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Lecture 10: Superconductors With Disorder

Up to now, all of our discussions have centered around superconductivity in an idealized, perfectly isotropic environment. Because such perfect order is never realized in the real world, it is important to extend the theory to systems with disorder. Surprisingly, we will find that the energy gap Δ and the transition temperature T_c are *not* affected by the presence of disorder. The effective superfluid density ρ_s will be strongly affected, however.

10.1 The Anderson Theorems

To study superconductivity in the presence of disorder, we will add a random potential $V_{\text{rnd}}(\vec{r})$ to the single-particle Hamiltonian such that

$$\hat{H}_0 = \frac{\hat{\vec{p}}^2}{2m} + V_{\text{rnd}}(\vec{r}) \tag{10.1}$$

Under this assumption, we will be able to extend the BCS theory to prove the following two statements:

Provided that disorder is not strong enough to cause the eigenstates of the single particle Hamiltonian to be localized,

(i) Disorder does *not* affect T_c.
(ii) Disorder does *not* affect Δ.

These statements were originally due to Anderson, and are commonly known as Anderson's Theorems. The key to understanding Anderson's Theorems is to think in terms of the exact single-particle eigenstates of $\hat{H}_0$

$$\hat{H}_0 |\, \alpha \,\rangle = \epsilon_\alpha |\, \alpha \,\rangle \tag{10.2}$$

Here, $|\, \alpha \,\rangle$ is an eigenstate of the disordered single-particle Hamiltonian for a *particular instance* of the random potential $V_{\text{rnd}}(\vec{r})$. Although in position space these wavefunctions may be very complicated functions of $\vec{r}$, there is no conceptual difficulty in defining such states. Practically speaking, they can be found approximately through standard PDE solving methods on a big computer.

Since the random potential destroys the continuous translational invariance of the system's Hamiltonian, momentum ($\vec{k}$) is no longer a good quantum number in a disordered system. Thus we no longer have the states of opposite momentum to pair-up like we did in our original BCS formulation. Anderson's contribution, however, was to suggest that in a more general setting we should look for *time-reversed states* to pair with each other.

10.1.1 Time Reversal Symmetry

What are time reversed states? To recall the idea behind time reversal symmetry, or microscopic reversibility, consider a system with Hamiltonian $\hat{H}(t)$ that is in the state $|\,\psi;t_0\,\rangle$ at time t_0. If we flow the system backwards in time through an infinitessimal interval dt, the Shrodinger equation tells us that the state at time $t_0 - dt$ should have been

$$|\,\psi;t_0-dt\,\rangle = \left[\mathbb{1} + i\frac{dt}{\hbar}\hat{H}(t_0)\right]|\,\psi;t_0\,\rangle \tag{10.3}$$

Now imagine reversing all momenta ($\hat{p_k} \to -\hat{p_k}$) and angular momenta ($\hat{L_k} \to -\hat{L_k}$) in the initial state $|\,\psi;t_0\,\rangle$. While this is easy to imagine classically where particles have definite positions and momenta, quantum mechanically it corresponds to changing the linear and angular momentum basis kets according to $|\,\vec{p}\rangle \to |-\vec{p}\rangle$ and $|\,J;\,J_z\,\rangle \to |\,J;\,-J_z\,\rangle$, respectively. Reversing the linear and angular momenta of all states and flowing the system forward in time to $t_0 + dt$ should not lead to any *observably* different consequences from the situation in which the original kets were simply flowed backwards to $t_0 - dt$. As is shown in standard quantum mechanics books, consistency of the theory requires that the coefficients of the basis kets be taken to their *complex conjugates*, i.e.

$$|\,\psi;t_0\,\rangle = \sum_{\vec{p}} c_{\vec{p}}|\,\vec{p}\rangle \tag{10.4}$$

$$\hat{T}\,|\,\psi;t_0\,\rangle = \sum_{\vec{p}} c^*_{\vec{p}}|-\vec{p}\rangle \tag{10.5}$$

By projecting these state onto a position eigenket $|\,\vec{r}\,\rangle$

$$\langle\,\vec{r}\,|\,\psi;t_0\,\rangle = \sum_{\vec{p}} c_{\vec{p}}(t_0)\langle\,\vec{r}\,|\,\vec{p}\rangle = \sum_{\vec{p}} c_{\vec{p}}(t_0)e^{i\vec{k}\cdot\vec{r}} \tag{10.6}$$

$$\langle\,\vec{r}\,|\,\hat{T}\,|\,\psi;t_0\,\rangle = \sum_{\vec{p}} c^*_{\vec{p}}(t_0)\langle\,\vec{r}\,|-\vec{p}\rangle = \sum_{\vec{p}} c^*_{\vec{p}}(t_0)e^{-i\vec{k}\cdot\vec{r}} \tag{10.7}$$

we see that the spatial orbital $\phi_T(\vec{r},t_0) = \langle\,\vec{r}\,|\,\hat{T}\,|\,\psi;t_0\,\rangle$ of the time reversed state is simply the complex conjugate of the original orbital $\phi(\vec{r},t_0) = \langle\,\vec{r}\,|\,\psi;t_0\,\rangle$.

Taking into account the reversal of angular momentum as well, the spin-orbital associated with the $\hat{H}_0$ eigenket $|\,\alpha\,\rangle$ and its time reversed partner are $\phi_{\alpha\,\uparrow}(\vec{r}) \xrightarrow{\hat{T}} \phi^*_{\alpha\,\downarrow}(\vec{r})$. For a system whose Hamiltonian is invariant under time-reversal (i.e. $\left[\hat{H}_0,\hat{T}\right] = 0$), these states are clearly degenerate. As a result, any linear combination of ϕ_α and ϕ^*_α is also an eigenstate of $\hat{H}_0$ with energy ϵ_α. In particular, this means that we can always arrange to pick *real*-valued orbitals, such that the time reversed pair is simply $\{\phi_{\alpha\,\uparrow},\phi_{\alpha\,\downarrow}\}$. This two-fold spin degeneracy, called the Kramers degeneracy, is always present in systems with time-reversal symmetry.

10.1.2 Effective BCS Hamiltonian

Recall our discussion of the BCS effective Hamiltonian in an isotropic medium from last semester. By applying the phonon mediated electron-electron interaction to the BCS wavefunction, we arrived at the reduced Hamiltonian

$$\hat{H}_{\text{red}} = \hat{H}_0 + \sum_{\vec{k},\vec{k}'} V_{\vec{k},\vec{k}'}c^\dagger_{\vec{k}'\,\uparrow}c^\dagger_{-\vec{k}'\,\downarrow}c_{-\vec{k}\,\downarrow}c_{\vec{k}\,\uparrow} \tag{10.8}$$

Since that reduced Hamiltonian was constructed by projecting onto the isotropic-medium BCS wave function written in terms of momentum eigenstates, we cannot take this reduced Hamiltonian as our starting point. Instead, we return to the more general phonon mediated electron-electron interaction

$$\hat{H}_{\mathrm{e-ph}} = \frac{1}{2} \sum_{\vec{k},\vec{k}',\vec{q},\sigma,\sigma'} V c^{\dagger}_{\vec{k}'+\vec{q},\,\sigma'} c^{\dagger}_{\vec{k}-\vec{q},\,\sigma} c_{\vec{k},\,\sigma} c_{\vec{k}',\,\sigma'} \tag{10.9}$$

Assuming $V = -V_0$ is independent of $\vec{k}$, $\vec{k}'$ etc, we can factor V outside the summation. Substituting

$$c^{\dagger}_{\vec{k}} = \int d\vec{r} e^{-i\vec{k}\cdot\vec{r}} \Psi^{\dagger}(\vec{r}) \tag{10.10}$$

into (10.9), we get

$$\begin{aligned}
&\sum \int d\vec{r}_1 d\vec{r}_2 d\vec{r}_3 d\vec{r}_4 e^{-i\vec{k}'+\vec{q}\cdot\vec{r}_1} e^{-i\vec{k}-\vec{q}\cdot\vec{r}_2} e^{i\vec{k}\cdot\vec{r}_3} e^{i\vec{k}'\cdot\vec{r}_4} \Psi^{\dagger}_{\sigma'}(\vec{r}_1)\Psi^{\dagger}_{\sigma}(\vec{r}_2)\Psi_{\sigma}(\vec{r}_3)\Psi_{\sigma}(\vec{r}_4)\\
&= \sum \int d\vec{r}_1 d\vec{r}_2 d\vec{r}_3 d\vec{r}_4 e^{-i\vec{q}\cdot(\vec{r}_2-\vec{r}_1)} e^{-i\vec{k}'\cdot(\vec{r}_1-\vec{r}_4)} e^{-i\vec{k}\cdot(\vec{r}_2-\vec{r}_3)} \Psi^{\dagger}_{\sigma'}(\vec{r}_1)\Psi^{\dagger}_{\sigma}(\vec{r}_2)\Psi_{\sigma}(\vec{r}_3)\Psi_{\sigma}(\vec{r}_4)\\
&= \sum_{\sigma,\sigma'} \int d\vec{r}_1 d\vec{r}_2 d\vec{r}_3 d\vec{r}_4 \delta(\vec{r}_2-\vec{r}_1)\delta(\vec{r}_1-\vec{r}_4)\delta(\vec{r}_2-\vec{r}_3) \Psi^{\dagger}_{\sigma'}(\vec{r}_1)\Psi^{\dagger}_{\sigma}(\vec{r}_2)\Psi_{\sigma}(\vec{r}_3)\Psi_{\sigma}(\vec{r}_4)\\
&= \sum_{\sigma,\sigma'} \int d\vec{r} \Psi^{\dagger}_{\sigma'}(\vec{r})\Psi^{\dagger}_{\sigma}(\vec{r})\Psi_{\sigma}(\vec{r})\Psi_{\sigma}(\vec{r})\\
&= 2 \int d\vec{r} \Psi^{\dagger}_{\uparrow}(\vec{r})\Psi^{\dagger}_{\downarrow}(\vec{r})\Psi_{\downarrow}(\vec{r})\Psi_{\uparrow}(\vec{r})
\end{aligned} \tag{10.11}$$

where in the last line we have explicitly put in the sum over σ and σ', used $\Psi_{\sigma}(\vec{r})\Psi_{\sigma}(\vec{r}) = \Psi^{\dagger}_{\sigma}(\vec{r})\Psi^{\dagger}_{\sigma}(\vec{r}) = 0$, and the Fermion creation/destruction operator anti-commutation relations.

Finally we can write the interaction in terms of the field creation/destruction operators

$$\hat{H}_{\mathrm{e-ph}} \approx -V_0 \int d\vec{r} \Psi^{\dagger}_{\uparrow}(\vec{r})\Psi^{\dagger}_{\downarrow}(\vec{r})\Psi_{\downarrow}(\vec{r})\Psi_{\uparrow}(\vec{r}) \tag{10.12}$$

where the approximation is that we assumed the interaction potential could be pulled outside the sum as a constant V_0.

Using relation (10.2) for the single-particle eigenstates of the disordered system, the total system Hamiltonian including the phonon mediated electron-electron coupling in second quantized form is

$$\hat{H} = \sum_{\alpha} \epsilon_{\alpha} \left(c^{\dagger}_{\alpha\uparrow} c_{\alpha\uparrow} + c^{\dagger}_{\alpha\downarrow} c_{\alpha\downarrow} \right) - V_0 \int d\vec{r} \Psi^{\dagger}_{\uparrow}(\vec{r})\Psi^{\dagger}_{\downarrow}(\vec{r})\Psi_{\downarrow}(\vec{r})\Psi_{\uparrow}(\vec{r}) \tag{10.13}$$

We can now proceed with the mean-field averaging procedure analogous to the one used in the clean superconductor case. Using the definition

$$\Delta(\vec{r}) = V_0 \langle \Psi_{\downarrow}(\vec{r})\Psi_{\uparrow}(\vec{r}) \rangle \tag{10.14}$$

we can rewrite the effective Hamiltonian as

$$\hat{H}_{\mathrm{eff}} = \sum_{\alpha} \epsilon_{\alpha} \left(c^{\dagger}_{\alpha\uparrow} c_{\alpha\uparrow} + c^{\dagger}_{\alpha\downarrow} c_{\alpha\downarrow} \right) - \frac{1}{2} \int d\vec{r} \left(\Delta(\vec{r}) \Psi^{\dagger}_{\uparrow}(\vec{r})\Psi^{\dagger}_{\downarrow}(\vec{r}) + \Delta^{*}(\vec{r})\Psi_{\downarrow}(\vec{r})\Psi_{\uparrow}(\vec{r}) \right) \tag{10.15}$$

With the change of basis

$$\Psi_\sigma(\vec{r}) = \sum_\alpha \phi_\alpha(\vec{r})\, c_{\alpha,\,\sigma} \tag{10.16}$$

we can rewrite the definition of $\Delta(\vec{r})$ in terms of the exact eigenstates of the disordered potential

$$\begin{aligned}\Delta(\vec{r}) &= V_0\, \langle\, \Psi_\downarrow(\vec{r})\Psi_\uparrow(\vec{r})\,\rangle \\ &= V_0 \sum_{\alpha,\,\beta} \phi_\alpha(\vec{r})\, \phi_\beta(\vec{r})\, \langle c_{\alpha\,\downarrow} c_{\beta\,\uparrow}\rangle \end{aligned}\tag{10.17}$$

In general, we expect that $\Delta(\vec{r})$ will be a very complicated function of $\vec{r}$. However, **in the presence of weak disorder**, the eigenstates $\phi_\alpha(\vec{r})$ are *extended in space*. At any particular location, (10.17) will contain a sum over a very large number of nearly random contributions from the different eigenstates. In this situation, $\Delta(\vec{r})$ sums up to a nearly constant value throughout space. Thus when disorder is weak, we can replace $\Delta(\vec{r})$ by its spatial average Δ

$$\Delta = \frac{1}{\Omega}\int d\vec{r}\Delta(\vec{r}) = \frac{V_0}{\Omega}\sum_{\alpha,\,\beta}\int d\vec{r}\, \phi_\alpha(\vec{r})\, \phi_\beta(\vec{r})\, \langle c_{\alpha\,\downarrow} c_{\beta\,\uparrow}\rangle \tag{10.18}$$

Recall that since $\phi_\alpha(\vec{r})$ and $\phi_\beta(\vec{r})$ are exact eigenstates of the single particle Hamiltonian, they obey the orthonormality condition

$$\int d\vec{r}\phi_\alpha(\vec{r})\phi_\beta(\vec{r}) = \delta_{\alpha\beta} \tag{10.19}$$

which leads to

$$\Delta = \frac{V_0}{\Omega}\sum_\alpha \langle c_{\alpha\,\downarrow} c_{\beta\,\uparrow}\rangle \tag{10.20}$$

and an effective Hamiltonian

$$\hat{H}_{\text{eff}} = \sum_\alpha \epsilon_\alpha \left(c^\dagger_{\alpha\,\uparrow} c_{\alpha\,\uparrow} + c^\dagger_{\alpha\,\downarrow} c_{\alpha\,\downarrow}\right) - \Delta c^\dagger_{\alpha\,\uparrow} c^\dagger_{\alpha\,\downarrow} - \Delta^* c_{\alpha\,\downarrow} c_{\alpha\,\uparrow} \tag{10.21}$$

Notice that the Hamiltonian is now block-diagonalized in α. In exactly the same way as before, we can now diagonalize the Hamiltonian separately for each α by the Bogoliubov transformation

$$\gamma_{\alpha\,\uparrow} = u_\alpha c_{\alpha\,\uparrow} + v_\alpha c^\dagger_{\alpha\,\downarrow} \tag{10.22}$$

$$\gamma^\dagger_{\alpha\,\downarrow} = v^*_\alpha c_{\alpha\,\uparrow} + u_\alpha c^\dagger_{\alpha\,\downarrow} \tag{10.23}$$

giving the familiar quasiparticle spectrum

$$E_\alpha = \sqrt{(\epsilon_\alpha - \mu)^2 + |\Delta|^2} \tag{10.24}$$

10.1.3 The Self-Consistent Equation

With the high degree of similarity between this more general formalism and the BCS relations obtained previously, it is straightforward to see that the same algebraic steps can be followed to arrive at the self-consistent equation for Δ:

$$\Delta = V_0 \sum_{\alpha} \frac{\Delta}{\sqrt{\xi_\alpha{}^2 + |\Delta|^2}} \left(1 - 2f(E_\alpha)\right) \tag{10.25}$$

where

$$\xi_\alpha = \epsilon_\alpha - \mu \tag{10.26}$$

The important terms in this sum occur for ξ_α less than the order of Δ. In this range, there is a high, nearly constant density of states $N(0)$, which at low temperatures ($f(E_\alpha) \approx 0$) allows us to convert the sum to an integral

$$\Delta = V_0 N(0) \int d\xi \frac{\Delta}{2E_\xi} \tag{10.27}$$

In equation (10.27), all references to the eigenstates and hence the potential have disappeared. This is exactly the self-consistent equation obtained by BCS theory. As a result T_c *and* Δ *are not affected by the presence of disorder.*

This is an extremely powerful and important result. As such, it is important to remember under what conditions it is valid. Our derivation was quite general, but relied on one key assumption. In order to get rid of the spatial dependence of $\Delta(\vec{r})$, we had to assume *weak disorder* such that the eigenstates $|\,\alpha\,\rangle$ are *extended in space.* This can be summarized in the condition

$$k_F \ell \gg 1 \tag{10.28}$$

where ℓ is the elastic scattering mean free path. Additionally, for the use of $N(0)$ as the density of states near μ to be valid, we need the disorder to not significantly affect the density of states.

10.1.4 Time Reversal Symmetry Breaking

Surprisingly, we found that disorder does not reduce T_c for a superconductor. However, terms in the Hamiltonian that break time reversal symmetry can have a devastating effect on T_c. Two such possibilities are

(i) Magnetic Impurity Scattering
(ii) Interaction With a Magnetic Field $\vec{H}$

In the case of (10.1.4), impurity scattering can lead to spin-flips which destroy the BCS-style pairing of time-reversed states. Even a tiny bit of magnetic impurities ($\frac{1}{\tau_{\text{mag}}} \approx \Delta_0$ can destroy superconductivity. For comparison, the analogous condition for elastic disorder scattering gives $\frac{1}{\tau_{\text{el}}} < \epsilon_F$.

Although a magnetic field also breaks time reversal symmetry, a superconductor is able to compensate for this by allowing the field to penetrate through local non-superconductive regions called vortices. Interestingly, disorder is actually beneficial in this case, as it provides a mechanism for pinning vortices in space. If the vortices are free to swim around throughout the superconductor, there can be dissipation which is undesirable.

10.1.5 Non-S-Wave Superconductors

Everything we have said about the Anderson Theorems and disorder so far is true only for *s-wave superconductors* for which

$$\Delta_{\vec{k}} = \langle c_{\vec{k}\uparrow} c_{-\vec{k}\downarrow} \rangle \tag{10.29}$$

is independent of $\vec{k}$.

If $\Delta_{\vec{k}}$ does have a dependence on $\vec{k}$, then the derivation will break down at the averaging step because impurities mess up the structure of momentum space. As a result, non-s-wave superconductors do not even tolerate elastic scattering; $\frac{1}{\tau_{\rm el}} \approx \Delta_0$ is the most that can be supported. Thus very clean samples are need to see non-s-wave superconducting behavior.

In the case of non-s-wave high T_c superconductors, we are saved by the fact that Δ_0 is quite large. This allows the high T_c materials to become superconducting with a reasonable amount of impurities. Furthermore, the CuO bond in copper-oxide materials is very strong, and naturally tends to prohibit impurities from entering into the lattice. Through intentional doping with zinc, the effect of impurities on T_c can be observed. For an s-wave superconductor, we would expect no change for small amounts of doping, while for high T_c materials we would expect a linear decrease of T_c with doping fraction.

10.2 Conductivity in Disordered Superconductors

So far, it seems as if disorder has almost no effect on superconductivity. While Δ and T_c are unaffected by weak disorder, however, the effective superfluid density ρ_s is affected strongly. Instead of considering ρ_s directly, we will focus on the absorption of electromagnetic radiation. That is, we will be interested in the quantity $\sigma'_{\perp}(\vec{q},\omega)$.

The situation is considerably simplified of $\vec{A}(\vec{r})$ and $\vec{j}(\vec{r})$ are *slowly varying* functions in *space*. This is the case in the limit $\vec{q} \to 0$. When working with light, we always work in this limit, but here we also apply this limit to thinking about the Meisner effect.

If the London penetration depth λ_L is much larger than the correlation length xi_0, then magnetic fields may penetrate a distance much larger than the length scale of variations of the order parameter. Thus this case also corresponds to the $\vec{q} \to 0$ limit.

10.2.1 Derivation of the Transverse Conductivity

Returning to the Kubo formula in the $\vec{q} \to 0$ limit,

$$\sigma'_{\perp(xx)}(\vec{q} \to 0, \omega) = \frac{\pi}{\omega}\frac{1}{\Omega}\overline{\sum_n \langle\, 0 \,|\int d\vec{r}\hat{j}^p_x(\vec{r})|\, n\,\rangle\langle\, n\,|\int d\vec{r}'\hat{j}^p_x(\vec{r}')|\, 0\,\rangle \delta\left(\omega - (E_n - E_0)\right)} \tag{10.30}$$

To proceed with the calculation, we need to write the paramagnetic current operator in second quantized form in the basis of exact single-particle eigenstates:

$$\int d\vec{r}\,\hat{j}^p_x(\vec{r}) = e \sum_{\alpha,\beta,\sigma} V_{\alpha\beta} c^\dagger_{\beta,\sigma} c_{\alpha,\sigma} \tag{10.31}$$

where

$$V_{\alpha\beta} = \frac{1}{m}\int d\vec{r}\phi^*_\beta(\vec{r})\frac{1}{i}\frac{\partial}{\partial x}\phi_\alpha(\vec{r}) \tag{10.32}$$

Next, we change to the Bogoliubov quasiparticle basis using the substitution

$$c_{\alpha\uparrow} = u_\alpha \gamma_{\alpha\uparrow} + v_\alpha \gamma^\dagger_{\alpha\downarrow} \tag{10.33}$$

After making this substitution and noting that the excited states $|\, n\,\rangle$ correspond to states containing single pairs of quasiparticle-hole excitations $|\,\alpha\,\beta\,\rangle$ of energy $E_\alpha + E_\beta$, we get

$$\sigma'_\perp(\vec{q} \to 0, \omega) = \frac{e^2}{\omega}\frac{\pi}{\Omega}\overline{\sum_{\alpha,\beta}\left(u_\alpha v_\beta - v_\alpha u_\beta\right)^2 |V_{\alpha\beta}|^2 \delta\left(\omega - (E_\alpha + E_\beta)\right)} \tag{10.34}$$

In the case of a clean superconductor, the analogous expression contained the coherence factor

$$p^2_{\vec{k},\vec{k}+\vec{q}} = \left(u_{\vec{k}} v_{\vec{k}+\vec{q}} - v_{\vec{k}} u_{\vec{k}+\vec{q}}\right)^2 \tag{10.35}$$

which is 0 for $\vec{q} \to 0$. With disorder present, however, the eigenstates are no longer the simple $\vec{k}$-states, and the coherence factor $p_{\alpha,\beta}$ does *not* vanish.

This lack of cancellation is not difficult to handle. Recall

$$u^2_\alpha = \frac{1}{2}\left(1 + \frac{\xi_\alpha}{E_\alpha}\right) \tag{10.36}$$

$$v^2_\alpha = \frac{1}{2}\left(1 - \frac{\xi_\alpha}{E_\alpha}\right) \tag{10.37}$$

Furthermore, note that u_α and v_α depend on α only through the combination $\frac{\xi_\alpha}{E_\alpha}$. In each term of the sum, we can insert the identity

$$1 = \int_\infty^\infty d\xi \delta(\xi - \xi_\alpha) \int_\infty^\infty d\xi' \delta(\xi' - \xi_\beta) \tag{10.38}$$

Furthermore, we can switch the order of summation and integration, and make the definition

$$f(\xi, \xi') \equiv \frac{1}{\Omega}\overline{\sum_{\alpha,\beta} |V_{\alpha\beta}|^2 \delta\left(\xi - \xi_\alpha\right) \delta\left(\xi' - \xi_\beta\right)} \tag{10.39}$$

With this definition, the conductivity simply becomes

$$\sigma'_\perp(\vec{q} \to 0, \omega) = \frac{e^2}{\omega}\pi \int_\infty^\infty d\xi \int_\infty^\infty d\xi' (u_\xi v_{\xi'} - v_\xi u_{\xi'})^2 f(\xi, \xi') \delta\left(\omega - (E_\xi + E_{\xi'})\right) \tag{10.40}$$

In this form, the conductivity is written as an integral over the reduced energies ξ and ξ'. All information about the actual eigenstates and the disorder is confined to the function $f(\xi, \xi')$. Furthermore, $f(\xi, \xi')$ is simply a function of the *normal metal* eigenstates, and does not require any knowledge of the superconducting behavior of the system. This function is in fact the very same function we encountered in the calculation of the conductivity of a disordered normal metal.

We can move past equation (10.40) by substituting in for u_ξ, etc using equations (10.36) and (10.37).

$$\left(uv' - vu'\right)^2 = \frac{1}{4}\left[\left(1 + \frac{\xi}{E}\right)^{1/2}\left(1 - \frac{\xi'}{E'}\right)^{1/2} - \left(1 - \frac{\xi}{E}\right)^{1/2}\left(1 + \frac{\xi'}{E'}\right)^{1/2}\right]^2 \tag{10.41}$$

$$= \frac{1}{4}\left[2\left(1 - \frac{\xi\xi'}{EE'}\right) - 2\left(1 - \frac{\xi^2}{E^2}\right)^{1/2}\left(1 - \frac{\xi'^2}{E'^2}\right)^{1/2}\right] \tag{10.42}$$

$$= \frac{1}{2}\left[1 - \frac{\xi\xi'}{EE'} - \frac{\Delta^2}{EE'}\right] \tag{10.43}$$

With this coherence factor expanded out in terms of ξ and E, the integral can now be performed directly once $f(\xi, \xi')$ is known.

10.2.2 Recovering the Normal Metal Result

Given this formula for the transverse conductivity, how can we return to the normal metal limit? If $\hbar\omega \gg \Delta$, then the energy gap will be inconsequential and we expect normal metal-like behavior. In fact, when $|\xi|, |\xi'| \gg \Delta$, with $\xi\xi' < 0$, $E \to \xi$ and the quantity $(uv' - vu') \to 1$. After a slight adjustment of the limits of integration to take advantage of the symmetry $\xi \leftrightarrow \xi'$, we recover the normal metal result obtained previously:

$$\sigma'_\perp \stackrel{\omega \gg \Delta}{\to} \sigma_n = \frac{e^2\pi}{\omega} \int_0^\infty d\xi \int_{-\infty}^0 d\xi' f(\xi, \xi') \delta\left(\omega - (\xi + |\xi'|)\right) \tag{10.44}$$

Previously, the prefactor 2 was due to the sum over degenerate spin states (Kramers Degeneracy). Here, since the Bogoliubov quasiparticles involve mixtures of spin states, the 2 is automatically accounted for.

10.2.3 More on Superconductor Conductivity

Using the normal metal result, we can now write the superconductor absorptivity in terms of the normal metal absorptivity and an integral

$$\begin{aligned}
\sigma'_\perp(\vec{q} \to 0, \omega) &= \frac{\sigma_n}{\omega} \int_{-\infty}^\infty d\xi \int_{-\infty}^\infty d\xi' \frac{1}{2} \left(uv' - vu'\right)^2 \delta(\omega - (E + E')) && (10.45)\\
&= \frac{\sigma_n}{\omega} \int_{-\infty}^\infty d\xi \int_{-\infty}^\infty d\xi' \frac{1}{4} \left(1 - \frac{\xi\xi'}{EE'} - \frac{\Delta^2}{EE'}\right) \delta(\omega - (E + E')) && (10.46)\\
&= 4\frac{\sigma_n}{\omega} \int_0^\infty d\xi \int_0^\infty d\xi' \frac{1}{4} \left(1 - \frac{\Delta^2}{EE'}\right) \delta(\omega - (E + E')) && (10.47)\\
&= \frac{\sigma_n}{\omega} \int_0^\infty d\xi \int_0^\infty d\xi' \left(1 - \frac{\Delta^2}{EE'}\right) \delta(\omega - (E + E')) && (10.48)
\end{aligned}$$

As a check of our sanity, we can look at the $\omega \gg \Delta$ limit of this integral. We expect that this should give us back the normal metal conductivity as before.

$$\begin{aligned}
\sigma'_\perp &= \frac{\sigma_n}{\omega} \int_0^\infty d\xi \int_0^\infty d\xi' \delta(\omega - (\xi + \xi')) && (10.49)\\
&= \frac{\sigma_n}{\omega} \int_0^\omega d\xi && (10.50)\\
&= \frac{\sigma_n}{\omega} \omega && (10.51)\\
&= \sigma_n && (10.52)
\end{aligned}$$

10.3 The London Penetration Depth

If we plot $\sigma'_\perp$ vs ω, we find that in the superconducting state $\sigma'_\perp = 0$ for $\omega < 2\Delta$, since no excitations are possible in this energy range. Above 2Δ, the curve rises rapidly to the normal metal value, and then decays away with a width on the order $1/\tau_{\rm el}$, the elastic scattering rate.

Such behavior is only visible experimentally if $1/\tau_{\rm el} \gg 2\Delta$, which means this is only the case for *disordered* superconductors. If the elastic scattering rate is very low compared with 2Δ, all of the spectral weight will simply collapse to a delta function as zero frequency.

Recall the Kubo formula

$$K = -i\omega\sigma \tag{10.53}$$

which implies

$$K'' = -i\omega\sigma' \tag{10.54}$$

Now, we would like to investigate the *real* part of the current response $K'(\omega = 0)$, as this is the quantity that related to the diamagnetic behavior of superconductors. Using Kramers-Kronig, we can relate the real and imaginary parts of K

$$\begin{aligned} K'(\vec{q} \to 0, \omega = 0) &= \int \frac{d\omega'}{\pi} \frac{K''(\omega')}{\omega'} && (10.55) \\ &= \int \frac{d\omega'}{\pi} \sigma'(\omega') && (10.56) \end{aligned}$$

The final expression is just the area under curve $\sigma'(\omega)$. Relating the normal metal and superconductor values,

$$\begin{aligned} K'_s - K'_n &= -\int d\omega' \left(\sigma'_s(\omega') - \sigma'_n(\omega')\right) && (10.57) \\ &\propto \sigma_n \Delta && (10.58) \\ &= \frac{ne^2\tau}{m}\Delta && (10.59) \\ &= \frac{ne^2}{m}(\tau\Delta) && (10.60) \end{aligned}$$

The proportionality comes from the fact that the difference between the superconductor and normal metal conductivity curves is the hole of width 2Δ and height σ_n that is removed in the superconductor conductivity curve due to the presence of the energy gap.

In the last line, we have separated the result into a product of the clean superconductor diamagnetism $\frac{ne^2}{m}$ and an addition factor of $\tau\Delta$ that arises from disorder. Returning to the linear response relation for the current, we get

$$\begin{aligned} \vec{j} &= -K\vec{A} && (10.61) \\ &= -\frac{ne^2}{m}(\tau\Delta)\vec{A} && (10.62) \\ &= -\frac{n_s e^2}{m}\vec{A} && (10.63) \end{aligned}$$

Thus we see that the effect of disorder is to change the superfluid density to

$$n_s = (\tau\Delta)n, \qquad \tau\Delta \ll 1 \tag{10.64}$$

Thus the effect of disorder is to dramatically reduce the superfluid density by the factor $\tau\Delta$. Accordingly, the current carried by the supefluid is also significantly reduced. A directly measurable consequence of this result is the increase of the London penetration depth

$$\lambda_L^{-2} = \frac{4\pi n_s e^2 \tau}{mc^2} \tag{10.65}$$

or

$$\lambda_L^{(\text{disorder})} = \lambda_L^{(\text{clean})} \cdot \frac{1}{\sqrt{\tau\Delta}} \tag{10.66}$$

The take home message of all this is that disorder reduces the superconductor's ability to cancel magnetic fields, allowing much greater penetration of the fields into the sample.