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PROFESSOR: OK. Let's start. So for several lectures, we've been talking about the XY model, which is a collection of two component spins, for example defined on a square lattice, but it could be any type of lattice.

And today we are going to compare and contrast that with the case of a solid. More specifically for simplicity, an isotropic solid. And a typical example of an isotropic solid in two dimension is the triangular lattice. So something like this.

What both of these systems have in common is that there is an underlying continuous symmetry that is broken. In the case of the XY model, at low temperatures all of the spins would be pointing more or less in the same direction. Potentially with fluctuations, let's say, go around the direction where they're supposed to be pointing-- being, let's say, here in the vertical-- characterized by some kind of an angle. Let's call it theta.

And essentially the uniform state at 0 temperature can be characterized by any theta. And because of the deformations, it's captured through these low energy Goldstone modes. And the probability of some particular deformation will be proportional at low temperature to the integral, let's say, in d dimensions of gradient, the change in theta within different spins squared. Of course, with some considerations as we discussed about topological defects, et cetera, being buried in this description.

Now, similarly, away from 0 temperature the atoms of a solid will fluctuate around the position that corresponds to the minimum. And so the actual picture that you would see at any finite temperature would be distorted with respect to the perfect configuration and there would be some kind of a vector [INAUDIBLE] u at each

location that would describe this fluctuation.

Again, there is no cost in making a uniform distortion theta that is the same across space. And similarly here, there is no cost in uniformly translating things. So these would both imply that this row can [? continue ?] [INAUDIBLE] leads to Goldstone modes.

And the corresponding energy cost in the case of the isotropically distorted system is typically returned as an integral involving elastic moduli, as we've seen. And the traditional way to write that is in this form where the same reason here that it wasn't theta that was appearing but gradient of theta, because a uniform theta does not make any cross. These strain fields are related derivatives of the distortion.

But now we remember that this u is a vector and this strain field uij it was one half of the derivative of uj in the i direction derivative of ui in the j direction symmetrized. Actually, probably better to write this symmetrically as beta h equals to this quantity.

And then the typical type of calculation that we did for the XY model was to go in terms of the independent spin wave modes. So we go to Fourier space. This becomes an integral ddq 2 pi to the d. The derivative here gave q squared. And then I had something like the Fourier transformed theta squared.

Now I can also more clearly represent this in the Fourier description. I will get something like 1/2 integral vdq 2 pi to the d. And a little bit of work shows that this form Fourier transforms to something that is proportional to mu. Again, it has to be proportionality to q squared because the energy should go to 0 as q goes to 0 for very long wavelength modes.

And so we'll have something like q squared mu tilde of q squared. This is a vector squared. u is a vector. q is a vector. And there is another component that gives you mu plus lambda. And consistent with rotational symmetry, we can have a term that is q dotted with u tilde of q, the whole thing squared.

And again, you can reason that you can describe the isotropic system in terms of just two elastic moduli is that the only rotational invariant quantities are q squared, u

squared, and q.u, that those are the only terms that you can [? find. ?]

Now the next thing. Once you know this form, you can immediately say something about the fluctuations. So you say that expectation value of theta tilde q theta tilde q prime, if I assume that this is a Gaussian theory and these are the only fluctuations that I'm considering, the answer is going to be 2 pi to the d delta function q plus q prime. And then I will get a factor of 1 over kq squared.

The corresponding thing here is a bit more complicated. Because as we said, this quantity u tilde is a vector. And I can look at the correlation between, say, the i-th component of that vector along mode q, the j-th component of that vector along mu 2 prime and ask what this is. And again, only the same values of q are coupled together, so we get the usual formula over here.

If this term was the only term in the story, that is if we were dealing with vectors q and u that are orthogonal to each other, then we would just get the 1 over q squared. And so there is actually a term that is like that. But because of this other term, there is the possibility that q and u are in the same direction, in which case, these two costs will add up and I will get something that is proportional to 2 mu plus lambda. And so then I will get 2 mu plus lambda q squared.

And here I will have qi qj. This becomes q to the 4th. Let me make sure that I did not write this incorrectly.

Yes, I did write it incorrectly. It turns out that there is a mu plus lambda out here too.

But essentially there's just a small complication that we get because of the fact that you have a vector u. But overall the scaling is something that goes like 1 over q squared, which is characteristics of Goldstone modes here with additional vectorial things to worry about. Let me make sure I separate these two.

Now one of the things that we have been concerned with is whether these fluctuations destroy long range order, so for long range order the information that we would like to have is that the entirety of the system is roughly pointing in the same direction, meaning that if I know that this beam here is pointed in the vertical

direction, if I go very far away, it is still more or less pointed out-- pointing in the same direction.

So for that, we could look, for example, as the degree which [? with ?] these two spins? Yes, question?

AUDIENCE: [INAUDIBLE] q squared [INAUDIBLE]?

PROFESSOR: This is 4. Is that what you're worried about?

AUDIENCE: Yeah. It should be 2, I guess.

PROFESSOR: No, because I have two powers of q out front.

AUDIENCE: Right.

PROFESSOR: So I think in the notes what I have is that I have actually written things this way. I put the factor of mu q squared out front. And then I had this as qi qj over q. And I think--

AUDIENCE: [INAUDIBLE] q squared [INAUDIBLE]?

PROFESSOR: q squared. Let me one more time check to make sure that that's not-- actually there's a minus sign. All right.

AUDIENCE: Wouldn't there be another factor of mu to compensate for the factoring out?

PROFESSOR: Yeah. So let's check this in the following fashion. So I can look at modes where u and q are perpendicular to each other. When u and q are perpendicular to each other, I don't have this there. And so then what I should have is just a cost that is 1 over mu q squared.

OK. So let's see if that comes about. What I can do is let's say imagine that I'm looking at a q that is oriented along the x direction. And I look at u's that are in the y direction. And I correlate two u's in the y direction.

So I will definitely get a delta ij here. I would have a 1. Here I would get a factor of nothing, because this term is absent. The q's that I'm looking at don't have any

component along the direction of u that I'm having. So this is absent and I will get a 1 over mu q squared, which is consistent with that.

Now let's look at the case where I look at u's that are aligned with the vector q. Let's say again in the x direction. Then, as far as the energy is concerned, both terms are present and the cost should be 2 mu plus lambda. So the answer that I should ultimately get should be 1 over 2 mu plus lambda.

So let's check. So here q squared-- and cancels with this q squared. All of the q's are in the direction of the indices that I'm looking at. So that cancels.

This is 1. So I will get 1 minus mu plus lambda over 2 mu plus lambda. So then I will get 2 mu plus lambda minus mu minus lambda. So I will get a mu in front that cancels this. And I will be left with just the 2 mu plus lambda. So we've checked that this is correct.

All right. So this is a proof, if you like, by induction. You knew that the only forms that are consistent are delta ij and qi qj over q squared. So we can give them two coefficients, a and b, and then adjust a and b so that these two limits that I considered are reproduced. So-- OK. I didn't have to resort to my notes if I was willing to spend the corresponding three minutes here.

So going back here, in order to see the degree of alignment of the two spins that we have over here, what we need to do is to do is to look something like cosine of, say, theta at location x minus theta at location x prime, which is non other than the real part of the expectation value of e to the theta at location x minus theta at location x prime.

And assuming that everything is governed by this Gaussian rate, the answer is going to obtain by computing this Gaussian as exponential of minus 1/2 the average of theta x minus theta x prime squared.

So then my task is to replace theta x and theta x prime in terms of these theta tilde in Fourier space. And then this will give me some expectation that involves this average. And when I complete that, eventually I need to Fourier transform 1 over q

squared. And we've seen that the Fourier transform of the 1 over q squared is the Coulomb interaction.

And so ultimately this becomes exponential of minus of 1 over k. The Coulomb interaction as a function of x minus x prime with the appropriate cutoff included in this expression.

And then the statement that we always make is that this entity at large distances, the Coulomb interaction saying three dimension, falls off as 1 over separation. So in three dimensions, this at large distances goes to a constant 1 over k is something that is proportional to temperature. So after doing enough temperature, this entity at large distances goes to a constant.

So there is some knowledge about the correlation between these things that is left. Whereas in two dimensions and below, when I go to sufficiently large distances the Coulomb interaction diverges. This whole thing goes to 0 and information is lost. So this is the in content that Goldstone modes destroy through long range order in two dimensions and below.

Except that we saw that in two dimensions, because the Coulomb interaction grows logarithmically, this correlation fall off according to this description only as a power law with an exponent connected to k. And then the discussion that we had for the XY model was that at high temperatures you have exponential decay in two dimensions. At low temperature you have power law decay. Although there is no true long range order, there has to be a phase transition. And we saw that this transition was described by the unbinding of these topological defects.

So we would like to state that something similar to that is going on over here. The only reason I went through this was to figure out what the analog of this entity should be in the case of the solid. So the analog of our theta is this vector u, the distortion.

What should be the analog of e to the i theta? And the things that you should keep in mind is that theta, I can't really tell it's difference between theta and theta plus 2

pi, theta of 4 pi, et cetera. So this e to the i theta captures that. If theta goes to theta plus a multiple of 2 pi, e to the i theta remains invariant.

So it's a good measure of ordering. It's better than theta itself.

So similarly, there's a better measure of deviation from perfect order here for the case of the lattice. You say, well, why should I worry about that since I can measure what u is from the perfect position?

Well, my answer is this point here. Did it come from here or from there? Should you use for u this or should you use that? You don't know.

So the u that you have the to use similar to the theta that you're using is somewhat arbitrary. And the arbitrariness of it is given by unit vectors along the original lattice. That is, u and any u that is increased or decreased by some lattice vector of the original lattice is the same thing. You can't take them apart in the same sense that theta and multiple of 2 pi you cannot.

So then that suggest that I can construct some kind of an order parameter, I call it row G, which is like this. This e to i something. It's e to the iu but u is a vector. And what I do I multiple by a G, which is an inverse lattice vector. So G is an inverse lattice vector.

And essentially the inverse lattice vectors are defined to have the property that when you multiply them with any of the original lattice vectors-- which are, let's say, parametrized by two integers, m and n-- the answer is going to be some multiple of 2 pi.

So you can see that irrespective of whether I choose u as distortion from one lattice position or any other lattice position that happens to be nearby, this measure is going to be giving you the same face.

So in the same way as before for the spins where I asked whether spins at large distances are correlated long range, I can do the same thing here and define a long range correlations for a solid by looking at e to the iG dotted with, let's say, u at

position x minus u at position x prime. And x and x prime are two points on the lattice.

And again, the u's that I have are Gaussian distributed. So this I will complete the square. I will have exponential minus 1/2. Well, OK, let me be a little bit careful with this. I have G alpha G beta over 2 because I have two factors of G dot u. I will write them using Einstein's summation convention as G alpha times u alpha component times the u beta component.

So once more what we can do is to then write these u's in terms of their Fourier vectors. So what I will have is exponential minus G alpha G beta over 2. I will need to integrate over q's. Because of these factors, I will get something like 2 minus 2 cosine of qx minus x prime.

So whenever you go to Fourier space, you will get e to the iq x minus e to the iq x prime. Different one for here with different q primes, but when you take the average, q prime is set to minus q so then you get this form as usual. And then I will have the average Fourier transform use, which is what I wrote over there. I will get a 1 over mu q square delta i delta alpha beta minus q alpha q beta q squared mu plus lambda over mu plus lambda.

There is a little bit of subtlety here. So I will do something this is not quite kosher but gives me the right answer. Certainly I can take this G alpha a beta inside here. When it multiplies delta alpha beta, I will simply get G squared. I really G alpha G alpha. Sum over alpha. It will give me G squared.

Here what I will get is a G dot q squared because I will have two Gq's divided by q squared.

Ultimately I have to integrate over all values of q, including all angle values. So there will be, as far as the angle is concerned, something like a cosine squared here. And the rather unconventional thing to do is to angular averages that first and write this as G squared over 2.

Once you do that, then essentially you can see that this structure just gives you a

constant. And what you will need to do is the usual Fourier transformation of 1 over q squared. So eventually you can see that the answer is going to be proportional to minus G squared. I will get a factor of 2 mu u from here.

Once I have written this as G squared over 2, I will get 1 minus 1/2 of this quantity. So that becomes 2 mu plus lambda-- or 4 mu plus lambda minus mu minus lambda. So I will get a 3 mu plus lambda divided by 2 2 mu plus lambda. And so there's a factor of 4 here. So that's just the numerics that goes out front.

The key to the whole thing will be the usual integration of 1 over q squared. Just as we had before, I will get again the same Coulomb interaction as a function of x minus x prime with appropriate lattice cut-off introduced so that the short distance behavior is controlled.

This whole thing has a name. It's called a Debye-Waller factor. Well, almost. In 3D.

And the point is that if I'm looking at this whole thing in three dimensions, at large enough separation this will go to a constant, just as we had before. And I will conclude that this entity, as I look at two points that are very far apart, at least that's sufficiently low temperature and all of my mus and lambdas are scaled inversely with temperature, so this whole thing is proportional to temperature. Essentially, you can see that this information at large distance goes to a finite constant, telling me that there's long range orders preserved.

Again, when I go to two dimensions, you would say that this, at large distances, grows logarithmically. And so the correlations will fall off, but fall of as a power law.

Now for the case of a solid, usually what you do in order to probe for the order is you shine x-rays and you see what comes out. So essentially, you will see that there will be some scattering. And the way that the wave vector of the light has changed is characterized by a vector q.

And we discussed this already, that the structure factor s of q is going to be related to expectation value of the phase that you get from the different points. So I will get a factor of e to the i q. the location of the different atoms, let's call them r, of x. I

have to do a sum over x. I have to scatter from all of the atoms. There's some overall form factor that if I'm not really interested, but this whole thing has to be first squished.

So once I square this, I will be getting averages between pairs of points. Because of translational symmetry I can relate to that sum over pairs of points to a center of mass that typically would pick up the number of points that I have. And then the expectation value of e to the something like sum over all points, e to the i q.r r of x minus r of 0 average. This average I can certainly take out here.

Yes?

STUDENT: Sorry. Is r what you're calling u before?

PROFESSOR: No. So r is the actual location of where the particle is location. So r, let's say for a particle that at zero temperature was sitting at lattice point-- that is, let's say, in two dimensions labeled by m and n-- it is shifted by an amount that is u that I would assign to that point. So the r's that I put here are the actual physical position of the atom. Which is composed of where it was sitting at 0 temperature plus a little bit more.

So when I do this, indeed I will get a sum over all positions. This could be a label mn, whatever, of e to the iq. The difference between the two r's would be, first of all, the difference between these two at 0 temperature, which would be some lattice vector of some variety. Let's call r 0 of x.

And that quantity has [INAUDIBLE] fluctuations. The fluctuations go into e to the iq.u of x minus mu of 0.

Now if I'm doing this at 0 temperature or very close to 0 temperature where this is either 1 at 0 temperature or not too much fluctuating at high temperatures, I will essentially be adding a lot of e to iqr's. These are essentially phases. They can be positive, negative, all over the complex plane. And typically you would expect that for a randomly chosen q, the answer from adding all of these things will be 0.

So the only time we're close to 0 temperature, you expect this to be something that is significant. Scattering is when q is one of these inverse lattice vectors.

So if I was really doing this at 0 temperature, this would be 1. And the answer that I would get is essentially I would get these delta function back peaks at the locations that correspond to the inverse lattice spacing. And that's presuming something that you have seen in crystallography or whatever. You take a solid and you scatter light from it and you will back points.

So what I expect is that when I have a q that is roughly close to G but not necessarily sitting exactly at that point, what I would get is a scattering that these proportional to an integral over the entire lattice of something like how far I am away from the right position.

And I should really write this as a sum over lattice points, but I can approximate this by an integral. The integral again has the property that it will give me 0 unless q is exactly sitting at G.

And then for here, I evaluate this quantity when q is close to G according to what I have calculated up there. So I will get exponential of minus G squared-- What do I have? Over 4 mu, three mu plus lambda, 2 mu plus lambda. The Coulomb interaction as a function of x.

Yes?

STUDENT: Are you missing an overall factor of m?

PROFESSOR: That's why I wrote the proportionality. Because what I'm really interested is what you see. And what you see is how things are varying as a function of q.

So I'll tell you what the answer is. So you do the Fourier transform from this. And in d equals to 3, what you find at 0 temperature, as we said, you will get Bragg spots at the locations of G's that correspond to the inverse lattice vectors, which actually themselves form a lattice.

So this is essentially the space of q. And these are the different values of G.

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Now what happens is that the strength of each one of these delta functions is then modulated by this factor evaluated at the corresponding G. And since it is proportional to G squared, you find that the spots that are close to the origin are well defined. As you go further and further away, the become very weaker. And at some point you cease to see them.

So that's the-- the Debye-Waller factor describes how these things vanish as you go further and further away. And essentially, you find that they go and diminish as e to the minus G squared. But clearly, this kind of scattering from a crystal is very different from the scattering that you expect from an liquid, which is essentially you will have a bright spot at q close to 0 and things that would symmetrically fall off, maybe with a couple of oscillations due to the size of the particles, at cetera.

Now what happens in d equals to 2 is kind of interesting. Because in d equals to 2, I'm doing a two dimensional integral, and this thing is growing as a power law. So basically what I end up having to deal with is something like e to the i q minus G x. And something that falls off as a over x to some power that I will simply cause eta G. So related to that combination.

And when you integrate this, just dimensionally you can see that it is an integral that scales as x to the power of 2 minus eta. x is inverse to q minus G. So this is going to scale as 1 over q minus g to the power of 2 minus eta.

So as you go along some particular direction and ask what you see, you find that will see peaks close to the locations of the lattice vectors. So let's say they are equally spaced along the particular axis that I'm picking.

In three dimension, you would have been seeing delta function at each one of these positions whose strength is governed by this Debye-Waller factor. In two dimensions, you will essentially see power laws close to each one of them. But as you go further, the exponent of the power law changes. So at some point you even cease to have a divergence. But you can essentially ignore anything like that [? happening. ?]

So you can see that because we don't have true long range order into two dimensions because of these logarithmic divergences, et cetera, the traditional picture that we have of a solid that is characterized by Bragg scattering and delta function peaks is modified, but still it is something that looks very different from the liquid.

So we expect that as we raise temperature there is a phase transition between scattering of this form and something that looks like liquid. And you can roughly guess that essentially it is going to be related with these etas becoming larger and larger so that these divergences disappear.

But what is the mechanism? So let's go back and see what the mechanism was that we discovered for the XY model. So we are going to look at d equals to 2 from now on. That discussion so far was general.

In d equals to 2, we said that we have topological defect. And the story here was that, as I have been discussing, the angle is undefined up to 2 pi. So it would configuration of angles that kind of radiate away from the origin.

And the idea was that if I take a circuit and integrate around this circuit-- let's call this the s gradient of theta-- the answer does not need to come back to 0. It will be some multiple of 2 pi . Some integer multiple.

And similarly here, we said that the value of u is undefined up to a lattice spacing. That leads to topological defects, which I find it very difficult to draw for the triangular lattice, so I'll draw it for the square lattice.

So at each one of these positions there's a particle sitting. And you look out here. You have a perfect square lattice. You look out here. You have a perfect square lattice.

But clearly it is not a perfect square lattice. And the analog of the circuit that I drew over here that encloses a singlularty-- there's clearly some kind of a defect sitting over here-- is that I can start with some point on the lattice and perform what is

called a Burgers Circuit.

What Burgers Circuit is is some walk that you would take on a lattice, which in a perfect lattice would bring you back to your starting point. So for example-- that was not a good starting point. Let me start from here.

Let's say I take four steps up-- one, two, three, four. I take five steps to the right. One, two, three, four, five. I reverse my four steps. So I go four steps down. One, two, three, four. I reverse my five steps and I go five steps left. One, two, three, four, five.

And you see that I did not end up where I was. I ended up here. And a failure to close my circuit has to be a lattice vector. And this is called d and it's called a Burgers Vector that characterize this defect, which is called a dislocation.

So the uncertainty that I have in assigning a value for u because of being on a lattice or deforming a lattice gives rise to these kinds of ambiguity that are similar to the topological defects that you had over here. Whereas the defects here were characterized by an integer, the defects here are characterized by a lattice vector.

And here the simplest things were, say, plus or minus. Clearly here on the square lattice the simplest things are plus b, minus b along the x direction and plus b minus b along the y direction. For example, on the triangular lattice it would be plus minus at any of the three directions that would define the lattice.

So calculating the distortion field here was actually an easy matter because we said that the gradient of theta was something like 1 over r. And so the gradient of theta, which was 1 over r, I could in fact write-- we saw as vector that is orthogonal to the vector that is going out of the plane. So we call that a z hat. It was also orthogonal to the direction of motion away from the defect, which we could characterize by looking at the gradient. The gradient vector for this distortion is clearly in the radial direction.

And the answer that we had here for one defect was ni log of r minus ri. And since I'm taking the gradient it doesn't matter. I can't put or not put the cut off. And if I had

multiple ones, I would simply sum over it.

So this was the contribution to the distortion that came from a collection of defects such as this. Now you can see that essentially in some continuum sense, all I'm doing here is similar. I can be taking a big circuit. And as I go along that big circuit, I can take a gradient-- So let's-- yeah. The s. gradient of the distortion field.

Distortion field is, of course, a vector. So it has components that I can write as the x component or the y component. So this alpha could be the x component of the distortion or the y component of the distortion. And once I complete the circuit, here the answer was 2 pi n. Here the answer is a lattice vector, which is this Burgers Vector that could be in any direction that you want.

Now as far as mathematics is concerned, this line and this line for each component are exactly the same. So the solution that I can write for gradient of u I can just copy from here for each component. So I can write that gradient of the u that is due to a collection of dislocations is something like z hat crossed with curl of a sum over all of the potential locations off my defect. Rather than n, I have b over 2. And it's a vector log of minus ri over [INAUDIBLE].

So if I have a collection of these dislocations at different places on my lattice-- locations ri, strengths, b alpha which is a vector-- then the distortion field that they would generate is given by this. It's very just following the answers that the [INAUDIBLE].

So then what we did for the case of the overall system-- in order to find how the different topological defects interact with each was to calculate beta h, which was an integral of gradient of theta squared. Yes?

STUDENT: So the b alpha, you have an index i [? also? ?]

PROFESSOR: b alpha have index i also. Like like the ni have index i. So maybe I write it in this fashion. Yeah.

So this is the cost that we have to evaluate, except that we have to be somewhat

careful with the meaning of this gradient of theta in that the gradient of theta, as we said, has a contribution that is from regular spin waves, the Goldstone modes, that can be deformed back to everybody pointing in the same direction without topological defect. Plus a contribution due to this topological defect that are categorized in that case by ni.

Now similarly here for the case of the solid, we have the beta h. Slightly more complicated integral d 2x. We have mu uij uij plus lambda over 2 uii ujj. And what we can do is to say that our strain field uij has a component that is like the Goldstone modes that we have been calculating so far, essentially treating everything as Gaussians. Except that I will write it as phi ij.

And then a part that would come from the distortion field u bar that I calculated. So I take that distortion field and then take derivatives of it to symmetrize appropriately to construct the strain that sums from them.

So we substituted this form over here. And we found that beta h that we got had a part that was simply relate to the Goldstone modes. This was the part that could be treating as a Gaussian.

And then we had a part that corresponded to the interactions among these defects. And that part we saw had the character of charges ni and nj plus minus 1 characterizing these topological defects having a Coulomb interaction [? between them ?].

And then, of course, this i less than j, we had to worry about what was happening when i was j. And that we considered to be the contributions of the core energy that, once exponentiated, we described as y. So it's beta h which is log of that would be log of y.

And actually this k bar, by the way, was simply 2 pi k. And it was 2 pi k because each one of these charges is 2 pi n. So there's 2 pi 2 pi. But the Coulomb interaction really should be log divided by 2 pi. And so k bar becomes 2 pi k times this.

So I can do the same thing over here. And what I find happens is that my beta h

gets decomposed as follows. There will be a part that is simply the original expression now for the field that is well behaved and has no dislocation piece.

And then you wouldn't be surprised because this structure is no different from the other structure. There's essentially a gradient of this distortion field squared. And you can see that if I take a second derivative of this distortion field or one derivative of this, effectively I will have two derivatives of a log, which will give me a delta function. So essentially these things do behave like what you would get from Coulomb type of potential. [INAUDIBLE] plus [INAUDIBLE] if the potential is 0.

So not surprisingly, I we get a minus k bar sum over pairs of where these dislocations are located. And then I will have b at location i, b at location j. And then I would have a log of ri minus rj with some cut off.

Except that these b's are actually vector. So this term is a dot product of the two b's. And it turns out that when you go through the algebra, there is another term, which is bi. ri minus rj, bj dotted with ri minus rj divided by ri minus rj squared.

So the charges that we had in our original theory were scalar quantities. These defects were characterized by a scalar value of n. And they were interacting with something that was like the ordinary Coulomb potential. Whereas now we are looking at a system that is characterized by charges that are vectors. And it turns out that what we have here is the vectorial analog of the Coulomb potential.

And again, if you think about an isotropic system and vectors, this is a vector that you can form, but b.r is another vector that you can form. And so both of them do appear. And once you go through the whole algebra and through the inverse Fourier transform, et cetera, you get this additional contribution to this vector Coulomb interaction.

That's really is the only difference. And of course, you will again get a sum over all locations of the core energies for creating these defects. And the value of k bar here is related to these parameters by mu mu plus lambda divided by pi 2u plus lambda.

So this combination controls what you have.

So the next thing that we did was to construct a perturbation. We essentially said that if I have these pairs of charges that can spontaneously appear at an energy cost but an entropy gain, they will effectively weaken the overall Coulomb potential. For example, if I have to test charges then there will be some polarization of the medium, some reorientation of these charges that appear, that weakens the effective charge here.

And we could perturbatively calculate what the correction was, except that we found that that correction, which involves an integration over the separation of these things. There was an integration that was potentially divergent for us no matter how small we made the core energies here.

And so what we did was eventually to construct an RG. And the RG was that the value of this parameter k bar-- actually more usefully its inverse that is related to temperature-- changed as a function of integrating out short distance degrees of freedom. It was something like 4 pi-- I think it was 4 pi cubed, it doesn't matter-- y squared.

And that the scaling of to core energy itself was determined by 2 minus pi k, which in terms of this k bar becomes 2 minus k bar over 2 times y. And we can do exactly the same thing over here, except for the complication of having to deal with the charges that are vectorial.

And you get that the strength of the Coulomb interaction, it's inverse that is related temperature gets re-normalized by exactly the same process. That is, there will be charges that will appear. All the mathematics you can see is clearly similar. And there will be a reduction, which then you can think of as an increase in temperature, which will be proportional to y squared. I don't know what the constant of proportionality is.

And that dy by dl-- so essentially I focus on the simplest type of dislocations that I would have of unit spacing-- has exactly the same [INAUDIBLE], 2 minus k bar over 2 y.

It turns out that there is actually a difference. And the difference is as follows. That while we did not calculate the next correction in this series, I said that this correction came from essentially what would a single dipole do, which would be some contribution that is order of y squared.

And I expect that there will be correction if I look at a configuration that has pairs of dipoles. And this will change this by order of y to the fourth and this by order of y cubed.

Whereas if I look at something like a triangular lattice, you can see that if I have two dislocations out there as test dislocations pointing out in opposite directions, which would have some kind of a interaction such as this. And I ask how that interaction is modified by the presence of dipoles of dislocation in the medium.

I could put a pair of dislocations integrate over the separation between them. I will get, essentially, the same type of mathematical structure that would ultimately give me this.

But then the next order term in this series is not for dislocations because I can have a neutral configuration of three dislocations, such as this. And so once you do that, you will find that the next correction here is order of y cubed and order of y squared here.

But generically, both of them have a phase diagram for RG flows that involves the inverse of the interaction of these charges. That is a temperature-like quantity. And there is a critical value of that inverse, which is 1/4. And what happens is that-- let's see, this other axis is y.

Anything smaller than 1/4 y tends to go to 0. Anything larger, y tends to get larger. And there is going to be some kind of a separatrix that he describes the transition between flows that go down here and the flows that go away.

And presumably, as I heat up my original system-- my lattice, different types of lattices-- at low temperature, I can figure out what this combination of mu an lambda

is. That gives me some value of k. And there is some kind of the core energy for the vertices that I can calculate.

So there will be some point in this phase diagram, which at low enough temperature I will go over here, which corresponds to a system that has an effective logarithmic interaction between dislocations.

As I change the temperature, I will proceed around some trajectory such as this because everything will change. And presumably, at some point I will intersect this and I will go to the other phase.

Now the characteristic of the other phase was that this parameter k was going to 0. It was essentially the logarithmic interaction was disappeared. And you can see that that parameter k going to 0 means that mu, the shear modulus, has to go to 0. So basically this phase out here is definitely characterized by 0 shear modulus.

So again, the two parameters mu and lambda, mu is the one that gives you the cost of trying to shear in a system. And essentially the presence of the resistance to shear is what defines a solid for you. So this is a solid to something that does not have rigidity transition.

It turns out that there is one subtlety here. I don't know whether it's good to mention or not. But so in the case of alpha this superfluidity, for example, we said that this coupling strength as you approach the transition rose to a universal value. In these units it will be 4. But that it approaches that universal value with a logarithmic singularity. Sorry, with a square root singularly. Sorry.

And we saw that this was experimentally observed in films of superfluid film. I also said that on the other side you have a finite correlations. The correlations will be decaying exponentially. And this correlation length has this very unusual signature, which is that it diverged as a square root of t minus tc in the exponent. It was giving rise to these essentially singularities.

While everything kind of looks identical between this RG and the other RG, it turns out that once you keep track of these corrections, these corrections actually are

important. And for the vectorial version of the Coulomb gas, you find that they shear modulus that is finite in this phase reaches its final value, which is not universal because the thing that the universal is that combination that involves both mu and lambda.

But rather than stc minus t to the 1/2, stc minus t to some exponent that is called mu bar, which is 0.36963 [INAUDIBLE]. So this is a kind of subtle thing that is buried in these recursion relationships once you go to a higher order. And accompanying that is a correlation length that also diverges with some behavior dominated by the same exponent.

So once people understood the original [? Koster ?] [INAUDIBLE] transition, it was kind of a very natural next step for them to think about these locations in a two dimensional solid and say that in the same manner that the unbinding of these defects got rid of the residual long range order that was left in the XY model. That this kind of ordering that we said exists for the two dimensional solid, this appears as a result of unbinding of dislocations and you get a liquid.

Turns out that that's not correct. And it was pointed out by Bert Halperin I think that if you look at a solid, there's two things that you. You have the translational order, but there's also orientational order. That is, you can look, for example, at a collection of spins here and the bonds are pointing along the x and y directions and you can look down here and the bonds are pointing in the x and y directions.

And from the picture that I drew over here, you can kind of see that once I insert this dislocation which corresponds to an additional line, the positions here are distorted. Once I have many of these inserted, it is kind of obvious that I will lose any idea of how the position of this lattice point and something out there is related.

It's not so obvious that inserting lots and lots of these lines will remove the knowledge that I have about the orientation of the bonds. And the insight that we will follow up-- and maybe it's probably a good idea to do it next time-- is that actually once the dislocations unbind, we will lose this kind of ordering. If I do this in two dimensions, these are power law decays at low temperatures.

Once the dislocations unbind, this becomes an exponential decay, as I would expect. But I can also define locally some kind of an orientation. Let's again call it theta. But theta is, let's say, the orientation of the bond that connects two neighboring spins.

Now what I can do is to look at the correlation of orientations at different positions. And it turns out that if I'm doing something like a triangular lattice, I can't just pick the angle and correlate the angle from one point to another point because let's say on the square lattice the angle itself, I don't know whether it was coming from a bond that was originally in the x direction or in the y direction. So this is unknown up to a factor of 90 degrees. In the triangular lattice it is an unknown up to a factor of 60 degrees.

So what I should do is I should define e to the 6 i theta at each location as a measure of the orientational order. You can see that, again, if I'm at 0 temperature, I can reorient the lattice any way that I like. This phase would be the same across the entire system.

So what I'm interested is to find out what is the expectation value of this object when I look at points that are further apart. And again, clearly if I'm at 0 temperature these thetas are the same. This will go to 1 at finite temperature. Because of the fluctuations it will start to move a bit.

And what we can show-- and you will do it next time-- is that as long as you are in the 0 temperature phase you will find that this goes to a constant, even in two dimensions. But once the dislocations unbind, what we will show is that it doesn't decay exponentially, but then it starts to decay as 1 over x to some power, let's call it eta theta. And then at some higher temperatures, it eventually starts to decay exponentially.

So there is the original solid in two dimension that we started that has true long range order in the orientations. And that is reflected in these Fourier transforms that we make. So we said that when we do the Fourier transform and look at the x-ray

scattering, let's say from a two dimensional crystal, you were seeing these kinds of pictures. And then these were becoming weaker as we went further and further out. But now you can see that this picture clearly has a very, very defined orientational aspect to it.

Now once we go to this intermediate phase, all of these peaks disappear. But what you find is that when you look at this, rather than getting a ring that is uniform, you will a see a ring that has very well- defined variation. One, two, three, four, five, six. Six-fold symmetry to it.

So this phase has no knowledge of where the particles are located, but knows the orientations. It's a kind of a liquid crystal. It actually has a name. It's called hexatic. But it's one of the family of different types of materials that have no translational order but some kind of orientational order that are hexatics.

So the transition between this hexatic phase to this fully disordered phase, it turns out in two dimension to be another one of these dislocation-- well, topology defect unbinding transition. So maybe we will finish with that next time around.