Review Problems

The test is 'closed book,' but if you wish you may bring a one-sided sheet of formulas. The intent of this sheet is as a reminder of important formulas and definitions, and not as a compact transcription of the answers provided here. The test will be composed entirely from a subset of the following problems **as well as those in problem sets 5 and 6**. Thus if you are familiar and comfortable with these problems, there will be no surprises!

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1. *Continuous spins:* In the standard $\mathcal{O}(n)$ model, n component unit vectors are placed on the sites of a lattice. The nearest neighbor spins are then connected by a bond $J\vec{s}_i \cdot \vec{s}_j$. In fact, if we are only interested in universal properties, any generalized interaction $f(\vec{s}_i \cdot \vec{s}_j)$ leads to the same critical behavior. By analogy with the Ising model, a suitable choice is

$$\exp\left[f(\vec{s}_i \cdot \vec{s}_j)\right] = 1 + (nt)\vec{s}_i \cdot \vec{s}_j,$$

resulting in the so called *loop model*.

(a) Construct a high temperature expansion of the loop model (for the partition function Z) in the parameter t, on a two-dimensional *hexagonal* (honeycomb) lattice.

(b) Show that the limit $n \to 0$ describes the configurations of a single self-avoiding polymer on the lattice.

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2. *Potts model I:* Consider Potts spins $s_i = (1, 2, \cdots, q)$, interacting via the Hamiltonian $-\beta\mathcal{H} = K\sum_{<ij>}\delta_{s_i,s_j}$.

(a) To treat this problem graphically at high temperatures, the Boltzmann weight for each bond is written as

$$\exp\left(K\delta_{s_i,s_j}\right) = C(K)\left[1 + T(K)g(s_i, s_j)\right],$$

with $g(s, s') = q\delta_{s,s'} - 1$. Find $C(K)$ and $T(K)$.

(b) Show that

$$\sum_{s=1}^{q} g(s, s') = 0 \ , \ \sum_{s=1}^{q} g(s_1, s)g(s, s_2) = qg(s_1, s_2) \ , \ \text{and} \ \sum_{s,s'}^{q} g(s, s')g(s', s) = q^2(q-1).$$

(c) Use the above results to calculate the free energy, and the correlation function $\langle g(s_m, s_n) \rangle$ for a one–dimensional chain.

(d) Calculate the partition function on the square lattice to order of T^4. Also calculate the first term in the low–temperature expansion of this problem.

(e) By comparing the first terms in low- and high–temperature series, find a duality rule for Potts models. Don't worry about higher order graphs, they will work out! Assuming a single transition temperature, find the value of $K_c(q)$.

(f) How do the higher order terms in the high–temperature series for the Potts model differ from those of the Ising model? What is the fundamental difference that sets apart the graphs for $q = 2$? (This is ultimately the reason why only the Ising model is solvable.)

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3. *Potts model II:* An alternative expansion is obtained by starting with

$$\exp\left[K\delta(s_i, s_j)\right] = 1 + v(K)\delta(s_i, s_j),$$

where $v(K) = e^K - 1$. In this case, the sum over spins *does not* remove any graphs, and all choices of distributing bonds at random on the lattice are acceptable.

(a) Including a magnetic field $h\sum_i \delta_{s_i,1}$, show that the partition function takes the form

$$Z(q, K, h) = \sum_{\text{all graphs}} \prod_{\text{clusters c in graph}} \left[v^{n_b^c} \times \left(q - 1 + e^{hn_s^c}\right)\right],$$

where n_b^c and n_s^c are the numbers of bonds and sites in cluster c. This is known as the *random cluster expansion*.

(b) Show that the limit $q \to 1$ describes a *percolation* problem, in which bonds are randomly distributed on the lattice with probability $p = v/(v+1)$. What is the percolation threshold on the square lattice?

(c) Show that in the limit $q \to 0$, only a single connected cluster contributes to leading order. The enumeration of all such clusters is known as listing *branched lattice animals*.

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4. *Potts duality:* Consider Potts spins, $s_i = (1, 2, \cdots, q)$, placed on the sites of a *square lattice* of N sites, interacting with their nearest-neighbors through a Hamiltonian

$$-\beta\mathcal{H} = K\sum_{<ij>} \delta_{s_i,s_j}.$$

(a) By comparing the first terms of high and low temperature series, or by any other method, show that the partition function has the property

$$Z(K) = qe^{2NK}\Xi\left[e^{-K}\right] = q^{-N}\left[e^K + q - 1\right]^{2N}\Xi\left[\frac{e^K - 1}{e^K + (q-1)}\right],$$

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for some function Ξ, and hence locate the critical point $K_c(q)$.

(b) Starting from the duality expression for $Z(K)$, derive a similar relation for the internal energy $U(K) = \langle \beta \mathcal{H} \rangle = -\partial \ln Z / \partial \ln K$. Use this to calculate the exact value of U at the critical point.

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5. *Anisotropic Random Walks:* Consider the ensemble of all random walks on a square lattice starting at the origin (0,0). Each walk has a weight of $t_x^{\ell_x} \times t_y^{\ell_y}$, where ℓ_x and ℓ_y are the number of steps taken along the x and y directions respectively.

(a) Calculate the total weight $W(x, y)$, of all walks terminating at (x, y). Show that W is well defined only for $\bar{t} = (t_x + t_y)/2 < t_c = 1/4$.

(b) What is the shape of a curve $W(x, y) = \text{constant}$, for large x and y, and close to the transition?

(c) How does the average number of steps, $\langle \ell \rangle = \langle \ell_x + \ell_y \rangle$, diverge as $\bar{t}$ approaches t_c?

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6. *Anisotropic Ising Model:* Consider the anisotropic Ising model on a square lattice with a Hamiltonian

$$-\beta \mathcal{H} = \sum_{x,y} \left(K_x \sigma_{x,y} \sigma_{x+1,y} + K_y \sigma_{x,y} \sigma_{x,y+1} \right);$$

i.e. with bonds of different strengths along the x and y directions.

(a) By following the method presented in the text, calculate the free energy for this model. You do not have to write down every step of the derivation. Just sketch the steps that need to be modified due to anisotropy; and calculate the final answer for $\ln Z / N$.

(b) Find the critical boundary in the (K_x, K_y) plane from the singularity of the free energy. Show that it coincides with the condition $K_x = \tilde{K}_y$, where $\tilde{K}$ indicates the standard dual interaction to K.

(c) Find the singular part of $\ln Z / N$, and comment on how anisotropy affects critical behavior in the exponent and amplitude ratios.

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7. *Müller–Hartmann Zittartz estimate* of the interfacial energy of the $d = 2$ Ising model on a square lattice:

(a) Consider an interface on the square lattice with periodic boundary conditions in one direction. Ignoring islands and overhangs, the configurations can be labelled by heights h_n for $1 \leq n \leq L$. Show that for an ansiotropic Ising model of interactions (K_x, K_Y), the energy of an interface along the x-direction is

$$-\beta \mathcal{H} = -2K_y L - 2K_x \sum_n |h_{n+1} - h_n| .$$

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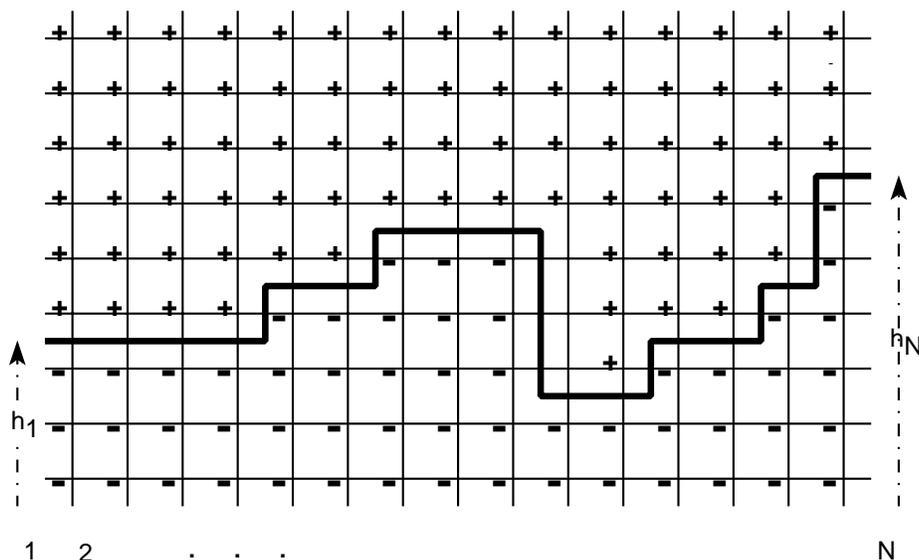
(b) Write down a column–to–column transfer matrix $\langle h|T|h'\rangle$, and diagonalize it.

(c) Obtain the interface free energy using the result in (b), or by any other method.

(d) Find the condition between K_x and K_y for which the interfacial free energy vanishes. Does this correspond to the critical boundary of the original 2d Ising model?

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8. *Anisotropic Landau Theory:* Consider an n–component magnetization field $\vec{m}(\mathbf{x})$ in d–dimensions.

(a) Using the previous problems on anisotropy as a guide, generalize the standard Landau–Ginzburg Hamiltonian to include the effects of spacial anisotropy.

(b) Are such anisotropies "relevant?"

(c) In La_2CuO_4, the Cu atoms are arranged on the sites of a square lattice in planes, and the planes are then stacked together. Each Cu atom carries a "spin", which we assume to be classical, and can point along any direction in space. There is a very strong antiferromagnetic interaction in each plane. There is also a very weak interplane interaction that prefers to align successive layers. Sketch the low–temperature magnetic phase, and indicate to what universality class the order–disorder transition belongs.

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9. *Anisotropic nonlinear σ model:* Consider unit n-component spins, $\vec{s}(\mathbf{x}) = (s_1, \cdots, s_n)$ with $\sum_\alpha s_\alpha^2 = 1$, subject to a Hamiltonian

$$\beta\mathcal{H} = \int d^d\mathbf{x} \left[\frac{1}{2T} (\nabla\vec{s})^2 + gs_1^2\right].$$

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For $g = 0$, renormalization group equations are obtained through rescaling distances by a factor $b = e^{\ell}$, and spins by a factor $\zeta = b^{y_s}$ with $y_s = -\frac{(n-1)}{4\pi}T$, and lead to the flow equation

$$\frac{dT}{d\ell} = -\epsilon T + \frac{(n-2)}{2\pi}T^2 + \mathcal{O}(T^3),$$

where $\epsilon = d - 2$.

(a) Find the fixed point, and the thermal eigenvalue y_T.

(b) Write the renormalization group equation for g in the vicinity of the above fixed point, and obtain the corresponding eigenvalue y_g.

(c) Sketch the phase diagram as a function of T and g, indicating the phases, and paying careful attention to the shape of the phase boundary as $g \to 0$.

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10. *Matrix models:* In some situations, the order parameter is a matrix rather than a vector. For example, in triangular (Heisenberg) antiferromagnets each triplet of spins aligns at 120°, locally defining a plane. The variations of this plane across the system are described by a 3×3 rotation matrix. We can construct a nonlinear σ model to describe a generalization of this problem as follows. Consider the Hamiltonian

$$\beta\mathcal{H} = \frac{K}{4}\int d^d\mathbf{x} \ \mathrm{tr}\left[\nabla M(\mathbf{x})\cdot\nabla M^T(\mathbf{x})\right] \quad,$$

where M is a *real, $N \times N$ orthogonal* matrix, and 'tr' denotes the trace operation. The condition of orthogonality is that $MM^T = M^TM = I$, where I is the $N \times N$ identity matrix, and M^T is the transposed matrix, $M^T_{ij} = M_{ji}$. The partition function is obtained by summing over all matrix functionals, as

$$Z = \int \mathcal{D}M(\mathbf{x})\delta\left(M(\mathbf{x})M^T(\mathbf{x}) - I\right)e^{-\beta\mathcal{H}[M(\mathbf{x})]} \quad.$$

(a) Rewrite the Hamiltonian and the orthogonality constraint in terms of the matrix elements M_{ij} $(i, j = 1, \cdots, N)$. Describe the ground state of the system.

(b) Define the symmetric and anti-symmetric matrices

$$\begin{cases} \sigma = \dfrac{1}{2}\left(M + M^T\right) = \sigma^T \\ \pi = \dfrac{1}{2}\left(M - M^T\right) = -\pi^T \end{cases} \quad.$$

Express $\beta\mathcal{H}$ and the orthogonality constraint in terms of the matrices σ and π.

(c) Consider *small fluctuations* about the ordered state $M(\mathbf{x}) = I$. Show that σ can be expanded in powers of π as

$$\sigma = I - \frac{1}{2}\pi\pi^T + \cdots.$$

Use the orthogonality constraint to integrate out σ, and obtain an expression for βH to fourth order in π. Note that there are two distinct types of fourth order terms. *Do not include* terms generated by the argument of the delta function. As shown for the nonlinear σ model in the text, these terms do not effect the results at lowest order.

(d) For an N-vector order parameter there are $N-1$ Goldstone modes. Show that an orthogonal $N \times N$ order parameter leads to $N(N-1)/2$ such modes.

(e) Consider the quadratic piece of $\beta\mathcal{H}$. Show that the two point correlation function in Fourier space is

$$\langle \pi_{ij}(\mathbf{q})\pi_{kl}(\mathbf{q}') \rangle = \frac{(2\pi)^d \delta^d(\mathbf{q}+\mathbf{q}')}{Kq^2} \left[\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} \right].$$

We shall now construct a renormalization group by removing Fourier modes $M^>(\mathbf{q})$, with $\mathbf{q}$ in the shell $\Lambda/b < |\mathbf{q}| < \Lambda$.

(f) Calculate the coarse grained expectation value for $\langle \text{tr}(\sigma) \rangle_0^>$ at low temperatures after removing these modes. Identify the scaling factor, $M'(\mathbf{x}') = M^<(\mathbf{x})/\zeta$, that restores $\text{tr}(M') = \text{tr}(\sigma') = N$.

(g) Use perturbation theory to calculate the coarse grained coupling constant $\tilde{K}$. Evaluate only the two diagrams that directly renormalize the $(\nabla\pi_{ij})^2$ term in $\beta\mathcal{H}$, and show that

$$\tilde{K} = K + \frac{N}{2} \int_{\Lambda/b}^{\Lambda} \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{1}{q^2} \quad .$$

(h) Using the result from part (f), show that after matrix rescaling, the RG equation for K' is given by:

$$K' = b^{d-2} \left[K - \frac{N-2}{2} \int_{\Lambda/b}^{\Lambda} \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{1}{q^2} \right] \quad .$$

(i) Obtain the *differential* RG equation for $T = 1/K$, by considering $b = 1 + \delta\ell$. Sketch the flows for $d < 2$ and $d = 2$. For $d = 2 + \epsilon$, compute T_c and the critical exponent ν.

(j) Consider a small symmetry breaking term $-h \int d^d\mathbf{x}\, \text{tr}(M)$, added to the Hamiltonian. Find the renormalization of h, and identify the corresponding exponent y_h.

Combining RG and symmetry arguments, it can be shown that the 3×3 matrix model is perturbatively equivalent to the $N = 4$ vector model at all orders. This would suggest

that stacked triangular antiferromagnets provide a realization of the $\mathcal{O}(4)$ universality class; see P. Azaria, B. Delamotte, and T. Jolicoeur, J. Appl. Phys. **69**, 6170 (1991). However, non-perturbative (topological aspects) appear to remove this equivalence as discussed in S.V. Isakov, T. Senthil, Y.B. Kim, Phys. Rev. B **72**, 174417 (2005).

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11. *The roughening transition:* Consider a continuum interface model which in $d = 3$ is described by the Hamiltonian

$$\beta \mathcal{H}_0 = -\frac{K}{2} \int d^2\mathbf{x} \, (\nabla h)^2 \quad,$$

where $h(\mathbf{x})$ is the interface height at location $\mathbf{x}$. For a crystalline facet, the allowed values of h are multiples of the lattice spacing. In the continuum, this tendency for integer h can be mimicked by adding a term

$$-\beta U = y_0 \int d^2\mathbf{x} \, \cos(2\pi h),$$

to the Hamiltonian. Treat $-\beta U$ as a perturbation, and proceed to construct a renormalization group as follows:

(a) Show that

$$\left\langle \exp\left[i \sum_\alpha q_\alpha h(\mathbf{x}_\alpha) \right] \right\rangle_0 = \exp\left[\frac{1}{K} \sum_{\alpha < \beta} q_\alpha q_\beta C(\mathbf{x}_\alpha - \mathbf{x}_\beta) \right]$$

for $\sum_\alpha q_\alpha = 0$, and zero otherwise. ($C(\mathbf{x}) = \ln|\mathbf{x}|/2\pi$ is the Coulomb interaction in two dimensions.)

(b) Prove that

$$\left\langle |h(\mathbf{x}) - h(\mathbf{y})|^2 \right\rangle = -\frac{d^2}{dk^2} G_k(\mathbf{x} - \mathbf{y}) \bigg|_{k=0},$$

where $G_k(\mathbf{x} - \mathbf{y}) = \left\langle \exp\left[ik\big(h(\mathbf{x}) - h(\mathbf{y})\big)\right]\right\rangle$.

(c) Use the results in (a) to calculate $G_k(\mathbf{x} - \mathbf{y})$ in perturbation theory to order of y_0^2. (Hint: Set $\cos(2\pi h) = \left(e^{2i\pi h} + e^{-2i\pi h}\right)/2$. The first order terms vanish according to the result in (a), while the second order contribution is identical in structure to that of the Coulomb gas described in this chapter.)

(d) Write the perturbation result in terms of an effective interaction K, and show that perturbation theory fails for K larger than a critical K_c.

(e) Recast the perturbation result in part (d) into renormalization group equations for K and y_0, by changing the "lattice spacing" from a to ae^ℓ.

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(f) Using the recursion relations, discuss the phase diagram and phases of this model.

(g) For large separations $|\mathbf{x} - \mathbf{y}|$, find the magnitude of the discontinuous jump in $\langle |h(\mathbf{x}) - h(\mathbf{y})|^2 \rangle$ at the transition.

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12. *Roughening and duality:* Consider a discretized version of the Hamiltonian in the previous problem, in which for each site i of a square lattice there is an integer valued height h_i. The Hamiltonian is

$$\beta \mathcal{H} = \frac{K}{2} \sum_{<i,j>} |h_i - h_j|^\infty \quad ,$$

where the "∞" power means that there is no energy cost for $\Delta h = 0$; an energy cost of $K/2$ for $\Delta h = \pm 1$; and $\Delta h = \pm 2$ or higher *are not allowed* for neighboring sites. (This is known as the restricted solid on solid (RSOS) model.)

(a) Construct the dual model either diagrammatically, or by following these steps:

(i) Change from the N site variables h_i, to the $2N$ bond variables $n_{ij} = h_i - h_j$. Show that the sum of n_{ij} around any plaquette is constrained to be zero.

(ii) Impose the constraints by using the identity $\int_0^{2\pi} d\theta e^{i\theta n}/2\pi = \delta_{n,0}$, for integer n.

(iii) After imposing the constraints, you can sum freely over the bond variables n_{ij} to obtain a dual interaction $\tilde{v}(\theta_i - \theta_j)$ between dual variables θ_i on neighboring plaquettes.

(b) Show that for large K, the dual problem is just the XY model. Is this conclusion consistent with the renormalization group results of the previous problem? (Also note the connection with the loop model.)

(c) Does the one dimensional version of this Hamiltonian, i.e. a 2d interface with

$$-\beta \mathcal{H} = -\frac{K}{2} \sum_i |h_i - h_{i+1}|^\infty \quad ,$$

have a roughening transition?

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