Assignment 7

Due: Wednesday, April 18, 2012

This homework assignment discusses various aspects of saturation. It extends the treatment in class with examples and asks you to do a more rigorous derivation of results obtained in class using the optical Bloch equations. The problems may look longer, mainly because we give a lot of guidance on how to approach them.

1. **Saturation Intensity**

   We define the saturation intensity of a laser for an optical transition as the intensity (power/area) at which a monochromatic beam excites the transition at a rate equal to one half of its natural line width. In this problem, we compute the saturation intensity for the principal transition in sodium, 590 nm.

   a) Express the Einstein A coefficient by the oscillator strength f, the fine structure constant α and the transition frequency ω. Estimate the lifetime of sodium by assuming an oscillator strength of unity.

   b) Find the saturation intensity for the principal transition in sodium. Treat the atom as a two-level system, neglecting fine and hyperfine structure.

2. **Saturation of Atomic Transitions**

   In class we discussed excitation of atoms via weak radiation. In this limit the atom scatters incident radiation at a rate proportional to the light intensity, corresponding to a fixed cross-section.

   We also discussed the excitation of atoms via strong radiation and showed that in this limit the atom performs Bloch oscillations between the ground and excited states. Since during these oscillations the mean excited state population population is at most $1/2$ and the excited state decays with rate Γ, the atom can scatter at most $\Gamma/2$ photons per unit time. To obtain a fixed scattering rate, as the radiation intensity increases, the photon-scattering cross-section decreases, becoming very low at high light intensities.

   This problem will motivate this saturation of atomic transitions by considering broadband excitation. The obtained results can be exactly extended to narrowband transitions.

   a) In the case of broadband excitation, the atom dynamics is correctly described by the Einstein rate equations. Consider a two-state atom with $R_{ge} = R_{eg}$ the stimulated absorption/emission rate and $A = \Gamma$ the spontaneous emission rate. Define the saturation parameter s as $s = 2R_{ge}/\Gamma$. Show that in equilibrium the ratio of the excited state to the ground state populations is $N_b/N_a = s/(s+2)$.

   b) Express the equilibrium spontaneous emission rate per atom AN_b in terms of Γ and s. Show that the cross-section for photon absorption bleaches out as $\sigma(s) = \sigma(s=0)/(1+s)$.

   c) Find the energy density $\langle w \rangle_{SAT}$ per unit frequency corresponding to $s = 1$. Explain why $\langle w \rangle$ is independent of the atomic dipole matrix element $\langle g| er |e\rangle$.

d) Use the relationship between Einstein's A and B coefficients to obtain an expression for $\langle w \rangle_{SAT}$ independent of the atomic dipole. For $s = 1$, what is the mean occupation number n per photon mode?

e) Suppose that the light is provided by a laser beam of intensity I_0 and Lorentzian lineshape centered at the atomic transition frequency ω_0 and of FWHM $\Gamma' \gg \Gamma$. What is the energy density of this beam per frequency interval at ω_0? What beam intensity I_s corresponds to $s = 1$?

f) Let ω_R be the Rabi frequency corresponding to a monochromatic beam with the same intensity I_0 as the broadband beam. Show that the stimulated broadband absorption rate can be written as $R = \omega_R^2/\Gamma'$. What is ω_R^2 corresponding to $s = 1$?

g) If you set $\Gamma' = \Gamma$, you get exactly the saturation intensity of a monochromatic laser beam and the Rabi frequency at saturation. Argue why.

3. **Optical Bloch Equations with Spontaneous Emission**

Consider a two level system driven with Rabi frequency ω_R with damping rate Γ. We denote the ground state and the excited state of the atom as $|a\rangle$ and $|b\rangle$. In this problem, we compute the population fraction in the excited state $|b\rangle$ at the limit $t \to \infty$.

a) Let us begin by guessing the population in the excited state $|b\rangle$ in the limit $t \to \infty$ at the large detuning $|\delta| \gg \Gamma, \omega_R$. We estimate $\rho_{bb}(t \to \infty)$ by two different approaches.

   i. For $\Gamma = 0$ (without spontaneous emission), what is the excited state fraction, $\rho_{bb}(t)$, given by the solution for undamped Rabi oscillations? What do you expect will happen if a weak damping term is added to account for spontaneous emission? Guess the result for ρ_{bb} in the limit $t \to \infty$ by assuming that the oscillatory term will damp out to the average value.

   ii. Compare your guess with the result obtained for ρ_{bb} in the lowest order perturbation theory, that is exactly how we obtained the AC Stark shift. Is it the same or not?
   For this, assume that the two states, $|a, 1\,\text{photons}\rangle$ and $|b, 0\,\text{photons}\rangle$, are coupled by $H_{int} = -\vec{d} \cdot \vec{E}$ with $\vec{E} = i\sqrt{\frac{2\pi\hbar\omega}{V}}\hat{\epsilon}(a - a^\dagger)$. Identify the Rabi frequency as $\hbar\omega_R = 2\sqrt{\frac{2\pi\hbar\omega}{V}}\hat{\epsilon} \cdot \vec{d}_{ab}\sqrt{n}$.

b) In order to consider the effect of spontaneous emission properly, we need to consider the time-evolution of the density matrix for the system: $\rho = \begin{pmatrix} \rho_{aa} & \rho_{ab} \\ \rho_{ba} & \rho_{bb} \end{pmatrix}$.

The density matrix ρ consists of two parts: the population fractions (ρ_{aa} and ρ_{bb}) and the coherence of the system (ρ_{ab} and ρ_{ba}). Here, let us denote the damping rate for the population fraction (ρ_{aa} and ρ_{bb}) as Γ_1 and the damping rate for the coherence (ρ_{ab} and ρ_{ba}) as Γ_2. Then, the evolution of the system, including spontaneous emission, can be completely determined by the following equation of motion for the density matrix:

$$\dot{\rho} = \frac{1}{i\hbar}[H, \rho] + \begin{pmatrix} \Gamma_1\rho_{bb} & -\Gamma_2\rho_{ab} \\ -\Gamma_2\rho_{ba} & -\Gamma_1\rho_{bb} \end{pmatrix}.$$

where $H = \frac{\hbar}{2}\begin{pmatrix} -\omega_0 & \omega_R e^{i\omega t} \\ \omega_R e^{-i\omega t} & \omega_0 \end{pmatrix}$ and $\rho = \begin{pmatrix} \rho_{aa} & \rho_{ab} \\ \rho_{ba} & \rho_{bb} \end{pmatrix}$ with $\rho_{ab} = \rho_{ba}^*$ and normalization condition $\rho_{aa} + \rho_{bb} = 1$.

The above equations of motion for the density matrix are called the *optical Bloch equations* whose solutions and physical properties will be discussed in depth in 8.422 (the second half of the atom physics course). In this problem, we only obtain the steady state solution without solving the optical Bloch equations directly.

   i. By making the the substitutions $\hat{\rho}_{ab} = \rho_{ab}e^{-i\omega t}$ and $\hat{\rho}_{ba} = \rho_{ba}e^{i\omega t}$, obtain the following equations of motion for each element in the density matrix:

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$$\dot{\rho}_{aa} = i\tfrac{\omega_R}{2}(\hat{\rho}_{ab} - \hat{\rho}_{ba}) + \Gamma_1\rho_{bb}$$
$$\dot{\rho}_{bb} = -i\tfrac{\omega_R}{2}(\hat{\rho}_{ab} - \hat{\rho}_{ba}) - \Gamma_1\rho_{bb}$$
$$\dot{\hat{\rho}}_{ab} = (-i\delta - \Gamma_2)\hat{\rho}_{ab} + i\tfrac{\omega_R}{2}(\rho_{aa} - \rho_{bb})$$
$$\dot{\hat{\rho}}_{ba} = (i\delta - \Gamma_2)\hat{\rho}_{ba} - i\tfrac{\omega_R}{2}(\rho_{aa} - \rho_{bb})$$

ii. Show that the steady state solution for arbitrary δ, Γ_1, Γ_2, and ω_R is:

$$\rho_{bb} = \frac{\omega_R^2}{2} \frac{\frac{\Gamma_2}{\Gamma_1}}{\delta^2 + \Gamma_2^2 + \frac{\Gamma_2}{\Gamma_1}\omega_R^2}$$

c) In part b), we denoted the damping rate for the population fraction as Γ_1 and the damping rate for the coherence as Γ_2. Accordingly, the result we obtained depends on both Γ_1 and Γ_2. Now we need to represent Γ_1 and Γ_2 in terms of the spontaneous emission rate Γ.

    i. Consider the case where there is no driving force ($H = 0$). Then the density matrix ρ evolves as follows:

$$\dot{\rho} = \begin{pmatrix} \Gamma_1\rho_{bb} & -\Gamma_2\rho_{ab} \\ -\Gamma_2\rho_{ba} & -\Gamma_1\rho_{bb} \end{pmatrix}.$$

Solve for the density matrix $\rho(t)$ at time t. Use $\rho_{aa}(0)$, $\rho_{ab}(0)$, $\rho_{ba}(0)$ and $\rho_{bb}(0)$ as initial conditions.

    ii. Let us suppose that the atom starts out in a superposition state

$$|\psi\rangle = (\alpha_a(0)|a\rangle + \alpha_b(0)|b\rangle) \otimes |0\rangle \tag{1}$$

where $\alpha_a(0)|a\rangle + \alpha_b(0)|b\rangle$ is the atomic state and $|0\rangle$ represents the vacuum. At time t, it will be in a state

$$|\psi\rangle = \alpha_a(t)|a\rangle \otimes |0\rangle + \alpha_b(t)|b\rangle \otimes |0\rangle + \sum_k c_k(t)|a\rangle \otimes |1_k\rangle$$

where $|n_k\rangle$ is a n-photon state in mode k.

    Represent the density matrix $\rho(t)$ in terms of $\alpha_a(t)$, $\alpha_b(t)$ and $c_k(t)$. By comparing this with the density matrix $\rho(t)$ obtained in i, show that

$$\Gamma_1 = \Gamma \qquad \Gamma_2 = \frac{1}{2}\Gamma \tag{2}$$

when there is no driving force. Explain why the off-diagonal element decay at half rate of the excited population.

d) In fact, the relations $\Gamma_1 = \Gamma$ and $\Gamma_2 = \frac{1}{2}\Gamma$ we have obtained in c) still hold in the presence of the driving force.

    i. By using this, rewrite the steady state solution for ρ_{bb}. Also, represent your result in terns of the saturation parameter $s = 2\omega_R^2/\Gamma^2$ which is obtained in Problem 2. (You have now rigorously derived an important result that was obtained in class from Fermi's golden rule!)

    ii. Finally, obtain the population fraction at the large detuning limit: $|\delta| \gg \Gamma, \omega_R$. Compare your result with your guess in a).

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