8.512 Theory of Solids II
Spring 2009

Lecture 6: Scaling Theory of Localization

6.1 Notion of dimensionless conductance.

$$g = \frac{G}{e^2/\pi\hbar} \quad \text{where} \quad G \text{ is (resistance)}^{-1}$$

$$G = \sigma L^{d-2} \qquad\qquad L = \text{dimension of system}$$

Recall Einstein's formula which relates conductivity to diffusion:

$$\sigma = 2e^2 \frac{N(0)}{\Omega} \cdot D$$

where

$$N(0) = \# \text{ of states/energy}$$

$$\Omega = \text{unit volume}$$

$$D = \text{diffusion coefficient}$$

Way to derive above equation is to calculate charge current density in two ways.

$$\boldsymbol{j} = -eD\boldsymbol{\nabla} \cdot n \qquad \cdots \qquad \text{current as diffusion of charge carriers}$$

$$\boldsymbol{j} = -\sigma\boldsymbol{\nabla}V \qquad \cdots \qquad V \text{ is local electrical potential}$$

$$= -\sigma \frac{dV}{dn} \cdot \boldsymbol{\nabla}n$$

Now, the local chemical potential is linearly related to V, i.e.

$$\mu = eV + \text{constant}$$

$$\therefore = \frac{dn}{dv} = e\frac{dn}{d\mu} = e \cdot \frac{2N(0)}{\Omega} \qquad \cdots \qquad 2 \text{ is for spin.}$$

$$\therefore j = -\frac{\sigma}{e} \frac{\Omega}{2N(0)} \cdot \boldsymbol{\nabla}n \tag{1}$$

Comparing this to Einstein's formula, one gets

$$\sigma = 2e^2 \frac{N(0)}{\Omega} \cdot D \tag{2}$$

$$\Rightarrow G = e^2 \frac{2N(0)}{\Omega} \cdot D \cdot L^{d-2} = \frac{2e^2}{\hbar} \cdot \frac{\hbar D}{L^2} \cdot N(0)$$

6.2 Thouless Energy

The last equation suggests to define a quantity called Thouless Energy, E_T

$$E_T = \frac{\hbar D}{L^2} = \frac{\hbar}{\tau_T} \tag{3}$$

Now,

$$N(0) \sim \frac{1}{\Delta} \qquad \cdots \qquad \Delta \text{ is level spacing}$$

$$\Rightarrow \quad G \sim \frac{2e^2}{\hbar} \frac{E_T}{\Delta} \qquad \Rightarrow \quad g \sim \frac{E_T}{\Delta}$$

Physical interpretation

Assume the one box problem is solved and we want to study the behavior of the system as it gets bigger. Each box has its own distribution of energy levels. As the two boxes are in contact, the wavefunctions that were initially localized in separate boxes would mix. For weak coupling, one can do perturbation theory if mixing is smaller than typical level spacing Δ. For weak mixing, the wavefunctions resemble the initial states and so one gets localized distribution. For strong coupling, the final state would be a complicated mixture of all initial states and we get extended states. Physically one can think of diffusion as lifetime to leak particle out of box. And so it's reasonable to think of Thouless energy as effective coupling. Thus the ratio E_T/Δ, which also represents the dimensionless conductance 'g,' characterizes the further scaling properties.

6.3 Sensitivity of energy eigenvalues to boundary conditions and its relation to E_T.

Here we derive another form of E_T. Usually one uses periodic boundary conditions

$$\psi_\alpha(x+L) = \psi_\alpha(x) \qquad \cdots \qquad \psi_\alpha \text{ is an eigenstate of the Hamiltonian.}$$

One can as well use the twisted boundary condition

$$\psi_\alpha(x+L) = e^{i\phi}\psi_\alpha(x) \qquad \cdots \qquad \phi \text{ is same for all } \alpha.$$

Such boundary conditions can arise if a magnetic field is present. Consider applying a uniform magnetic field (magnetic field is confined in flux tube) along the z direction. And we study the behavior of the eigenstates sitting on a cylindrical surface whose axis is parallel to the magnetic field.
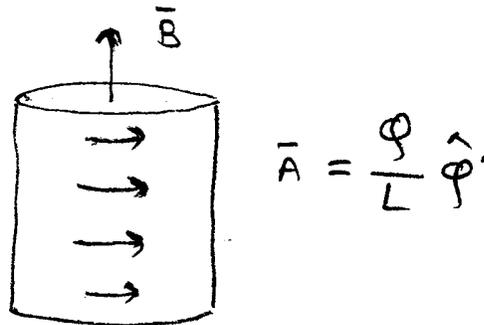
$$\frac{1}{2m}\left(\frac{\boldsymbol{\nabla}}{i} + \frac{e\boldsymbol{A}}{c}\right)^2 \psi_\alpha + V(\boldsymbol{r})\psi_\alpha = E_\alpha \psi_\alpha$$

$$\psi \to \psi' \exp\left[-i\int_o^x \boldsymbol{A}\cdot d\boldsymbol{\ell}\right]$$

$$\Rightarrow \frac{1}{2m}\left(-\nabla^2\right)\psi_\alpha'(\boldsymbol{r}) + v(\boldsymbol{r})\psi_\alpha'(\boldsymbol{r}) = E_\alpha\psi_\alpha'(\boldsymbol{r})$$

Thus the gauge transformation has removed $\boldsymbol{A}$ from the Schrodinger equation, but now $\psi_\alpha'(\boldsymbol{r})$ has different boundary conditions.

$$\begin{aligned}
\psi'(x+L) &= \psi'(x)\exp\left[i\oint_o^L \boldsymbol{A}\cdot d\boldsymbol{\ell}\right] \\
&= \psi'(x)\exp\left[i\phi_B\right]
\end{aligned}$$

where ϕ_B is magnetic flux. We will compute ΔE_α using perturbation theory.

$$H' = \frac{e}{c}V_x A = V_x \frac{\phi}{L}$$

To first order in H', $\Delta E_\alpha = 0$ as $<H'>\sim<V_x>=0$.

For perturbation theory second order in $\boldsymbol{A}$, we have two terms: the diamagnetic term

$$\frac{e^2}{c^2}\frac{A^2}{2m}$$

which gives a constant shift to all energy levels

$$\frac{1}{2}\frac{1}{m} = \phi^2$$

The paramagnetic term gives, to second order in A (or ϕ),

$$\sum_{\beta\pm\alpha}\frac{|<\beta|V_x A|\alpha>|^2}{E_\alpha - E_\beta} = \frac{\phi^2}{L^2}\sum_{\beta\neq\alpha}\frac{|<\beta|V_x|\alpha>|^2}{E_\alpha - E_\beta}$$

$$\therefore \frac{\partial^2 E_\alpha}{\partial\phi^2} = \frac{1}{mL^2} + \frac{2}{L^2}\sum_{\alpha\neq\beta}\frac{|<\beta|V_x|\alpha>|^2}{E_\alpha - E_\beta} \tag{4}$$

Thus one gets the same matrix element $<\beta|V_x|\alpha>$ that appeared in derivation of Kubo formula.

Typical variation in energy levels can look like this. As first order shift is zero, all curves have zero slope at $\alpha = 0$. All low lying states don't undergo much variation as they occur in potential valleys and hence localized.

One can do a sanity check on Eq.(4). On general grounds, there is equal probability for any level to go up or down and so average variation in E_α's should be zero. By using Thomas-Reich-Kunz F-sum rule, one can indeed show that

$$\sum_\alpha \frac{\partial^2 E_\alpha}{\partial\phi^2} = 0$$
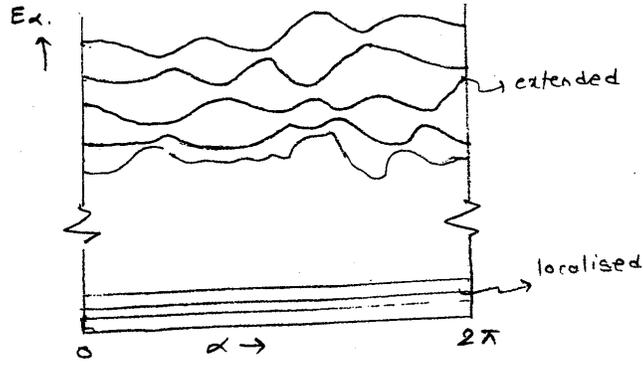
Now, $|<\beta|V_x|\alpha>|^2 \sim \overline{V^2}$. Fluctuation of such terms is dominated by the term with maximum weight, i.e., by smallest $E_\alpha - E_\beta$

$$\text{variance} \left(\sum_{\beta \neq \alpha} \frac{|<\beta|V_x|\alpha>|^2}{E_\alpha - E_\beta} \sim \frac{\overline{V^2}}{\Delta} \right)$$

This is the average fluctuation in the curvature.

Thus typically,

$$\left| \frac{\partial^2 E_\alpha}{\partial \phi^2} \right| \simeq \frac{1}{L^2} \frac{\overline{V^2}}{\Delta} \sim E_t$$

$$E_t \sim \frac{1}{L^2} \frac{\overline{V^2}}{\Delta} \sim \frac{1}{L^2} \overline{V^2} \cdot N(0)$$

Now,

$$\sigma = 2e^2 \pi \frac{N^2(0)}{\Omega} \cdot \overline{V^2} = 2e^2 \frac{N(0)}{\Omega} D$$

By comparing Boltzmann & Einstein formula for σ

$$\Rightarrow D = \pi \overline{V^2} N(0) \Rightarrow E_t = \frac{D}{L^2} \sim E_T$$

Thus for large E_T/Δ, one gets extended states and hence large conductance, while for small E_T/Δ, one gets localized states and hence small conductance. Thus we have established the relevance of Thouless energy for conductance.

6.4 Application to $1 - \Delta$ System
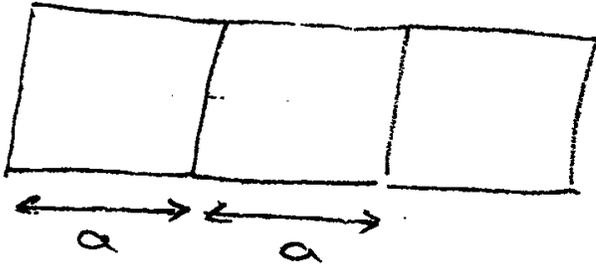
$$L = n \cdot a.$$

One expects Ohm's low behavior for small length.

$$g(L) \sim \frac{1}{L} \quad \text{as} \ \ L \ \text{increases}, g(L) \ \text{decreases}$$

as

$$g(L) < 1 \Rightarrow \frac{E_T}{\Delta} < 1$$

and so the states are localized.

$$\Rightarrow \quad \text{For large } L, \quad g(L) \sim e^{-L/\xi}$$

$$\cdots \xi \ \text{is the localization length}$$

Due to inelastic scattering the eigenvalues become ill-defined because of dephasing. Let us define a quantity τ_ϕ as the time over which a state looses the information about its phase. τ_ϕ for various scattering mechanisms can be given by

$$\frac{1}{\tau_{ee}} \sim \frac{T^2}{\epsilon_F} \qquad \frac{1}{\tau_{e-phonon}} \sim \frac{T^3}{\omega_D^2}$$

Let us define L_ϕ as the length over which the dephasing occurs.

$$L_\phi = \sqrt{D_o \tau_\phi}$$

So our argument works if

$$L_\phi >> \ \text{width of the sample.}$$

Otherwise we are considering a system to which our derivation based on Quantum Mechanics ideas doesn't apply and the system should rather be treated classically. And in such a situation, one used Ohm's law. So to observe non-metallic behavior of copper wire, one needs to have a very small wire thickness or very low temperature as L_ϕ increases with decrease in temperature.

Thus one expects the following behavior of conductivity of a one-dimensional system.
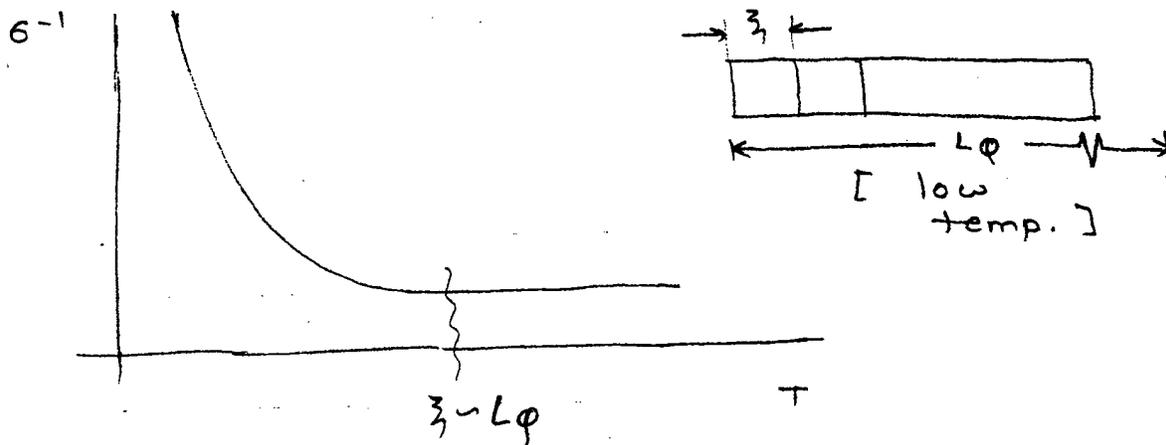
For $L_\phi > \xi$, we can think of conductivity in terms of hopping of electrons over a distance scale ξ in time τ_ϕ. This gives rise to a diffusion constant of ξ^2/τ_ϕ and resistivity which is proportional to $\tau_\phi \simeq T^{-p}$. [reference, book by Y. Imry]