rho d\rho dz = \pi n\frac{\hbar^2}{m}\int_0^L \int_\xi^b \frac{d\rho}{\rho}dz = \pi n\frac{\hbar^2}{m}\ln\left(\frac{b}{\xi}\right)L \tag{13}$$

(According to Eq.(12) the density can be approximated by a constant for $\rho > \xi$, while the depletion of density in the vortex core, at $\rho \leq \xi$, cuts the log divergence at small ρ.) the contribution to the energy due to the core, which can be estimated using the trial function (12), turns out to be approximately $\pi n\frac{\hbar^2}{m}\ln 1.464L$, which is smaller than our kinetic energy estimate (13).

The energy of the vortex in a jar rotating with velocity Ω is $E_v(\Omega) = E_v - \Omega M$, where M is the angular momentum of the vortex,

$$M_v = \int mn\,\mathbf{r}\times\mathbf{v}\,d^3r = \int_0^L \int_0^b n(\rho)m\rho v(\rho)2\pi\rho d\rho dz = \pi b^2\hbar nL \tag{14}$$

The vortex becomes energetically favorable at

$$\Omega > \Omega_c^{(1)} = E_v/M_v = \frac{\hbar}{mb^2}\ln\left(\frac{b}{\xi}\right) \tag{15}$$

Note the inverse square dependence of $\Omega_c^{(1)}$ on the radius b, which means that it is easier to produce vortices in a larger jar.

If the rotation velocity is larger than $\Omega_c^{(1)}$ and keeps increasing, one can reach the next critical value $\Omega_c^{(2)}$ at which the second vortex appears, and then, at some higher value $\Omega_c^{(3)}$, the third vortes enters the jar, and so on.

At high rotation speed, when there are many vortices, one can estimate their number N from the requirement that the total circulation due to the vortices, $\frac{\hbar}{m}N$, matches the circulation of a uniformly rotating fluid, $\oint \mathbf{v}\cdot d\mathbf{r} = \Omega\pi b^2$, which gives a linear dependence

$$N(\Omega) \simeq \frac{m}{\hbar}\pi b^2\Omega \tag{16}$$

Of course, since N is integer, in reality the number of vortices increases discretely, in steps, on average following the proportionality relation (16).

1.3 Trapped gases.

Bose condensation of confined gasses differs somewhat from BEC in a uniform system that we discussed so far. Most importantly, the BEC transition is accompanied by an abrupt change of density distribution. This is due to the fact that the lowest energy quantum state in which atoms condense is peaked at the trap center and has spatial extent much less than the size of thermal cloud at temperatures slightly above T_{BEC}.

In the experiments on BEC in trapped gasses, atoms are confined by a magnetic trap, which can be described by a harmonic potential $U(r) = \frac{1}{2}m\omega^2\mathbf{r}^2$. The ground state is a gaussian wavepacket $\psi_0(r) \propto \exp(-r^2/2l_\omega^2)$ of width $l_\omega = \sqrt{\hbar/m\omega}$.

In an ideal Bose gas, in the absence of interactions, in the BEC state at $T = 0$ all the atoms populate the state ψ_0. One notes that the density of this state, at the peak, $n \simeq N/l_\omega^3$, can be extremely high when the number of atoms is large. In the presence of interactions, on can easily reach the limit when the interaction energy per particle is much larger than the level spacing in the trap, $\lambda n \gg \hbar\omega$. For that, the number of atoms should exceed $N_c = \hbar\omega l_\omega^3/\lambda$. However, for realistic parameters the value N_c can be $10^3 - 10^4$, which is much less than the typical atom numbers N in the experiments.

To understand the BEC state at a larger number of atoms, one can start with the Gross-Pitaevskii energy functional and look for a non-uniform state $\varphi(x)$ that minimizes the energy,

$$E(\varphi) = \int \left(\frac{\hbar^2}{2m} |\nabla\varphi(x)|^2 + ((U(x) - \mu)|\varphi(x)|^2 + \frac{1}{2}\lambda|\varphi(x)|^4 \right) dx \qquad (17)$$

with the particle number $N = \int |\varphi|^2 dx$ being fixed by a chemical potential μ.

Let us argue that one can discard the gradient term in the energy functional, since the expected condensate size is much larger than the oscillator ground state width l_ω, for which the kinetic and potential energy terms are approximately equal. Indeed for a condensate of size R, the kinetic, potential, and interaction energies can be estimated as $\frac{\hbar^2}{2m}N/R^2$, $\frac{m\omega^2}{2}NR^2$, and $\lambda N^2/R^3$, respectively (since the gas density $n \simeq N/R^3$). Comparing the potential and interaction energy, obtain the condensate size $R \simeq (\lambda N/m\omega^2)^{1/5}$. At large $N \gg N_c$, the value R is much larger than l_ω that satisfies $\frac{\hbar^2}{2m}l_\omega^{-2} = \frac{m\omega^2}{2}l_\omega^2$. Hence the kinetic energy is small, $\frac{\hbar^2}{2m}R^{-2} \ll \frac{m\omega^2}{2}R^2$, which justifies ignoring it in the estimate of R.

Without the kinetic energy term, the functional be written in terms of the density $n = |\varphi|^2$ only, $E(n) = \int \left(((U(x) - \mu)n + \frac{1}{2}\lambda n^2 \right) dx$. After taking the minimum, one has

$$\mu = U(x) + \lambda n \qquad (18)$$

One can arrive at this result by making a local density approximation, i.e. treating each small part of the BEC cloud as a uniform system. For the latter, as we already know, the relation between the chemical potential and density is given by Eq.(18). In addition, the chemical potential in equilibrium must be constant throughout the system. This condition fixes the density distribution $n(x)$ so that the term λn in Eq.(18) compensates the potential $U(x)$ variation in space, which gives

$$n(x) = \begin{cases} (\mu - U(x))/\lambda\,, & U < \mu \\ 0\,, & U > \mu \end{cases} \tag{19}$$

We note that the argument used to find the density distribution is similar to that of the Thomas-Fermi theory of many-electron atoms, based on a local density approximation for electrons moving in an effective potential that is determined selfconsistently from an elecrostatic problem. The local density approximation in trapped BEC, along with Eqs.(18),(19), is often referred to as Thomas-Fermi approximation.

For a central-symmetric harmonic trap potential, by relating BEC radius parameter R with the chemical potential via $\mu = \frac{m\omega^2}{2}R^2$, from Eq.(19) we obtain density distribution of the form

$$n(r) = \frac{m\omega^2}{2\lambda}\left(R^2 - r^2\right)\,, \quad r < R \tag{20}$$

The particle number can be related with R (and thus with μ) as follows:

$$N = \int_0^R \frac{m\omega^2}{2\lambda}\left(R^2 - r^2\right) 4\pi r^2 dr 2\pi \left(\frac{1}{3} - \frac{1}{5}\right)\frac{m\omega^2}{\lambda}R^5 = \frac{4\pi}{15\lambda}m\omega^2 R^5 \tag{21}$$

which gives a relation between the BEC radius and the number of particles,

$$R = \left(\frac{15\lambda}{4\pi m\omega^2}N\right)^{1/5} \tag{22}$$

For large $N \gg N_c$, the radius R is much larger than the BEC healing length $\xi = \hbar/\sqrt{\lambda n m}$ estimated for typical density $n = N/\frac{4\pi}{3}R^3$, which determines the scale of spatial nonlocality in BEC correlations. This means that the Thomas-Fermi approximation is indeed a local density approximation.

The above discussion summarizes the situation at $T = 0$. Let us briefly discuss how the BEC transition affects density distribution. At temperatures above the transition, the gas in the trap forms a cloud of width R_T that can be estimated from $\frac{1}{2}m\omega^2 R_T^2 \simeq T$. At $T < T_{BEC}$, condensate appears forming a much more narrow peak of radius (22) that coexists with the broad thermal distribution. As temperature goes down and becomes very small, the thermal component in the density distribution disappears, and one obtains the zero-temperature state (20).

1.4 Finite T effects: quasiparticle lifetime.

Decay of quasiparticles due to elastic scattering

$$\mathcal{H}_{int} = \frac{\lambda}{2}\sum_{k_1+k_2=k_3+k_4} a^+_{k_4}a^+_{k_3}a_{k_2}a_{k_1} \tag{23}$$

Golden Rule for transition rate:

$$W_{i\to f} = \frac{2\pi}{\hbar} \sum_f |\langle f|\mathcal{H}_{int}|i\rangle|^2 \delta(E_f - E_i) \tag{24}$$

In a normal Bose gas, at $T > T_{BEC}$, the rate of scattering out of the state $|i\rangle$ is

$$\text{out}: \qquad \frac{df_i}{dt} = -\frac{2\pi}{\hbar} \sum_{ij,mn} |M_{ij,mn}|^2 f_i f_j (1+f_m)(1+f_n)\delta(\epsilon_i + \epsilon_j - \epsilon_m - \epsilon_n) \tag{25}$$

while the rate of scattering in $|i\rangle$ is

$$\text{in}: \qquad \frac{df_i}{dt} = \frac{2\pi}{\hbar} \sum_{ij,mn} |M_{mn,ij}|^2 (1+f_i)(1+f_j) f_m f_n \delta(\epsilon_i + \epsilon_j - \epsilon_m - \epsilon_n) \tag{26}$$

with $M_{ij,mn} = M_{mn,ij} = \lambda$, $f_i = \langle a_i^+ a_i \rangle$, etc. The resulting rate is the sum of the in-rate and out-rate $df_i/dt = df_i/dt|_{in} + df_i/dt|_{out}$,

$$\frac{df_i}{dt} = -\frac{2\pi}{\hbar} \sum_{ij,mn} |M_{ij,mn}|^2 \left(f_i f_j (1+f_m)(1+f_n) - (1+f_i)(1+f_j) f_m f_n \right) \delta(\epsilon_i + \epsilon_j - \epsilon_m - \epsilon_n) \tag{27}$$

Features:

- The rate $\frac{df_i}{dt}$ vanishes in equilibrium, since $1 + f_j = e^{\beta\epsilon_j} f_j$, etc.
- For near-equilibrium distribution, $\frac{df_i}{dt} = -\frac{1}{\tau}(f_i - f_i^{(0)})$ with $\frac{1}{\tau} = \pi a^2 n v_T$ the classical scattering rate
  (recall: $\lambda = 4\pi\hbar^2 a/m$, $v_T = \sqrt{2T/m}$)
- Despite scattering, quasiparticles are well defined: $\epsilon(\mathbf{k}) \gg \frac{1}{\tau}$

In the BEC state, scattering is *stimulated* by the presence of the condensate:

$$a_n,\ a_n^+ \to a_0,\ a_0^+ = \sqrt{N} \tag{28}$$

which gives the rate of scattering out of the state $|i\rangle$ as

$$\text{out}: \qquad \frac{df_i}{dt} = -\frac{2\pi}{\hbar} \sum_{ij,m} |M_{ij,m}|^2 f_i f_j (1+f_m)\delta(\epsilon_i + \epsilon_j - \epsilon_m) \tag{29}$$

and the rate of scattering in $|i\rangle$,

$$\text{in}: \qquad \frac{df_i}{dt} = \frac{2\pi}{\hbar} \sum_{ij,m} |M_{m,ij}|^2 (1+f_i)(1+f_j) f_m \delta(\epsilon_i + \epsilon_j - \epsilon_m) \tag{30}$$

with $M_{ij,mn} = M_{mn,ij} \propto \lambda\sqrt{N}$.

The resulting rate, $df_i/dt = df_i/dt|_{in} + df_i/dt|_{out}$, is

$$\frac{df_i}{dt} = -\frac{2\pi}{\hbar} \sum_{ij,m} |M_{ij,m}|^2 \left(f_i f_j (1+f_m) - (1+f_i)(1+f_j) f_m \right) \delta(\epsilon_i + \epsilon_j - \epsilon_m) \tag{31}$$

Due to the presence of Bose condensate, scattering rate is enhanced at $T < T_{BEC}$.

1.5 Finite T effects: two-fluid hydrodynamics, I & II sound.

Momentum can be carried both by the condensate and excitations:

$$\mathbf{j} = \rho \mathbf{v}_s + \mathbf{j}_{ex} = \rho \mathbf{v}_s + \int \mathbf{p} f_{\mathbf{p}} \frac{d^3 p}{(2\pi\hbar)^3}$$

Normal fluid is described by quasiparticle distribution

$$f_{\mathbf{p}} = \frac{1}{\exp\left(\beta\left(\epsilon_{\mathbf{p}} - \mathbf{p}\cdot(\mathbf{v}_n - \mathbf{v}_s)\right)\right) - 1}$$

with the quasiparticle energy Doppler-shifted due to relative motion of the normal gas and superfluid. The momentum due to the normal component is

$$\mathbf{j}_{ex} = \int \mathbf{p} f_{\mathbf{p}} \frac{d^3 p}{(2\pi\hbar)^3} = \rho_n(|\mathbf{v}_n - \mathbf{v}_s|)\,(\mathbf{v}_n - \mathbf{v}_s)$$

At small velocities, have

$$\rho_n = \int \frac{\mathbf{p}^2}{3}\left(-\partial f_{\mathbf{p}}/\partial \epsilon_{\mathbf{p}}\right) \frac{d^3 p}{(2\pi\hbar)^3} = \begin{cases} (2\pi^2/45\hbar^3 c^5) T^4 & T \ll T_\lambda \\ \rho (T/T_{BEC})^{3/2} & T_\lambda \ll T < T_{BEC} \end{cases}$$

where $T_\lambda = \lambda n$ is the Bogoliubov sound – free particle crossover energy, and $c = \sqrt{\lambda n/m}$ is the sound velocity.

With $\rho_s = \rho - \rho_n$, the total momentum density can be expressed as

$$\mathbf{j} = \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n$$

To describe dynamics, one needs separate equations for ρ_s, $\mathbf{v}_s$ and ρ_n, $\mathbf{v}_n$. We will not discuss the two-fluid hydrodynamics in full generality. Instead, we describe a particular phenomenon, the II sound, a collective mode that appears in the two-fluid regime. In this mode, the relative fraction of the normal and superfluid component oscillates and can propagate in a sound-like fashion.

We consider a uniform system, in the absence of external potential. The total mass and mass current density obey the continuity relation

$$\partial_t \rho + \nabla \cdot \mathbf{j} = 0 \quad (32)$$

and the momentum conservation law

$$\partial_t \mathbf{j} = -\nabla p \quad (33)$$

After eliminating $\mathbf{j}$ from (32) and (33), one obtains

$$\partial_t^2 \rho - \nabla^2 p = 0 \quad (34)$$

For superfluid velocity, one can write

$$m\partial_t \mathbf{v}_s = -\nabla\mu \tag{35}$$

This relation, derived above from the Gross-Pitaevskii equation, is in fact very general, and is true for any superfluid. It follows from the relation between the phase of superfluid order parameter and the chemical potential, $\hbar\partial_t\theta = -\mu$, discussed above. In this form it was first introduced by Josephson in the theory of superconductivity.

Use the thermodynamic Gibbs relation $Nd\mu = Vdp - SdT$ with $\rho = Nm/V$, $\sigma = S/Nm$, to relate the gradient $\nabla\mu$ with the pressure and temperature gradients:

$$\nabla\mu = \frac{m}{\rho}\nabla p - \sigma m\nabla T \tag{36}$$

From Eqs.(36),(35),(33), combined with $\mathbf{j} = \rho_s\mathbf{v}_s + \rho_n\mathbf{v}_n$, obtain

$$\partial_t\left(\mathbf{v}_n - \mathbf{v}_s\right) = -\sigma\frac{\rho}{\rho_n}\nabla T \tag{37}$$

In a sound wave, gas compression is adiabatic, with entropy conserved:

$$\partial_t\left(\rho\sigma\right) + \nabla\left(\rho\sigma\mathbf{v}_n\right) = 0 \tag{38}$$

(The entropy is carried by the normal component only!) After linearizing and combining with the mass conservation Eq.(32), have

$$\partial_t\sigma = \sigma\frac{\rho_s}{\rho}\nabla\left(\mathbf{v}_s - \mathbf{v}_n\right) \tag{39}$$

Combined with Eq.(37), this yields

$$\partial_t^2\sigma = \frac{\rho_s}{\rho_n}\sigma^2\nabla^2 T \tag{40}$$

Collective modes are obtained by considering small oscillations of density, pressure, temperature and entropy, of the form $\exp\left(i\mathbf{qr} - i\omega t\right)$. It is convenient to choose density and temperature as independent variables. Linearizing Eqs. (34), (40) in $\delta\rho$, δT, obtain

$$\omega^2\delta\rho - q^2\left[\left(\frac{\partial p}{\partial\rho}\right)_T\delta\rho + \left(\frac{\partial p}{\partial T}\right)_\rho\delta T\right] = 0 \tag{41}$$

$$\omega^2\left[\left(\frac{\partial\sigma}{\partial\rho}\right)_T\delta\rho + \left(\frac{\partial p\sigma}{\partial T}\right)_\rho\delta T\right] - q^2\frac{\rho_s}{\rho_n}\sigma^2\delta T = 0 \tag{42}$$

In terms of sound velocity $u = \omega/q$, the equations have a solution if

$$(u^2 - c_1^2)(u^2 - c_2^2) - u^2 c_3^2 = 0 \tag{43}$$

with

$$c_1^2 = \left(\frac{\partial p}{\partial \rho}\right)_T , \quad c_2^2 = \frac{\rho_s \sigma^2 T}{\rho_n \tilde{C}} , \quad c_3^2 = \left(\frac{\partial p}{\partial T}\right)_\rho^2 \frac{T}{\rho \tilde{C}} \tag{44}$$

The constants c_1 and c_2 are the isothermal sound velocity and the velocity of temperature waves at constant density, while $\tilde{C} = T\left(\partial\sigma/\partial T\right)_\rho$ is the specific heat at constant volume, per unit mass. The expression for constant c_3 of the form (44) was obtained from Eqs. (41),(42) by using Maxwell relation

$$\left(\frac{\partial p}{\partial T}\right)_\rho = \left(\frac{\partial S}{\partial V}\right)_T = -\rho^2 \left(\frac{\partial \sigma}{\partial \rho}\right)_T \tag{45}$$

The I and II sound velocities, obtained from Eq.(43), are

$$u_{\mathrm{I,II}}^2 = \frac{1}{2}\left(c_1^2 + c_2^2 + c_3^2\right) \pm \frac{1}{2}\sqrt{\left(c_1^2 + c_2^2 + c_3^2\right)^2 - 4c_1^2 c_2^2} \tag{46}$$

So far, the treatment was completely general, applicable to any Bose system, irrespective the interaction strength and form.

Using the result, one can look at several regimes. In a weakly nonideal gas, at low temperatures, $T \ll T_\lambda = \lambda n$, the I sound velocity coincides with that of Bogoliubov quasiparticles at low energies,

$$u_{\mathrm{I}} = \sqrt{\lambda n/m}, \tag{47}$$

while the II sound velocity is $\sqrt{3}$ times lower,

$$u_{\mathrm{II}} = u_I/\sqrt{3} = \sqrt{\lambda n/3m}, \tag{48}$$

The velocity u_{II} decreases as a function of temperature.

In ^{4}He the II sound represents mostly a temperature wave, and to excite/detect it people had to use oscillatory thermal sources and heat sensors.