Path integrals for many particle systems

If the particles are all distinguishable, then extension to many particles is completely trivial - simply integrate over the trajectories of the coordinates $(x_1, \ldots, x_N)$.

What if the particles are identical?

To bring out the issues associated with "statistics" of identical quantum particles, consider the case of just 2 particles with coordinates $(\vec{x}_1, \vec{x}_2)$.

($\vec{x}_{1,2}$ are vectors in d-dimensions).

Consider the amplitude for an initial configuration to return to the same configuration after a time t.

To also clearly separate issues of statistics from more ordinary short ranged interactions, assume that there is a strong "hard-core" repulsion when the particles are very close to each other so that they always stay a non-zero distance apart.

~~Thus~~ We can discuss the motion in terms of center-of-mass $\vec{R}$

& relative coordinates $\vec{r}$.

Again issues of statistics & the effect of exchange only enter the relative coordinate $\vec{r}$ — so we will focus on this entirely & ignore the CM coordinate $\vec{R}$.

Note that $\vec{r} = 0$ is excluded by the hard-core restriction so $\vec{r} \in R^d - \{0\}$.

Paths where the initial & final 2-particle configurations are the same have either $\vec{r}_{fin} = \vec{r}_{in}$

or $\vec{r}_{fin} = -\vec{r}_{in}$.

Generally the amplitude for any particular path P

$$A[P] \propto e^{iS[P]}$$

where $S[P]$ is the classical action evaluated along the path.

For all paths that can be continuously deformed into one another, the constant of proportionality is the same.

However for 2 paths that cannot be continuously deformed into one another, the ~~amplitudes~~ proportionality constants can (in principle) differ by an overall phase.

Write $A[P] = \left(e^{i\theta_P}\right)\left(e^{iS[P]}\right)$

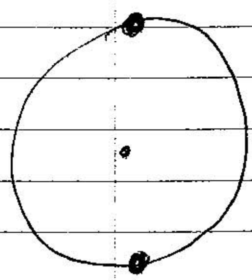
where θ_P is fixed ~~for~~ within each class of paths that can be ~~be~~ continuously deformed into one another.

~~For $d=3$ (or higher)~~ Further if a path $P_3 = P_2 P_1$

i.e it is composed out of the 2 individual return paths P_2 & P_1, then

$$e^{i\theta_{P_3}} = e^{i\theta_{P_1}} e^{i\theta_{P_2}}$$

For $d=3$ (or higher), there are only 2



classes of paths that cannot be continuously deformed into one another; paths for which $\vec{r}_f = \vec{r}_i$ form one class & paths with $\vec{r}_f = -\vec{r}_i$ form another class.

As the overall phase choice is arbitrary, choose $\theta_p = 0$ for paths where $\vec{r}_f = \vec{r}_i$, & let $\theta_p = \theta$ for paths where $\vec{r}_f = -\vec{r}_i$.

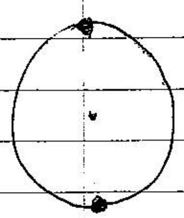
Consider paths which go from north pole to South pole & then back again to north pole, we get

$$e^{2i\theta} = 1 \Rightarrow e^{i\theta} = \pm 1.$$

$e^{i\theta} = +1$ describes bosons

$= -1$ describes fermions.

In $d=2$ situation is different



Characterize $\vec{r}$ by polar coordinates

(r, ϕ).

The important variable is ϕ.

Return paths have $\phi = 0$ or π (mod 2π).

However every path where ϕ starts from 0 & winds around several times to reach $m\pi$ is distinct (for distinct m)

$\Rightarrow$ There are an infinite # of classes of paths that cannot ~~all~~ be deformed into ~~one~~ another (each class is labelled by the winding m).

If for $m = 0$ we choose $\theta = 0$, and for $m = 1$, set some θ, then $\theta_m = m\theta$.
$\in [0, 2\pi)$

Thus we can have a perfectly consistent quantum theory where exchange of 2 particles ($m = 1$) gives a phase $e^{i\theta}$ which is neither 0 nor 1 in $d = 2$.

Such particles are known as "anyons".

Returning to bosons / fermions, and to the case of arbitrary # N of total # of particles,

partition fn $Z = \mathrm{Tr}\left(e^{-\beta H}\right)$

$$= \int \prod_{i=1}^{N} dx_i \left\langle \{x_1, \ldots, x_N\} \right| e^{-\beta H} \left| \{x_1, \ldots, x_N\} \right\rangle$$

$$= \frac{1}{N!} \int \prod_{i=1}^{N} dx_i \sum_P \mathrm{sgn}(P) \left\langle x_{P_1} \ldots x_{P_N} \right| e^{-\beta H} \left| x_1, \ldots, x_N \right\rangle.$$

where P now denotes a permutation.

Can now proceed as before (break up $e^{-\beta H}$ into $\left(e^{-\epsilon H}\right)^M$, insert resolution of identity $\mathbb{1} = \int dx_1 \cdots dx_N \left| x_1 \cdots x_N \right\rangle \left\langle x_1, \cdots x_N \right|$, etc) to get

$$Z = \frac{1}{N!} \sum_P \mathrm{sgn}(P) \int_{\substack{x_1(\beta) = x_{P_1}(0) \\ \vdots \\ x_N(\beta) = x_{P_N}(0)}} \left[\mathcal{D}x_1 \, \mathcal{D}x_2 \cdots \mathcal{D}x_N\right] e^{-\int_0^\beta d\tau \sum_{i=1}^{N} \left[\frac{m}{2}\left(\frac{dx_i}{d\tau}\right)^2 + U(\{x_i\})\right]}$$

(assuming the Hamiltonian is

$$H = \sum_i p_i^2/2m + U(\{x_i\}) \; .)$$

<u>Coherent state path integral</u>.

Focus on bosons (deal with fermions at a later stage when necessary).

First review coherent states - these provide a useful basis for states in Fock space.

They are eigenstates of annihilation operators.

Consider a single particle state $|\alpha\rangle$ & the corresponding destruction operator a_α.

Define $|\phi\rangle$ such that

$$a_\alpha|\phi\rangle = \phi_\alpha|\phi\rangle$$

First consider states where exactly 1 level α is occupied.
Then ~~Let~~ $|\phi\rangle = \sum_{n_\alpha} \phi_{n_\alpha} |n_\alpha\rangle$

$$a_\alpha|\phi\rangle = \sum_{n_\alpha} \phi_{n_\alpha} \sqrt{n_\alpha}\, |n_\alpha - 1\rangle$$

$$= \phi_\alpha \sum_{n_\alpha} \phi_{n_\alpha} |n_\alpha\rangle$$

$$\Rightarrow \phi_\alpha \phi_{n_\alpha - 1} = \sqrt{n_\alpha}\, \phi_{n_\alpha}$$

$$\Rightarrow \phi_{n_\alpha} = \frac{\phi_\alpha^{n_\alpha}}{\sqrt{n_\alpha !}} \quad \text{so that} \quad |\phi\rangle = \sum_{n_\alpha} \frac{\phi_\alpha^{n_\alpha}}{\sqrt{n_\alpha !}} |n_\alpha\rangle$$

In general expand $|\phi\rangle = \sum_{\{n_\alpha\}} \phi_{\{n_\alpha\}} |\{n_\alpha\}\rangle$

Following same line of reasoning as above, conclude

$$\phi_{\{n_\alpha\}} = \prod_\alpha \frac{\phi_\alpha^{n_\alpha}}{\sqrt{n_\alpha!}} \quad \text{so that}$$

$$|\phi\rangle = \sum_{\{n_\alpha\}} \left(\frac{\phi_1^{n_1}}{\sqrt{n_1!}} \frac{\phi_2^{n_2}}{\sqrt{n_2!}} \cdots \right) |\{n_\alpha\}\rangle$$

Now use $|\{n_\alpha\}\rangle = \left(\frac{a_1^{\dagger n_1}}{\sqrt{n_1!}} \frac{a_2^{\dagger n_2}}{\sqrt{n_2!}} \cdots \right) |0\rangle$ to

write

$$|\phi\rangle = \sum_{\{n_\alpha\}} \left(\frac{(\phi_1 a_1^\dagger)^{n_1}}{n_1!} \frac{(\phi_2 a_2^\dagger)^{n_2}}{n_2!} \cdots \right) |0\rangle$$

$$= e^{(\phi_1 a_1^\dagger + \phi_2 a_2^\dagger + \cdots)} |0\rangle$$

$$\boxed{|\phi\rangle = e^{\sum_\alpha \phi_\alpha a_\alpha^\dagger} |0\rangle}$$

$$\boxed{\langle\phi| = \langle 0| e^{\sum_\alpha \phi_\alpha^* a_\alpha}}$$

The overlap $\langle \phi | \phi' \rangle$ of 2 coherent states is

$$\langle \phi | \phi' \rangle = \sum_{\{n_\alpha\}} \sum_{\{n'_\alpha\}} \frac{\left(\phi^{*\,n_{\alpha_1}}_{\alpha_1} \phi^{*\,n_{\alpha_2}}_{\alpha_2} \cdots\right)}{\sqrt{n_{\alpha_1}!\, n_{\alpha_2}!\, n_{\alpha_3}! \cdots}}$$

$$\frac{\left(\phi'^{\,n_{\alpha_1}}_{\alpha_1} \phi'^{\,n_{\alpha_2}}_{\alpha_2} \cdots\right)}{\sqrt{n'_{\alpha_1}!\, n'_{\alpha_2}! \cdots}} \langle \{n_\alpha\} | \{n'_\alpha\} \rangle$$

$$= \sum_{\{n_\alpha\}} \frac{\left(\phi^*_{\alpha_1} \phi'_{\alpha_1}\right)^{n_{\alpha_1}}}{n_{\alpha_1}!} \frac{\left(\phi^*_{\alpha_2} \phi'_{\alpha_2}\right)^{n_{\alpha_2}}}{n_{\alpha_2}!} \cdots$$

$$= e^{\sum_\alpha \phi^*_\alpha \phi'_\alpha}$$

The coherent states are an (over) complete basis.

$$\int \prod_\alpha \left(\frac{d\phi^*_\alpha d\phi_\alpha}{2\pi i}\right) e^{-\sum_\alpha \phi^*_\alpha \phi_\alpha} |\phi\rangle\langle\phi| = 1$$

[Proof:

$$\text{LHS} = \int \prod_\alpha \left(\frac{d\phi_\alpha^* d\phi_\alpha}{2\pi i} \right) e^{-\sum_\alpha |\phi_\alpha|^2}$$

$$\sum_{\{n_\alpha, n'_\alpha\}} \prod_\alpha \frac{\phi_\alpha^{n_\alpha} (\phi_\alpha^*)^{n'_\alpha}}{\sqrt{n_\alpha! \, n'_\alpha!}} |\{n_\alpha\}\rangle\langle\{n'_\alpha\}|$$

$$= \sum_{\{n_\alpha\}} \int \prod_\alpha \left[\frac{d\phi_\alpha^* d\phi_\alpha}{2\pi i} e^{-|\phi_\alpha|^2} \frac{|\phi_\alpha|^{2n_\alpha}}{n_\alpha!} \right] |\{n_\alpha\}\rangle\langle\{n_\alpha\}|$$

$$= \sum_{\{n_\alpha\}} |\{n_\alpha\}\rangle\langle\{n_\alpha\}| = 1 .]$$

For any operator $\hat{O}$, $\operatorname{tr} \hat{O} = \operatorname{tr} \int \frac{d\phi^* d\phi}{2\pi i} e^{-|\phi|^2} |\phi\rangle\langle\phi| \hat{O}$

$$= \int \prod_\alpha \frac{d\phi_\alpha^* d\phi_\alpha}{2\pi i} e^{-\sum_\alpha |\phi_\alpha|^2} \langle \phi | \hat{O} | \phi \rangle .$$

Consider any "normal-ordered" operator

Eg: $\hat{O} = a^{\dagger}_{\alpha} a_{\beta}$

$$\langle \phi | a^{\dagger}_{\alpha} a_{\beta} | \phi \rangle = \phi^{*}_{\alpha} \phi_{\beta} \langle \phi | \phi \rangle = \phi^{*}_{\alpha} \phi_{\beta} \, e^{|\phi|^2}$$

In general if $\hat{O} = \left(\prod_{\alpha} \left(a^{\dagger}_{\alpha}\right)^{M_\alpha} \right) \left(\prod_{\beta} a_{\beta}^{N_\beta} \right)$

$$\langle \phi | \hat{O} | \phi \rangle = \prod_{\alpha} \left(\left(\phi^{*}_{\alpha}\right)^{M_\alpha} \left(\phi_{\alpha}^{N_\alpha}\right) \right) \langle \phi | \phi \rangle$$

Partition function $Z = \mathrm{tr}\left(e^{-\beta \hat{H}}\right)$

$$= \int \prod_{\alpha} \frac{d\phi^{*}_{\alpha} \, d\phi_{\alpha}}{2\pi i} \, e^{-\sum_{\alpha} |\phi_{\alpha}|^2} \langle \phi | e^{-\beta \hat{H}} | \phi \rangle$$

As before develop a path integral ~~dss~~ representation for Z.

Proceed as usual

$$Z = \int \frac{d\phi^* d\phi}{2\pi i} \, e^{-\phi^*_\alpha \phi_\alpha} \langle \phi | \left(e^{-\epsilon H}\right)^N | \phi \rangle$$

$N\epsilon = \beta$, $\epsilon \to 0$, $N \to \infty$.

$$= \int \frac{d\phi_0^* d\phi_0}{2\pi i} \prod_{j=1}^{N} \frac{d\phi_j^* d\phi_j}{2\pi i} \, e^{-\sum_{j=0}^{N} \phi^*_{j\alpha}\phi_{j\alpha}}$$

$$\langle \phi_0 | e^{-\epsilon H} | \phi_N \rangle \langle \phi_N | e^{-\epsilon H} | \phi_{N-1} \rangle$$

$$\cdots \langle \phi_{j+1} | e^{-\epsilon H} | \phi_j \rangle \cdots$$

$$\langle \phi_1 | e^{-\epsilon H} | \phi_0 \rangle .$$

$$\langle \phi_{j+1} | e^{-\epsilon H} | \phi_j \rangle \simeq \langle \phi_{j+1} | (1 - \epsilon H) | \phi_j \rangle + o(\epsilon^2)$$

Assume $H = H(\{a^\dagger_\alpha\}, \{a_\alpha\})$ is normal ordered.

Then $\langle \phi_{j+1} | H | \phi_j \rangle = \langle \phi_{j+1} | \phi_j \rangle H(\phi^*_{j+1}, \phi_j)$

$$= e^{+\phi^*_{j+1}\phi_j} H(\phi^*_{j+1}, \phi_j)$$

$$\therefore \quad \langle \phi_{j+1} | e^{-\epsilon H} | \phi_j \rangle = e^{+\phi^*_{j+1}\phi_j} \left(1 - \epsilon H(\phi^*_{j+1}, \phi_j)\right)$$

$$\simeq e^{+\phi^*_{j+1}\phi_j - \epsilon H(\phi^*_{j+1}, \phi_j)}$$

$$+ \, O(\epsilon^2).$$

$$\therefore \quad Z = \int_{[\phi_{N+1} = \phi_0]} \prod_{j=0}^{N} \frac{d\phi^*_j \, d\phi_j}{2\pi i} \; e^{-\sum_{j=0}^{N} \phi^*_{j+1}(\phi_{j+1} - \phi_j) - \epsilon \sum_{j=0}^{N} H(\phi^*_{j+1}, \phi_j)}$$

Now introduce "continuum" notation when $\epsilon \to 0$, $N \to \infty$.

Write $\phi_{j+1} - \phi_j = \epsilon \dfrac{\partial \phi}{\partial \tau}$ to write

$$Z = \int_{\phi(\beta) = \phi(0)} [D\phi] \; e^{-\int_0^\beta d\tau \left(\phi^* \frac{\partial \phi}{\partial \tau} + H(\phi^*, \phi)\right)}$$

= functional integral over ϕ with periodic boundary conditions.

Comments: To be really precise, the ~~meaning of~~ continuum path integral is really a "short-hand" notation for the full discrete time integrals.

~~Strictly speaking~~ In discrete time,

$$\sum_{j=0}^{N} \phi^*_{j+1}\left(\phi_{j+1}-\phi_j\right)$$ may be written in terms of

its the Fourier components of ϕ:

$$\phi_j = \frac{1}{\beta}\sum_{\omega_n} e^{-i\omega_n \tau}\phi(\omega_n)$$

with $\omega_n = 2\pi n/\beta$, and $\tau = j\epsilon$

Then $$\sum_{j=0}^{N} \phi^*_{j+1}\left(\phi_{j+1}-\phi_j\right) = \frac{1}{\beta}\sum_{\omega_n}\phi^*(\omega_n)\left(1-e^{i\omega_n\tau}\right)\phi(\omega_n)$$

In the continuum expression we have effectively replaced ~~[illegible]~~ $1-e^{i\omega_n \tau}$ by $i\omega_n\tau$.

Clearly this is legitimate only if $\omega_n\tau \ll 1$

i.e. if the "frequencies" of interest $\omega_n \lesssim \frac{1}{\text{temporal lattice spacing}}$.

As in evaluation of Z for simple harmonic oscillator, expect that for full calculation of Z, all ω_n contribute & we must be careful about sticking to the precise lattice definition of the path integral.

However if for a general many body $H = H_0 + H_{int}$

where H_0 is the "free" quadratic part

& H_{int} is the interacting part, we calculate

$$\frac{Z}{Z_0} = \frac{\mathrm{tr}\left(e^{-\beta H}\right)}{\mathrm{tr}\left(e^{-\beta H_0}\right)}, \quad \text{then expect that}$$

this will only be dominated by $\omega_n \ll 1/\epsilon$ so the calculation can be formulated directly in the continuum without worrying about carefully taking the limit of the lattice action.

Similarly in calculating averages

$$\langle \hat{O} \rangle = \frac{\mathrm{Tr}\left(\hat{O} e^{-\beta \hat{H}}\right)}{\mathrm{Tr}\left(e^{-\beta \hat{H}}\right)}$$

, expect can typically directly work in the continuum.

<u>Real time path integral</u>

It is equally easy to derive a coherent state path integral for the evolution operator e^{-itH} in real time.

The result is

$$\langle \phi_f \, t_f | U | \phi_i \, t_i \rangle = \int_{\phi(t_i)=\phi_i}^{\phi(t_f)=\phi_f} D\phi^* D\phi \; e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt \left[i\hbar \phi^* \frac{\partial \phi}{\partial t} - H(\phi^*, \phi) \right]}$$

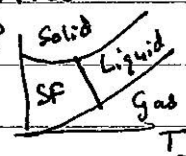
Phenomenon of superfluidity

At ordinary (atmospheric) pressure, He-4 remains a liquid down to $T = 0°K$.

This is due to a combination of 2 things:

- (a) the He-4 atom is inert as it has a filled shell $\Rightarrow$ the residual interaction between two He-4 atoms is very weak.

- (b) He-4 is a very light atom $\Rightarrow$ kinetic energy of zero point motion is large.

However at low $T \simeq 2$ K, He-4 nevertheless undergoes a phase transition from the higher temp. ordinary liquid to a lower temp. extraordinary liquid called the superfluid. [Phase dia: 

]

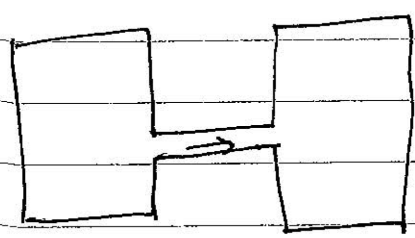
In the low-T phase, He-4 can flow through fine capillaries without ~~[illegible]~~ any friction (i.e no viscosity).

There are a number of other wierd phenomena which can be summarized by the following description:

In the low-T phase, the fluid is a mixture of two components – a "normal" fluid and a "superfluid".

The superfluid component has zero viscosity and Zero entropy.

(As an example of a phenomenon which suggests this, consider two containers of He-4 connected together by a fine capillary



A B

If $P_A > P_B$, the superfluid component will flow from A to B.

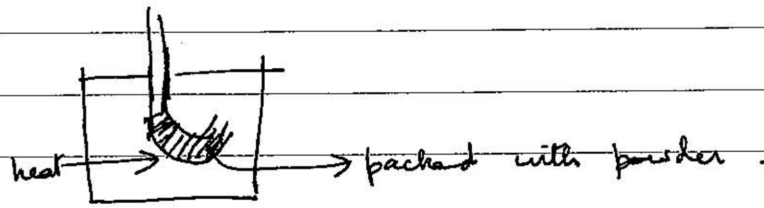
As this ~~superflow~~ component carries no entropy, the entropy/mass of A will increase & that of B will decrease

$\Rightarrow$ A will heat up and B will cool down which is actually observed.

The inverse effect is also observed.

Consider heating A relative to B – the temp. gradient causes a pressure gradient forcing the superfluid component to ~~keep~~ flow from A to B.

If the set up is like



then the increased pressure in the heated region can cause superflow up the tube to create a fountain).

In searching for a theoretical understanding & description of superfluidity, we first note 2 things:

(a) The phenomenon happens at low-T where we expect that the quantum statistics of the He-4 atoms must play an important role.

(b) He-4 atoms are bosons — if we completely ignore the repulsion between He-4 atoms, we get the BEC phenomenon where there is indeed the formation of a zero entropy condensate.

(Note however BEC $\neq$ superfluidity).

Here we take a "top-down" approach and attempt to describe the properties of a ~~s~~ system of interacting bosons at low temperature.

We will ~~ass~~ analyse a particular physical situation which will enable a "painless" description of the low-T superfluid phase using ideas of broken symmetry.

The universal properties of the resulting phase generalize to other physical situations.