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PROFESSOR:

OK, let's start. So looking for a way to understand the universality of phase transitions, we arrived at the simplest model that should capture some of that. That was the Ising model, where at each site of a lattice-- and for a while I will be talking in this lecture about the square lattice-- you put a binary variable. Let's call it σ_i that takes two values, minus plus 1, on each of the N sites. So we have a total of 2^N possible configurations. And we subject that to an energy cost that tries to make nearest neighbors to be the same. So this symbol stands for sum over all pairs of nearest neighbors on whatever lattice you have. Here the square lattice.

And the tendency for them to be parallel as opposed to anti-parallel is captured through this dimensionless energy divided by Kt parameter K . So then in principle, as we change K we could also potentially add a magnetic field. There could be a phase transition in the system. And that should be captured by looking at the partition function, which is obtained by summing over the 2^N possibilities of this weight that is $e^{-\sum_{ij} K \sigma_i \sigma_j}$. So that's easier said than done. And the question was, well, how can we proceed with this?

And last lecture, we suggested two routes for looking at this system. One of them was to start by looking at a low-temperature expansion. And here, let's say, we would start with one of two possible ground states. Let's say all of the spins could, for example, be pointing in the plus direction. That is certainly the largest contribution that you would have to the partition function at 0 temperature. And that contribution is all of the bonds being satisfied. Each one of them giving a factor of e^K .

Let's say we are on a square lattice. On a square lattice, each site has two bonds associated with it. So this will go like $2N$. Since we have N sites, we will $2N$ bonds.

There is actually, of course, two possibilities. We can have either all of them plus or all of them minus. So there is a kind of trivial degeneracy of 2, which doesn't really make too much of a difference. And then we can start looking at excitations around this. And so we said that the first type of excitation is somewhere on the lattice we make one of these pluses into a minus. And once we do that, we have made 4 bonds that go out of this minus site unhappy. So the cost of going from plus k to minus K , which is minus $2k$, in this case is repeated four times.

And this particular excitation can be placed in any one of N locations on the lattice. So this was this kind of excitation. And then we could go and have the next excitation where two of them are flipped. And we would have a situation such as this. And since this dimer can point in the x - or the y -direction on the square lattice, it has a degeneracy of 2.

I have e to the minus $2K$. And how many bonds have I broken?

One, two, three, four, five, six. Times 6. And I can go on. So this was a situation such as this.

A general term in the series would correspond to creating some kind of an island of minus in this sea of pluses. And the contribution would be e to the minus $2K$ times the perimeter of this island. So that would be a way to calculate the low-temperature expansion we discussed last time around.

We also said that I could do a high temperature expansion. And for this, we use the trick. We said I can write the partition function as a sum over all these 2 to the N configuration. Yes.

AUDIENCE: Is there a reason we don't have separate islands?

PROFESSOR: Oh, we do. We do. So in general, in this picture I could have multiple islands. Yes. And what would be interesting is certainly when I take the log of Z , then the log of Z here I would have NK . I would have $\log 2$. And then there would be a bunch of terms.

And what we saw was those bunches of terms, starting with that one, can be captured into a series where the n comes out front and the terms in the series would be functions of e to the minus $2K$. And indeed, when we exponentiate this, this would have single islands only. The exponential will have multiple islands that we have for the partition functions.

So these terms, as we see, are higher powers of n because you would be able to move them in different directions. When you take the log, only terms that are linear and can survive. So this sum over bond configurations I can write as a product over bonds. And the factor of e to the $K \sum_i \sum_j$, we saw that we could capture slightly differently. So e to the $K \sum_i \sum_j$ I can write as hyperbolic cosine of K plus hyperbolic tanh of $K \sum_i \sum_j$. And this t is going to be my symbol for hyperbolic tanh of K , so that I don't have to repeat it all over the place.

So this is just rewriting of that exponential, recognizing that it has two possibilities. We took the $\cosh K$ to the outside. And if I'm, again, on the square lattice, I have $2N$ bonds so there will be $2N$ factors of $\cosh K$ that I will take into the outside. And then I would have a series, which would be these terms that I can start expanding in powers of t .

And the lowest order I have 1. And then we discussed what kinds of terms are allowed. We saw that if I take just one factor of $t \sum_i \sum_j$, then I have a σ_i sitting here and σ_j sitting here. I sum over the two possibilities of σ being plus or minus. It will give me 0. So there is a contribution order of t if I expand that, but summing over σ_i and σ_j would give me 0. So the only choice that I have is that this σ that is sitting out here I should square.

And I can square it by putting a bond that corresponds to this σ and that σ . This became σ squared. I don't have to worry about that. Then I can complete this so that that's squared and this so that that's squared. And so this was a diagram that contributed N this quantity t to the fourth. Well, what's the next type of graph that I could draw?

I could do something like this. And that is, again, something that I can orient along

the x-direction or along the y-direction. So there is a factor of 2 from the orientation. And that's how many factors of t. It is 6. And so in general, what do I have?

I will have to draw some kind of a graph on the lattice where at each site, I either have 0, 2, or potentially 4. There is no difficulty with 4 bonds emanating from a site because that sums to the fourth. Basically, those kinds of things actually will give me then the sum over-- possibilities of sigma will give me 2 to the number of sites.

And then these graphs, basically will get a factor which is 2 to the number of bonds in graph. And then again, you can see that here I could have multiple loops, just like we discussed over there. Multiple loops will go with larger factors of N.

The thing that we are interested in is $\log Z$, which is $N \log 2 \cos^2 K$. And then I have to take the log of this expression, and I'll call it g of t . Yes.

AUDIENCE: How did we get rid of [INAUDIBLE] bonds?

PROFESSOR: OK. So this $e^{K \sum_i \sigma_i}$ has two possible values. It is either e^{K} or e^{-K} . And I can write it, therefore, as $\frac{e^K + e^{-K}}{2} + \frac{e^K - e^{-K}}{2} \sum_i \sigma_i$, right?

AUDIENCE: My question was--

PROFESSOR: Yeah, I know. And then, this became $\cosh K + \sum_i \sigma_i$. So when I draw my lattice, for the bond that goes between sites i and j , in the partition function I have this contribution. This contribution is completely equivalent to this. And this is two terms. The $\cosh K$ we took outside. The two terms is either 1 or $\sigma_i \sigma_j$. There isn't anything that is multiple lines.

And as I said, you can make separately an expansion in powers of K . This corresponds to nothing, or going forward and backward, or going forward, backward, et cetera, if you make an expansion in powers of K . This term corresponds to going once.

Essentially, this captures all of the things that stay back to the same site and this is a re-summation of everything that steps you forward. So everything that steps you

forward and carries information from this sites to that site has been appropriately taken care of. And it occurs once. And that's one reason.

And again, if you have three things coming at a particular site, it's a $\sum_i q_i$. So these are completely the only thing that happen. But in principle, this is one diagrammatic series. This is one diagrammatic series. But you stare at them a little bit and you'll see why I put the same g for both of them.

On the square lattice, they're identify series. See that series had N something to the fourth power, $2N$ something to the sixth power, N something to the fourth power, $2N$ something to the sixth power. The first two terms are identical. You can convince yourself that all the terms will be identical also, including something complicated, such as if I were to draw-- I don't know. A diagram such as this one, which has 2 or 4 per site.

I can convert that to something that had spins plus out here, then it minus here, minus here, plus here. And it's a completely consistent diagram that I would have had in the low-temperature series. So you can convince yourself that there is a one-to-one correspondence between these two series. They are identical for the square lattice.

As we will discuss, this is a property of the square lattice. So you have this very nice symmetry that you conclude that the partition function per site, the part that is interesting, either I can get it from here as K plus this function of e to the minus $2K$ or from the high temperatures series as $\log 2$ hyperbolic cosine squared K . Plus exactly the same function of $\tanh K$.

This part of it, actually, I don't really care because these are analytical functions. I expect this model to have a phase transition. The low-temperature and the high-temperature behavior should be different. It should be order at low temperature, [INAUDIBLE] at high temperature. There should be a phase transition and a singularity between those two cases. The singular part must be captured here. So these are the singular functions.

So the singular part of the free energy has this interesting property that you can evaluate it for some parameter K , which is large in the low-temperature phase, or some parameter t , which is small in the high-temperature phase. And the two will be related completely to each other. They are essentially the same thing. So this property is called duality.

And so what I have said, first of all, is that there is a relationship between, say, the coupling $\tanh K$ and a coupling that I can separately call K tilde, such that if I evaluate the high-temperature series at K , it is like evaluating the low-temperature series at K tilde, where K tilde of K is minus $1/2$ log of the hyperbolic tanh at K .

I can plot for you what this function looks like. So this is K . This is what K tilde of K looks like. And it is something like this.

Basically, strong coupling, or low temperature, gets mapped to weak coupling, or high temperature, and vice versa. So it's kind of like this, that there is this axes of the strength of K going from low temperature, strong, to high temperature, weak.

And what I have shown is that if I start with somewhere out here, it is mapped to somewhere down here. If I start from somewhere here, it would be mapping to somewhere here. So one question to ask is, well, OK, I start from here. I go here. If I put that value of k in here, do I go to a third point or do I come back to here?

And I will show you that it is, in fact, a mapping that goes both ways. And a way to show that is like this.

Let me look at the hyperbolic sine of $2K$. Hyperbolic sines of twice the angles, or twice the hyperbolic sines of K hyperbolic cosine of K . All my answers are in terms of $\tanh K$, so I can make this sine to become a \tanh by dividing by \cosh , and then making this \cosh squared. So that is $2 \text{ hyperbolic tanh of } K$. And then, there's various ways to sort of remember the identity for hyperbolic cosine squared minus sine squared is 1.

If I divide by c squared, it becomes $1 - t$ squared is 1 over c squared. So the hyperbolic cosine squared here is the inverse of $1 - \text{hyperbolic tanh squared of}$

K.

Now, for $\tanh K$, we have this identity, $\frac{1 - e^{-2K}}{1 + e^{-2K}}$. 1 minus the square of that.

If I multiply both sides by e^{-2K} in the numerator and denominator by e^{2K} , hopefully you recognize this as $\frac{1 - e^{-2K}}{1 + e^{-2K}}$. So the identity here that was kind of not very transparent, if I had made the change of variables to the hyperbolic sine of twice the angle, had this simple form.

So then the symmetry between K and $2K$ is immediately obvious. You pick one value of K , or $\sinh K$, and then the inverse. And the inverse of the inverse, you are back to where you are. So it is clear that this is kind of like an $x \rightarrow 1/x$ mapping.

$x \rightarrow 1/x$ mapping would also kind of look exactly like this. In fact, if instead of K and $2K$ I had plotted hyperbolic sine versus hyperbolic sine of twice the angle, it would have been just the $1/x$ curve.

Now, just to give you another example, if I had the function $f(x)$, which is $x + x^2$. This function, if I divide by x^2 , becomes $x^{-1} + x$. So this is, again, $f(x^{-1})$.

So if I evaluate this function for any value like 5, then I know the value exactly for $1/5$. If I evaluate it for 200, I know it for $1/200$, and vice versa. Our g function is kind of like that.

Now, this function you can see that starts to go linearly increases with x , and then eventually it comes down like this must have one maximum. Where is the maximum?

It has to be 1. I don't have to take derivatives of everything, et cetera. If there is one point which corresponds to the maximum, it's the point that maps to itself.

Now, this function for the Ising model, I know it has a phase transition. Or, I guess it

has a phase transition. So there is one point, hopefully one point, at which it becomes singular. I don't know, maybe it's three point. But let's say it's one point at which it becomes singular. Then I should be able to figure its singularity by precisely the same argument.

So the function that corresponds to x going to $1/x$ is this hyperbolic sine. So if there is a point which is the unique point that corresponds to the singularity, it has to be the point that is self-dual-- maps on to itself, just like 1. So $\sinh(2Kc)$ should be 1. And what is this?

Hyperbolic sine we can write as $\frac{e^{2Kc} - e^{-2Kc}}{2}$. We can manipulate this equation slightly to $e^{4Kc} - 2e^{2Kc} - 1 = 0$, which is a quadratic equation for e^{2Kc} . So I can immediately solve for e^{2Kc} is--

This has a 2, so I can say $1 \pm \sqrt{2}$. So that is a square root of 2.

The exponential better be positive, so I can't pick the negative solution. And so we know--

AUDIENCE: Isn't it [INAUDIBLE]?

PROFESSOR: Where did I-- OK. Multiply by e^{2Kc} .

AUDIENCE: Taking that as correct, I didn't check it--

PROFESSOR: Yes. OK. $x^2 - 2x + c = 0$. Well, actually, this is-- so x is $1 \pm \sqrt{1 - c}$. Our c is negative. So it's 1 ± 1 .

So the critical coupling that we have is $\frac{1}{2} \log(1 + \sqrt{2})$. So we know that the critical point of the Ising model occurs at this value, which you can put on your calculator. And it is something like this.

There is one assumption. Of course, there is essentially only one singularity. And there is one singularity in this free energy. But if it is, we have solved it for this case.

So I think to emphasize this was discovered around '50s by Wannier, this idea of duality. And suddenly, you had exact solution for something like the square Ising model. Question is, how much information does it give you?

First of all, the property of self-duality is that of the square lattice. So if I had done this on the triangular lattice, you would've seen that the low-temperature and high-temperature expansions don't match.

It turns out that in order to construct the dual of any lattice, what you have to do is to put, let's say, points in the center of the units that you have and see what lattice these centers make.

So when you try to do that for the triangular lattice, you will see that the centers form, actually, hexagonal lattice, and vice versa. However, there is a trick using duality that you can still calculate critical points of hexagonal and triangular lattice. And that you will do in one of the problem sets.

OK, secondly. Again, that trick allows you to go beyond square lattice, but it turns out that for reasons that we will see shortly, it is limited. And you can only do these kinds of dualities to yourself for two-dimensional lattices. And what these kinds of mappings in general give for two-dimensional lattices is potentially, but not always, the critical value of K_c .

And again, one of the things that you will see is that you can do this for other models in two dimension. For example, the Potts model we can calculate the critical point through this kind of procedure. However, it doesn't tell you anything about the nature of the singularity.

So essentially, what we've shown is that on the K -axis, there is some point maybe that describes the singularity that you are going to have. But the shape of this singularity, the exponent can be anything. And this mapping does not tell you anything about that.

It does tell you one thing. We also mentioned that the ratio of amplitudes above and

below for various singular quantities is something that is universal because of these mappings from high temperatures to low temperatures. Although I don't know what the nature of the singularity is, I know that the amplitude ratio is [INAUDIBLE]. So there is some universal information that one gains beyond the non-universal location of the critical point, but not that much more. OK. Any questions?

AUDIENCE: Is it possible to extract from this line a differential equation for g ?

PROFESSOR: Yes. And indeed, that differential relation you will use in one of the problem sets that I forgot to mention, and will be used to derive the value of the derivative, which is related to the energy of the system at the critical point. But you are right.

This is such a beautiful thing that maybe we can try to force it to work in higher dimensions. So let's see if we were to try to go with this approach for the 3D Ising model what would happen. So what did we do?

We wrote the low-temperature series, high-temperature series and compared them. Again, let's do the cubic lattice, which I will not really attempt to draw. That's the system that we want to calculate. So let's do the low T -series.

Our partition function is going to start with the state where every spin is, let's say, up. Three bonds per site on the cubic lattice. So it's $3NK$. Again, the trivial degeneracy of 2 for the two possible all plus or all minus states.

The first excitation is to flip a spin. So any one of N sites could have been flipped, creating a cube. A cube has 6 faces that go out, so there is essentially 6 bonds that are broken. So basically, there is this minus that is in a box surrounded by plus. And as you can see, $6 \text{ plus minus } 1$ that go out of that.

The next one would be when we have 2 minuses. And that can be oriented three ways in three dimensions. e^{-2K} . $1, 2 \text{ times } 4. 8 \text{ plus } 2. \text{ Times } 10.$ And so the general term in the series I have to draw some droplet of minuses in a sea of pluses. And then I would have e^{-2K} times the boundary or the area of this droplet.

Actually, droplets because there could be multiple droplets as we've seen. There's no problem with that.

If I do the high T, follow exactly the procedure I had described before, partition function is going to be 2 to the number of sites. Cosh K to the power of the number of bonds and there are 3 bonds per site. So there's 3N there. And then we start to draw our diagrams.

The first diagram is just exactly like what we had before. I have to make a square. And this square can be placed on any face of the cube. And there are 3 faces that are equivalent.

The next type of diagram that I can draw has 6 bonds in it. So this could be an example of that. And if you do the counting, there are 18N of those. And so you go. And the generic term in the series is going to be some kind of a loop. Again, even number of bonds per site is the operative term. And then I have t to the power of the number of bonds making this closed loop. Or loops.

So you stare at the series and you see immediately that there is no correspondence like we saw before. The coefficients here are 1N, 3N. Here, there are 1, 3, and 18N. Powers are 6, 10. Here are 4, 6. There's no correspondence between these two. So there is nothing that one could say.

But you say, I really like this. So maybe I'll phrase the question differently. Can I consider some other model whose high-temperature expansion reproduces this low-temperature expansion of the Ising model? So this is the question, can we find a model whose high T expansion reproduces low T of 3D Ising model?

So rather than knowing what the model is, now we are going to kind of work backward from this graphical picture that we have. So what would have been the analog thing over here, let's say that I had this picture of droplets in the 2D Ising model.

I recognize that I need to make these perimeters out of something. And I know that I can make these things that are joined together to a procedure such as the one that

we have over here. But the unit thing, there it was the elements that I had along the perimeter. What is the corresponding unit that I have for the low temperature series of the Ising model?

I have e to the minus $2K$ to the power of the number of faces. So first thing is unit has to be a face. So basically, what I need to do is to have a series, which is an expansion in terms of faces, and then somehow I can glue these faces together, like I glued these bonds together. So we found our unit.

The next thing that we need is some kind of a glue to put all of these LEGO faces together. So how did we join things together here?

We had these sigmas that were sitting by themselves. And then putting two sigmas together, I ensured that when I summed over sigma, I had to glue two of the T 's together. Can I do the same thing over here?

If I put the sigmas on the corners of these faces, you can see it doesn't work because here I have three. So I'm forced to put the sigmas on the lines that join the faces. So what I need to do is, therefore, to have a variable such as this where I have these sigmas sitting on the-- let's call this a plaquette, p . And this plaquette will be having around it four different bonds.

And if I have the product of these four bonds-- again, these sigmas being minus plus 1-- I am forced to glue these sigmas in pairs. And I can join these things together, these squares, to make whatever shape that I like that would correspond to the shapes that I have over there. So what I need to do is to have for each face a factor of this. So this is the analog of this factor that I have over here.

And then what I need to do is to do a product over all plaquettes. And I sum over all sigma tildes. And this would be the partition function of some other system.

In this other system, you can see that if I make its expansion, there will be a one-to-one correspondence between the terms in the expansion of this partition function and the 3D Ising model partition function. Again, this kind of term we have seen. If I had put factors of cosh here, which don't really do much, I can re-express as e to

the something-- $k \tilde{\sigma}_1 \sigma_2 \sigma_3 \sigma_4$. Essentially every time you see $1 + t$ times some binary variable, you can rewrite it into this fashion.

So what we have come up with is the following. That in order to construct the dual of the three-dimensional Ising model, what we do is you go all over your cube. On each bond of it, you put a variable that is minus plus 1. So previously for the Ising model, the variables were sitting on the sites. So Ising, these were site variables. Whereas, this dual Ising, these are the bond variables that are minus plus 1.

In the case of the Ising, the interactions were the product of sites making on a bond. Whereas, for the dual Ising, the interactions are around the face. There's four of them that go around the face.

But whatever this new theory is, we know that its free energy because of this relation is related to the free energy of the three-dimensional Ising model. Also, we know that the three-dimensional Ising model has a phase transition between a disordered phase and the magnetized phase at low temperature. There is a singularity.

As I span the parameter K of the three-dimensional Ising model, there is a K_c . Now, I can find out what that K_c is because I don't have self-duality. But I know that as I span the parameter K of the Ising model, I'm also spanning the parameter $K \tilde{\sigma}$ of this new theory. And since the original model has a phase transition, this new model must also have a phase transition. So there exists a K_c for both models.

You say, OK. Fine. But there is some complicated kind of Ising model that you have devised and it has a phase transition. What's the big deal?

Well, the big deal is that this model is not supposed to have a phase transition because it has a different type of symmetry. The symmetry that we have for the Ising model is a global symmetry. That is, the energy of a particular state is the energy of the state in which all of the spins are reversed. Because the form of the energy is bilinear. If I take all of the sigmas from one configuration and make them minus in that configuration, the energy will not change. But I have to do that

globally. It's a global symmetry.

Now, this model has a local symmetry because what I can do is I can pick one of the sites. And out of this site, there are six of these bonds that are going out on which there is one of these σ tilde.

If I pick this site and I change the sign of all of these six that emanate from this site, the energy will not change. Because the energy gets contributions from faces. And you can see that for any one of the faces, there are two σ 's that have changed. So the energy, which is the product of all four of them, has not changed.

So this model has a different form, which is a local symmetry. And in fact, it is very much related to gauge theories. It's a kind of discrete version of the gauge theories that you have seen in electromagnetism. Since there are two possibilities, it's sometimes called a Z_2 gauge theory.

Now, the thing about the gauge theories is that there is a theorem which states that local or these gauge theories, gauge symmetries, cannot be spontaneously broken. So for the case of the Ising model, we have this symmetry between σ going to minus σ . But yet, we know that if I go to low temperature, I will have a state in which globally all of the spins are either plus or minus. So there is a symmetry broken state which is what we have been discussing.

Now, the reason that that cannot take place in these gauge theories, I will just sketch what is happening. Essentially, we have been thinking in terms of this broken symmetries by putting an infinitesimal magnetic field. And we saw that, basically, if I'm at temperatures of $1/k$'s that are below some critical value, then if h is plus, everybody would be plus. If h is minus, everybody would be minus.

And the reason as you approach h goes to 0 from one site that you don't get average of 0 is because the difference between the energy of this state and that state as you go to 0 temperature is proportional to N times h . So although h is going to 0 with N being very large, the influence of infinitesimal h is magnified enormously.

Now, for the case of these local gauge theories, you cannot have a similar

argument. Because if I pick this spin, let's say one of these bond spins. And let's say is its average-- what is its average? As I said, the h going to 0.

Well, the difference between a state in which it is plus or the state in which it is minus is, in fact, $6h$. Because all I need to do is to pick a site that is close to that bond and flip all of the spins that are close to that. All of the K 's are equally satisfied. The difference between that state and the one where there is a flip is just $6h$. So that remains finite as h goes to 0. There is no barrier towards flipping those spins. So there is no broken symmetry in this system. So this can be proven very nicely and rigorously. So we have now two statements about this Ising version of a gauge theory.

First of all, we know that at low temperatures, still the average value of each bond is equally likely to be plus or minus. From that perspective of local values of these bond spins, it is as disordered as the highest-temperature phase. Yet, because of its duality to the three-dimensional Ising model, we know that it undergoes some kind of a singularity going from high temperatures to low temperatures. So there is probably some kind of a phase transition, but it has to be very different from any of the phase transitions that we have discussed so far because there is no spontaneous symmetry breaking. So what's going on?

Now, later on in the course, we will see another example of this that is much less exotic than a gauge theory, but it has the same kind of principle applicable to it. There will be a phase transition without local symmetry breaking in something like a superfluid in two dimensions.

So one thing that that phase transition and this one have in common is, again, the lack of this local-order parameter from symmetry breaking. And both of them share something that was pointed out by Wegner once this puzzle emerged, which is that one has to look at some kind of a global-- well, I shouldn't even call it global. It is something that is called a Wilson loop.

So the idea is the following. That we have these variable σ tilde that are sitting on the bonds of a lattice. Now, the problem is that with this local transformations, I

can very easily make this sigma go to minus sigma. So that is not a good thing to consider. However, what was the problem?

Let's say I pick this site. All of the sigmas that went out of that site I changed to minus themselves. That was the gauge transformation, but it became minus.

But if I multiply the sigma with another sigma that goes out of that site, then I have cured that problem.

If I changed this to minus itself, this changes, this changes. The product remains [INAUDIBLE]. But then I have the problem here. So what I do is I make a long loop.

I look at the expectation value. So this Wilson loop is the product of sigma tilde around a loop.

And what I can do is I can look at the average of that quantity. The average of that quantity is something that is clearly invariant to this kind of gauge transformation. So the signatures of a potential phase transition could potentially be revealed by looking at something like this.

But clearly, that is a quantity that is always also going to be positive. So the thing that I am looking at is not that this quantity is, say, positive in one phase and 0 in the other phase. It is on how this quantity depends on the shape and characteristics of these loop. So what I can do is I can calculate this average, both in high temperatures and low temperatures, and compare them.

So if I look at-- let's starts with, yeah, high temperatures. The high-temperature expansion. So I want to calculate the expectation value of the product of sigma tilde, let's call it i , where i belongs to some kind of a loop c . So c is all of these bonds. I want to calculate that expectation value. How do I calculate that expectation value?

Well, I have to sum over all configurations of this product with a weight. What is my weight?

My weight is this factor of product over all plaquettes of $1 + t \text{ sigma tilde sigma}$

tilde sigma tilde for the plaquette. So this is the weight that I have. I can put the hyperbolic cosine or not put it, it doesn't matter.

But then this weight has to be properly normalized. It means that I have to divide by something in the denominator, which basically does not include the quantity that I am averaging.

So the graphs that occur in the denominator are the things that we have been discussing. Essentially, I start with 1. The next term is to put these faces together to make a cube, and then more complicated shapes. That, essentially, every bond is going to be having some kind of a complement. Well, but what about the terms in the numerator?

For the terms in the numerator, I have these factors of sigma tilde that are lying all around this loop. And I'm summing over the two possible values. So in order that summing over this does not give me 0, I better make sure that there is a complement to that.

The complement to that can only come from here. So for example, I would put a face over here that ensures that that is squared and that is squared. But then I have this one. Then I will put another one here. I will put another one, et cetera.

And you can see that the lowest-order term that I would get is this factor that characterize each one of them. Raised to the power of the area of this loop c .

You can ask higher-order terms. You can kind of build a hat on top of this. This you can put anywhere, again, along this area of this thing. So the next correction in this series you can see is also going to be something that will be 1 plus t to the fourth times the area.

The point is that as you add more and more terms, you preserve the structure that the whole thing is going to be proportional to the area of loop times some function of this parameter t .

So what we know is that if I take the expectation value of this entity, then it's

logarithm will be proportional in high temperatures to the area of this loop. Now, what happens if I try to do a low T-series for the same quantity?

So I have to start with a configuration at low temperature that minimizes the energy. One configuration clearly is one where all of the sigmas are plus. And that will give me a term.

If I am calculating the partition function in the denominator, there will be a term that will be proportional to e to whatever this K tilde is per face. And there are three faces of a cube. There are N cubed, so there will be $3K$ tilde N for the configuration that is all plus.

But this is not the only low-temperature configuration. That is what we were discussing. Because I can pick a site out of N site and the 6 bonds that go out of it, I can make minus themselves. And the energy would be exactly the same. So whereas the Ising model, I had a multiplicity of 2. Here, there is a multiplicity of 2 to the N . So that's the lowest term in the low-temperature expansion of the partition function. I'm doing the partition function, which is the denominator first. Then, what can I do?

Then, let's say I start with a configuration where all of them are pluses. There are, of course, 2 to the N gauge copies of that. So whatever I do to this configuration, I can do the analog in all of the others. But let's keep the copy where all of the sigmas are plus, and then I flip one of the sigmas to minus.

Then, it's essentially-- think of a cube. There is a line that was plus and I made it minus. There are four faces going out of that that were previously plus K , now become minus K . So I will have $2K$ tilde times 4 because of these four things that are going out.

And the bond I can orient in x -, y -, or z -direction. So there are $3N$ possibilities. And so I could have a series such as this in the denominator where subsequent terms would be to put more and more minus in this particular [INAUDIBLE].

Now, let's see how these series would affect the sum that I would have to do in

order to calculate this expectation value. For any one of these configurations, since I am kind of looking at the ground state, let's say they're all pluses. Clearly, the contribution to this product will be unity. That does not change.

But now, let's think about the configurations in which one of the bonds is made to flip. As long as that bond does not touch any of the bonds that are part of the loop, the value of the loop will remain the same. So let's say that the loop has P sites. So this is number of bonds in c . Let's call it P_c .

For these, with this weight, the value of Wilson loop would still be plus. But for the times where I have picked one of these to become minus, then the product becomes minus. So for the remaining P_c times of this factor, e to the minus $2K$ tilde times 4, rather than having plus I will have minus. So you can see that-- what is this?

N should be up here. That the difference between what is in the numerator and what is in the denominator of this low-temperature series has to do with the bonds that have been sitting as part of the Wilson loop. And if I imagine that this is a small quantity and write these as exponentials, you can see that this is going to start with e to the minus $2K$ tilde times 4 times the perimeter of this cluster. Of this loop.

And you can go and look at higher and higher order terms. The point is that in high temperature, the property of the shape of the loop that determines this expectation value is its area. Whereas, in the low-temperature expansion, it is its perimeter. So you could, for example, calculate the-- for a large loop, the log of this quantity and divide it by the perimeter. And in the low-temperature phase, it would be finite. In the high-temperature phase, it would go to 0 because the area scale is bigger than the perimeter. So we have found something that is an analog of an older parameter and can distinguish the different phases, and it is reflected in the way that the correlations take place.

Let's try to sort of think about some potential physics that could be related to this. Let's start with the gauge theory aspect of this.

Well, the one gauge theory that you probably know is quantum electrodynamics, whose action you would write in the following way. The action would involve an integration over space as well as time.

And by appropriate rescalings, you can write the energy that is in the electromagnetic field as $\int d^4x \frac{1}{2} F_{\mu\nu} F^{\mu\nu}$, where A is the 4-vector potential out of which you can construct the electric field and magnetic field.

And the reason this is a gauge theory is because if I take A and add to it some function of ϕ of x and t , as long as I take the derivative of this function, you can see that the change here would be $\partial_\mu \phi - \partial_\nu \phi$. There is really no change. And we know that basically you can choose whatever value of this phase-- this gauge fixing potential over here.

Now, this is the electromagnetic field by itself. If you want to couple it to something like matter or electrons, you write something like $\bar{\psi} \gamma^\mu \psi \partial_\mu A$, which is some derivative, and then you have $e \bar{\psi} \psi A$. If there's a mass to this object, you would put it here. This would be something like $\bar{\psi} \psi$. This would be something that would describe the coupling of this electromagnetic field to some charged particle, such as the electron.

And this entire thing satisfies the gauge symmetry provided that once you do this, you also replace ψ with $e^{i\phi} \psi$. So the same ϕ , if it appears in both, essentially the change in A that you would have from here will be compensated by the change that you would get from the derivative acting on this phase factor. And so the whole thing is not affected.

What we have constructed in this model is kind of an Ising analog of this theory. Because the Hamiltonian that we have, which carries the weight after exponentiate of the different configurations, has a part which is the sum over all of the plaquettes of this $\sigma_{\mu\nu} \sigma_{\mu\nu}$, the four bonds around the plaquette.

We could put some kind of a coupling here if we want to. And the analog of this

transformation that we have-- well, maybe it will become more apparent if I add the next term, which is the analog of the coupling to matter. If I put a spin and then σ_{ij} .

So again, imagine that we have our cubic lattice, or some other lattice, in which we have these variables σ that are sitting on the bonds. And the first term is the product around the face. And the second term I put these variables s that are sitting here.

And I have made a coupling between these two s 's. So if the σ 's were not there, I could make an Ising model with s 's being plus or minus, which are coupled across nearest neighbors.

What I do is that the strength of that coupling I make to be plus or minus, depending on the value of the gauge field, if you like. This Ising gauge field that is sitting over there.

Now, the analog of these symmetries that we have for QED is as follows. I can pick a particular s_i and change its value to minus. And I can pick all of the σ 's that go out of that i to the neighbours and simultaneously make them minus. And then this energy would not change.

First of all, let's say if I pick this site and change its face from being plus or minus, then the bonds that-- the σ 's that go out of it will change their values to minus themselves. Since each one of the faces contains two of them, the value of the energy from here is not changed. Since the couplings to the neighboring s 's involve the σ 's that sit between them, and I have flipped both s and σ , those do not change irrespective of what I do with the face of all the other ones. So I have made an Ising, or binary version, of this transformation, constructed a model that has, except being Ising symmetry, a lot of the properties that you would have for this kind of action.

Now again, continuing with that, this difference between why we see an area rule or a perimeter rule has some physical consequence that is worth mentioning. And this,

again, has not much to do with the main thrust of this course, but just a matter of overall education. It is useful to know.

So in this picture, where one of the dimensions corresponds to time, imagine that you create a kind of a Wilson loop which is very long in one direction that I want to think of as being time. And the analog of this action that we have discussed is to create a pair of charges, separate them by a distance x , propagate them for a long time t , and then [INAUDIBLE]. And ask, what is the contribution of a configuration such as this to their action on average?

And so you would say that if particles are at a distance x , they are subject to some kind of a potential v of x . And if this potential has been propagated in time for a length or a duration that I will call T , that the effect that it has on the system is to have an interaction such as this in the action propagated over a time T . So I should have something like this. So this should somehow be related to this average of the Wilson loop in manner to not be made precise, but very rough just to get the general idea.

And so what we have said is that the value of this Wilson loop has different behaviors at high T and low T in its dependence on shape. And that high temperatures, it is proportional to the area. So it should be proportional to xT . Whereas, in low temperature, it is proportional to perimeter. So it should be proportional to x plus T .

So if I read off the form of V of x from these two dependents, you will see that V of x goes proportionately to x in one regime. And once I divide by T , the leading coefficients-- so this is essentially I want to look at the limit where T is becoming very large. So this thing when I divide by T goes to a constant. And I can very roughly interpret this as the interaction between particles that are separated by x via this kind of theory.

And I see that for particles that are separated by x in that kind of theory, there is a weak coupling-- high temperature corresponds to weak coupling-- where the further apart I go, the potential that is bringing them together becomes linearly stronger. So

this is what is called confinement. Whereas, in the other limit of low temperatures, or strong coupling, what you find is that the interaction between them essentially goes to a constant. The potential goes to a constant, so the force would go to 0. So they are asymptotically free.

So if I start with this kind of theory and try to interpret it in the language of quantum field theory as something that describes interaction between particles, I find that it has potentially two phases. One phase in which the particles of the theory are strongly bind together, like quarks that are inside the nucleus. And you can try to separate the quarks, but they would snap back. You can't have free quarks.

And then there is another phase where essentially the particles don't see each other. And indeed, quarks right inside the nucleus are essentially free. We can sort of regard them as free particles. So this theory actually has aspects of what is known as confinement and asymptotic freedom within quantum chromodynamics.

The difference is that in this theory, there is a phase transition and the two behaviors are separated from each other. Whereas in QCD, it's essentially a crossover from one behavior to another behavior without the phase transition.

So we started by thinking about these Ising models. And we kind of branched into theories that describe loops, theories that describe droplets, theories that describe gauge couplings, et cetera. So you can see that that nice, simple line of the partition function that I have written for you has within it a lot of interesting complexity.

We kind of went off the direction that we wanted to go with phase transitions, so we will remedy that next time, coming back to thinking about how to think in terms of the Ising model, and try to do more with understanding the behavior and singularities of this partition function.